A note on t-Cayley hypergraphs

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Abstract
In this paper we study some properties of t-Cayley hypergraph in terms of algebraic properties. This did not attract much attention in the literature.

Keywords
Hypergraph, t- hypergraph, k-transitive, mn-transitive.

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1. Introduction
A hypergraph $H$ is a pair $(V(H); E(H))$, where $V(H)$ is a finite nonempty set and $E(H)$ is a finite family of nonempty subsets of $V(H)$. The elements of $V(H)$ are called vertices and the elements of $E(H)$ are called edges. Two vertices in a hypergraph are adjacent if there is a hyperedge which contains both vertices [1].

A path of length $k$ in a hypergraph $(V(H); E(H))$ is an alternating sequence $(v_1, e_1, v_2, e_2, ..., v_k, e_k, v_{k+1})$ in which $v_i \in V(H)$ for each $i = 1, 2, ..., k+1, e_i \in E(H)$, $\{v_i, v_{i+1}\} \subseteq e_i$ for $i = 1, 2, ..., k$ and $v_i \neq v_j$ and $e_i \neq e_j$ for $i \neq j$.

A hypergraph is connected if for any pair of vertices, there is a path which connects these vertices [1]. Let $G$ be a group, $\Omega$ a subset of $G \setminus \{1\}$ and $t$ an integer satisfying $2 \leq t \leq \max\{o(\omega) : \omega \in \Omega\}$. In [8], M. Buratti introduced $t$-Cayley Hypergraph $H = t - \text{Cay}[G; \Omega]$ as follows:

$$V(H) := G \quad \text{and} \quad E(H) = \{g, g\omega, ..., g\omega^{t-1} : g \in G, \omega \in \Omega\}.$$ 

He proved that $t$-Cayley Hypergraphs are vertex transitive and regular. Moreover, he obtained a necessary and sufficient condition for $t$-Cayley Hypergraphs to be connected.

In [6] H. Galeana Sanchez and Cesar Hernandez-Cruz introduced the concepts of $k$- transitivity and $k$-path transitivity in Cayley digraphs. A digraph $G$ is $k$- transitive if the existence of a path $(x_0, x_1, ..., x_k)$ of length $k$ in $G$ implies that $x_0$ and $x_k$ are adjacent. A digraph is called $k$- path transitive if whenever there is a $xy$ path of length less than or equal to $k$ and a $yz$ path of length less than or equal to $k$, then there exists a $xz$ path of length less than or equal to $k$.

Anil Kumar V. and Mohanan T. generalised the concept of $k$-transitivity as follows[3]: Let $m$ and $n$ be two positive integers such that $m > n$. A digraph $G$ is $(m, n)$- transitive if whenever there is a directed path of length $m$ from $x$ to $y$ there is a directed path of length $n$ from $x$ to $y$.

In this paper we study some graph theoretic properties in terms of algebraic properties.

2. Main Results
Let $G$ be a group with identity element 1 and let $\Omega$ be a subset of $G \setminus \{1\}$. We define

$$A := \{w^n : w \in \Omega, n = 1, 2, ..., t - 1\} \setminus \{1\}.$$ 

A $t$-Cayley Hypergraph $H = t - \text{Cay}[G; \Omega]$ is complete if and only if $G = A$.

Proof. First, assume that $H$ be a complete hypergraph. Then for $x \in G$, 1 and $x$ are adjacent. This implies that $1, x \in e = \{gs^i : 0 \leq i \leq t - 1\}$, for some $g \in G$, and some $s \in \Omega$. This implies there exists $p, q \in \{0, 1, ..., t - 1\}$ such that $1 = gs^p$ and $x = gs^q$. Observe that

$$x = gs^p, s^{q-p} = s^{q-p} \in A.$$ 

Since $x \in G$ is arbitrary, $G \subseteq A$. Obviously, $A \subseteq G$. Therefore, $G = A$.
Conversely, assume that \( G = A \). We want to show that \( H \) is complete. Let \( x, y \in G \). Then \( y = xz \) for some \( z \in G \). Since \( G = A \), \( z \in A \). Then \( z = w' \), for some \( w \in \Omega \) and \( r \in \{1, 2, \ldots, t - 1\} \). This implies that \( y = xw' \). This means that \( x, y \) belongs to an edge \( e = \{xw' : 0 \leq i \leq t - 1\} \). Therefore \( x \) and \( y \) are adjacent. This completes the proof of the theorem.

A hypergraph \( G \) is a hasse – diagram if \( G \) is connected and for any path \( x_0, x_1, \ldots, x_n \), \( n \geq 2 \) from \( x_0 \) to \( x_n \) in \( G \), \( x_0 \) and \( x_n \) are not adjacent.

\( H \) is a hasse-diagram if and only if \( H \) is connected and \( A \cap A^n = \emptyset \) for \( n \geq 2 \).

**Proof.** First, assume that \( H \) is a hasse-diagram. Let \( x \in A^n, n \geq 2 \). There exists \( w_1, w_2, \ldots, w_n \in A \) where \( w_1, w_2, \ldots, w_n \in \Omega \) and \( r_1, r_2, \ldots, r_n \in \{1, 2, \ldots, t - 1\} \) such that \( x = w_1 w_2 \ldots w_n \). Clearly \( w_1 w_2 \ldots w_{i-1} w_i w_{i+1} \ldots w_n \) is a path of length \( n \) from \( 1 \) to \( x \). So \( H \) is a hasse-diagram and \( x \) are not adjacent. That is, there exists no edge \( e = \{gw^t : 0 \leq i \leq t - 1\} \), \( g \in G \), \( w \in \Omega \) such that \( 1, x \in e \). This implies that \( x \neq 1 \) for any \( x \in \Omega \), \( r \in \{0, 1, \ldots, t - 1\} \) which gives \( x \notin A \). That is \( x \notin A^n \) implies \( x \notin A \). Therefore \( A \cap A^n = \emptyset \) for \( n \geq 2 \).

