
β-open set and β-closed set in fuzzy topological space are introduced based on work of Mubarki [7]. Also some properties and some theorems are investigated.

The present paper (X, τ), (Y, σ) (or simply X, Y) represents non-empty fuzzy topological spaces. Let μ be a fuzzy subset of a space X. The fuzzy closure of μ, fuzzy interior of μ, fuzzy δ-closure of μ and the fuzzy δ-interior of μ are denoted by cl(μ), int(μ), clδ(μ) and intδ(μ) respectively. The fuzzy δ-interior of fuzzy subset μ of X is the union of all fuzzy regular open sets contained in μ. A fuzzy subset μ is called fuzzy δ-open [8] if μ = intδ(μ). The complement of fuzzy δ-open set is called fuzzy δ-closed (i.e., μ = clδ(μ)).

2. Fuzzy Topological Space

Definition 2.1. A family τ ⊆ ⅅX of fuzzy subsets is called a fuzzy topology (in the sense of Chang [2]) for X if it satisfies the following conditions:

(i) 0X, 1X ∈ τ.
(ii) λ, μ ∈ τ, then λ ∨ μ ∈ τ.
(iii) λi ∈ τ for each i ∈ I, then ∨ieiλi ∈ τ.

Members of τ are called fuzzy open subsets and the complement of fuzzy open subsets is called fuzzy closed subsets on fuzzy topological space (X, τ).

Definition 2.2. A fuzzy set μ of (X, τ) is called as follows [9–13]:

(i) Fuzzy semi open if μ ≤ cl(int(μ)).
(ii) Fuzzy α-open if μ ≤ cl(int(μ)).
(iii) Fuzzy pre-open (fuzzy pre-closed) if μ ≤ cl (int(μ)) (cl (int(μ)) ≤ μ).
(iv) Fuzzy regular open if μ = cl (int(μ)) (μ = int (cl(μ)) closed).
(v) Fuzzy e-open if μ ≤ cl (intδ(μ)) ∨ int (clδ(μ)) (μ ≥ cl (intδ(μ)) ∧ int (clδ(μ)) closed).
3. Fuzzy $\beta^*$-open Sets in Fuzzy Topological Space

In this section a new open set in fuzzy topological space is introduced.

**Definition 3.1.** A fuzzy subset $\mu$ of a fuzzy topological space $(X, \tau)$ is said to be $\beta^*$-open set if

$$
\mu \leq \text{cl} \left( \text{int}(\text{cl}(\mu)) \right) \lor \text{int} (\text{cl}_\delta(\mu)).
$$

**Example 3.2.** $X = \{x, y, z\}$ and the fuzzy topology

$$
\tau = \left\{ 0, 1, \{x_0, y_0, z_0, 1\}, \{x_0, y_0, z_0, 1\}, \{x_0, y_0, z_0, 1\} \right\}
$$

and

$$
\tau^c = \left\{ 0, 1, \{x_0, y_0, z_0, 1\}, \{x_0, y_0, z_0, 1\}, \{x_0, y_0, z_0, 1\}, \{x_0, y_0, z_0, 1\} \right\}.
$$

Let $\mu = \{x_0, y_0, z_0, 1\}, \text{cl} (\text{int}(\text{cl}(\mu))) \lor \text{int} (\text{cl}_\delta(\mu))$

$$
= \{x_0, y_0, z_0, 1\}.
$$

i.e., $\mu$ is $\beta^*$-open set.

**Remark 3.3.** From the definitions I obtain the following diagram holds for each a subset of $\mu$ of $X$.

```
Fuzzy regular open
↓
Fuzzy $\delta$-open
↓
Fuzzy open $\rightarrow$ Fuzzy semi open $\rightarrow$ Fuzzy $\gamma$-open $\rightarrow$ Fuzzy $\beta$-open
↓
Fuzzy $\delta$-pre open $\rightarrow$ Fuzzy $\beta^*$-open
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**Result 3.4.**

(i) Fuzzy $\beta^*$-open is fuzzy $\delta$-preopen if

$$
\text{cl} (\text{int}(\text{cl}(\mu))) = 0.
$$

(ii) Fuzzy $\beta^*$-open is $\beta$-open if

$$
\text{int}(\text{cl}_\delta(\mu)) = 0.
$$

**Proposition 3.5.** If $\mu$ is fuzzy $\delta$-pre open and $\lambda$ is fuzzy $\beta$-open then $\mu \lor \lambda$ is fuzzy $\beta^*$-open.

**Proof.** Obvious from Definition 3.1.

**Proposition 3.6.** Let $(X, \tau)$ be a fuzzy topological space. Then the union of any two fuzzy $\beta^*$-open sets is an $\beta^*$-open set.

**Proof.** Let $\mu_1, \mu_2$ be two fuzzy $\beta^*$-open sets,

$$
\mu_1 \leq \text{cl} (\text{int}(\text{cl}(\mu_1))) \lor \text{int} (\text{cl}_\delta(\mu_1))
$$

and

$$
\mu_2 \leq \text{cl} (\text{int}(\text{cl}(\mu_2))) \lor \text{int} (\text{cl}_\delta(\mu_2))
$$

(by Definition 3.1)

Then we have,

$$
\mu_1 \lor \mu_2 \leq \text{cl} (\text{int}(\text{cl}(\mu_1))) \lor \text{int} (\text{cl}_\delta(\mu_1))
$$

and

$$
\mu_1 \lor \mu_2 \leq \text{cl} (\text{int}(\text{cl}(\mu_2))) \lor \text{int} (\text{cl}_\delta(\mu_2)).
$$

Since, the arbitrary union of fuzzy $\beta^*$-open sets is fuzzy $\beta^*$-open set.

**Theorem 3.7.** Let $(X, \tau)$ be a fuzzy topological space and let $\{\mu_a\}_{a \in F}$ be the collection of fuzzy $\beta^*$-open sets in fuzzy topological space $X$, then $\bigvee_{a \in F} (\mu_a)$ is fuzzy $\beta^*$-open set.

**Proof.** Let $F$ be the collection of fuzzy $\beta^*$-open sets in fuzzy topological space $(X, \tau)$.

For each $\alpha \in F$, $\mu_a \leq \text{cl} (\text{int}(\text{cl}(\mu_a))) \lor \text{int}(\text{cl}_\delta(\mu_a))$

Thus,

$$
\bigvee_{a \in F} (\mu_a) \leq \bigvee_{a \in F} \text{cl} (\text{int}(\text{cl}(\mu_a))) \lor \text{int}(\text{cl}_\delta(\mu_a)).
$$

Since, the arbitrary union of fuzzy $\beta^*$-open sets is fuzzy $\beta^*$-open set.

**Theorem 3.8.** Let $(X, \tau)$ and $(X, \sigma)$ be any two fuzzy topological spaces such that $X$ is product related to $Y$. Then the product $\mu_1 \times \mu_2$ of a fuzzy $\beta^*$-open set $\mu_1$ of $X$ and fuzzy $\beta^*$-open set $\mu_2$ of $Y$ is fuzzy $\beta^*$-open set of the fuzzy product space $X \times Y$.

**Proof.** Let $\mu_1, \mu_2$ are two fuzzy $\beta^*$-open sets of $X$ and $Y$ respectively, From Definition 3.1,

$$
\mu_1 \leq \text{cl} (\text{int}(\text{cl}(\mu_1))) \lor \text{int}(\text{cl}_\delta(\mu_1))
$$

and

$$
\mu_2 \leq \text{cl} (\text{int}(\text{cl}(\mu_2))) \lor \text{int}(\text{cl}_\delta(\mu_2)).
$$

Then we have,

$$
\mu_1 \times \mu_2 \leq \text{cl} (\text{int}(\text{cl}(\mu_1))) \lor \text{int}(\text{cl}_\delta(\mu_1))
$$

and

$$
\mu_1 \times \mu_2 \leq \text{cl} (\text{int}(\text{cl}(\mu_2))) \lor \text{int}(\text{cl}_\delta(\mu_2)).
$$

Thus

$$
\mu_1 \times \mu_2 \leq \text{cl} (\text{int}(\text{cl}(\mu_1 \times \mu_2))) \lor \text{int}(\text{cl}_\delta(\mu_1 \times \mu_2)).
$$

$\mu_1 \times \mu_2$ is fuzzy $\beta^*$-open in the fuzzy product space $X \times Y$.

4. Fuzzy $\beta^*$-Closed Sets in Fuzzy Topological Space

In this section a new closed set in fuzzy topological space is introduced.
Definition 4.1. A fuzzy subset \( \mu \) of a fuzzy topological space \((X, \tau)\) is said to be \( \beta^* \)-closed set if
\[
\mu \geq \text{int}(\text{cl}(\text{int}(\mu))) \land \text{cl}(\text{int}_\delta(\mu)).
\]

Example 4.2. \( X = \{x, y, z\} \). And the fuzzy topology
\[
\tau = \{0, 1\}, \{x_{0.2}, y_{0.1}, z_{0.1}\}, \{x_{0.6}, y_{0.9}, z_{0.9}\}, \{x_{0.4}, y_{0.3}, z_{0.7}\}, \{x_{0.8}, y_{0.9}, z_{0.9}\}
\]

and
\[
\tau^* = \{0, 1\}, \{x_{0.8}, y_{0.9}, z_{0.9}\}, \{x_{0.6}, y_{0.9}, z_{0.9}\}, \{x_{0.4}, y_{0.1}, z_{0.1}\}, \{x_{0.4}, y_{0.5}, z_{0.3}\}, \{x_{0.2}, y_{0.1}, z_{0.1}\}.\]

Let \( \mu = \{x_{0.6}, y_{0.9}, z_{0.9}\} \),
\[
\text{int}(\text{cl}(\text{int}(\mu))) \land \text{cl}(\text{int}_\delta(\mu)) = \{x_{0.6}, y_{0.9}, z_{0.9}\}.
\]
\( \mu \) is \( \beta^* \)-closed set.

Result 4.3.
(i) Fuzzy \( \beta^* \)-closed is fuzzy \( \delta \)-semi open if
\[
\text{cl}(\text{int}(\mu)) = 0.
\]
(ii) Fuzzy \( \beta^* \)-closed is \( \alpha \)-open if \( \text{cl}(\text{int}_\delta(\mu)) = 0 \).

Proposition 4.4. If \( \mu \) is fuzzy \( \delta \)-semi open and \( \lambda \) is fuzzy \( \alpha \)-open then \( \mu \land \lambda \) is fuzzy \( \beta^* \)-closed.

Proof. Obvious from Definition 4.1.

Proposition 4.5. Let \((X, \tau)\) be a fuzzy topological space. Then the intersection of two fuzzy \( \beta^* \)-closed sets is a \( \beta^* \)-closed set in the fuzzy topological space \((X, \tau)\).

Proof. Let \( \mu_1, \mu_2 \) be two fuzzy \( \beta^* \)-closed sets.
\[
\mu_1 \geq \text{int}(\text{cl}(\text{int}(\mu_1))) \land \text{cl}(\text{int}_\delta(\mu_1)).
\]
\[
\mu_2 \geq \text{int}(\text{cl}(\text{int}(\mu_2))) \land \text{cl}(\text{int}_\delta(\mu_2)).
\]
Then we have,
\[
\mu_1 \land \mu_2 \geq \text{int}(\text{cl}(\text{int}(\mu_1))) \land \text{cl}(\text{int}_\delta(\mu_1)) \land \text{int}(\text{cl}(\text{int}(\mu_2))) \land \text{cl}(\text{int}_\delta(\mu_2)).
\]
Therefore, \( \mu_1 \land \mu_2 \) is fuzzy \( \beta^* \)-closed set.

Theorem 4.6. Let \((X, \tau)\) be a fuzzy topological space and let \( \{\mu_\alpha\}_{\alpha \in \mathcal{F}} \) be the collection of fuzzy \( \beta^* \)-closed sets in fuzzy topological space \( X \), then \( \bigwedge_{\alpha \in \mathcal{F}} (\mu_\alpha) \) is fuzzy \( \beta^* \)-closed set.

Proof. Let \( \mathcal{F} \) be the collection of fuzzy \( \beta^* \)-closed sets in fuzzy topological space \((X, \tau)\).

For each \( \alpha \in \mathcal{F} \), \( \mu_\alpha \geq \text{int}(\text{cl}(\text{int}(\mu_\alpha))) \land \text{cl}(\text{int}_\delta(\mu_\alpha)) \).
Thus,
\[
\bigwedge_{\alpha \in \mathcal{F}} (\mu_\alpha) \geq \text{int}(\text{cl}(\text{int}(\bigwedge_{\alpha \in \mathcal{F}} (\mu_\alpha)))) \land \text{cl}(\text{int}_\delta(\bigwedge_{\alpha \in \mathcal{F}} (\mu_\alpha))).
\]
Since, the arbitrary intersection of fuzzy \( \beta^* \)-closed sets is fuzzy \( \beta^* \)-closed set.

Theorem 4.7. Let \((X, \tau)\) and \((X, \sigma)\) be any two fuzzy topological spaces such that \( X \) is product related to \( Y \). Then the product \( \mu_1 \times \mu_2 \) of a fuzzy \( \beta^* \)-closed set \( \mu_1 \) of \( X \) and fuzzy \( \beta^* \)-closed set \( \mu_2 \) of \( Y \) is fuzzy \( \beta^* \)-closed set of the fuzzy product space \( X \times Y \).

Proof. Let \( \mu_1 \), \( \mu_2 \) are two fuzzy \( \beta^* \)-closed sets of \( X \) and \( Y \) respectively. From Definition 4.1,
\[
\mu_1 \geq \text{int}(\text{cl}(\text{int}(\mu_1))) \land \text{cl}(\text{int}_\delta(\mu_1)).
\]
And
\[
\mu_2 \geq \text{int}(\text{cl}(\text{int}(\mu_2))) \land \text{cl}(\text{int}_\delta(\mu_2)).
\]
Then we have,
\[
\mu_1 \times \mu_2 \geq \text{int}(\text{cl}(\text{int}(\mu_1))) \land \text{cl}(\text{int}_\delta(\mu_1)) \land \text{int}(\text{cl}(\text{int}(\mu_2))) \land \text{cl}(\text{int}_\delta(\mu_2)).
\]
\( \mu_1 \times \mu_2 \) is fuzzy \( \beta^* \)-closed in the fuzzy product space \( X \times Y \).

5. Conclusion

In this paper, a new class of open and closed sets in fuzzy topological space, namely \( \beta^* \)-open and \( \beta^* \)-closed sets is introduced. Then some new examples and theorems in separation axioms on fuzzy topological space are developed.

References

On $\beta^*$-open and $\beta^*$-closed sets in fuzzy topological space — 294/294