Conversely suppose that \( H \) is connected and \( \Omega \cap A^n = \emptyset \) for \( n \geq 2 \). Let \( x, y \in G \). Then there exists a path, say, \( x = x_0, x_1, \ldots, x_n \) such that \( x \neq y \) from \( x \) to \( y \). This implies that \( x = x_0, x_1, \ldots, x_{n-1}, x_n \in \{g \in \Omega : 0 \leq k \leq t - 1\}, i = 1, 2, \ldots, n \). Observe that \( x = x_{i-1} w_i \) for some \( r_i \in \{1, 2, \ldots, t - 1\} \). Then

\[
y = x_n = x w_1 w_2 \ldots w_n \tag{2.1}
\]

If \( x \) and \( y \) are adjacent, then there exist \( g \in \Omega \) and \( w \in \Omega \) such that \( x, y \in \{gw^k : 0 \leq k \leq t - 1\} \). Then

\[
y = x w^{k_0} \tag{2.2}
\]

for some \( k_0 \in \{1, 2, \ldots, t - 1\} \). From (2.1) and (2.2), \( w^{k_0} = w_1 w_2 \ldots w_n \), which implies \( w \in A^n \). This implies that \( w \in A \) and \( A \cap A^n = \emptyset \), which is a contradiction to the assumption that \( A \cap A^n = \emptyset \) for \( n \geq 2 \). Hence \( x \) and \( y \) are not adjacent. Thus \( H \) is a hasse-diagram. This completes the proof.

The hypergraph \( H \) is \( k \)-transitive if and only if \( A^k \subseteq A \).

**Proof.** Assume that \( H \) is \( k \)-transitive. Let \( x \in A^k \). Then there exists \( w_1, w_2, \ldots, w_k \in A \) where \( w_1, w_2, \ldots, w_k \in \Omega \) and \( r_1, r_2, \ldots, r_k \in \{1, 2, \ldots, t - 1\} \) such that \( x = w_1 w_2 \ldots w_k \). Obviously, \( w_1, w_2, \ldots, w_k \) is a path from \( 1 \) to \( x \) of length \( k \). Since \( H \) is \( k \)-transitive, \( 1 \) and \( x \) are adjacent. Then there exist \( w \in \Omega \) such that \( x = w \). For some \( r \in \{1, 2, \ldots, t - 1\} \) which implies that \( x \neq w \). Hence \( A^k \subseteq A \).

Conversely suppose \( A^k \subseteq A \). Let \( x, y \in V(H) \) be such that there exists a path of length \( k \) from \( x \) to \( y \), say, \( x = x_0, x_1, \ldots, x_k = y \). Then we obtain \( y = x, w_1 w_2 \ldots w_k \) for some \( w \in A \), \( r_1 \in \{1, 2, \ldots, t - 1\}, i = 1, 2, \ldots, k \). Since \( A^k \subseteq A \), \( w_1, w_2, \ldots, w_k \in A \). Then there exist \( w \in \Omega \) and \( r \in \{1, 2, \ldots, t - 1\} \) such that \( y = w_1 w_2 \ldots w_k \). This gives, \( y = xw' \) which clearly implies \( x \) and \( y \) are adjacent. Hence \( H \) is \( k \)-transitive.
Conversely, assume \((A \cup A^2 \cup \ldots \cup A^k)^2 \subseteq A \cup A^2 \cup \ldots \cup A^k\). Let there exist a path from \(x\) to \(y\) of length \(i \leq k\) and a path from \(y\) to \(z\) of length \(j \leq k\). Then there exists \(w_1, w_2, \ldots, w_i, v_1, v_2, \ldots, v_j \in \Omega\) and \(r_1, r_2, \ldots, r_i, p_1, p_2, \ldots, p_j \in \{1, 2, \ldots, t - 1\}\) such that \(y = xw_1^r w_2^r \ldots w_i^r v_j^p \ldots v_2^p \ldots v_j^p = xa_1 a_2\), where \(a_1 \in A^i\) and \(a_2 \in A^j\). This implies \(z = xa_0\) where \(a_0 = a_1 a_2 \in (A \cup A^2 \cup \ldots \cup A^k)^2\). Then by assumption \(a_0 \in A \cup A^2 \cup \ldots \cup A^k\) and hence \(a_0 \in A^p\), for some \(p \leq k\). Then \(z = xa_0^p u_1^p \ldots u_p^p\), \(u_1, u_2, \ldots, u_p \in \Omega, q_1, q_2, \ldots, q_p \in \{1, 2, \ldots, t - 1\}\). This implies that there exist a path from \(x\) to \(z\) of length \(p \leq k\). Hence \(H\) is \(k\)-path transitive.

\(H\) is \((m, n)\)-transitive if and only if \(A^m \subseteq A^n\).

**Proof.** Suppose \(H\) is \((m, n)\)-transitive. Let \(x \in A^m\). Then there exist \(w_1, w_2, \ldots, w_m \in \Omega\) and \(r_1, r_2, \ldots, r_m \in \{1, 2, \ldots, t - 1\}\) such that \(x = w_1^r \ldots w_m^r\). Then \(1, w_1, w_1^2 \ldots w_2, w_1^2 \ldots w_2^2 \ldots \ldots w_m^r\) is a path from \(1\) to \(x\) of length \(m\). Since \(H\) is \((m, n)\)-transitive, there exist a path \(1 = x_0, x_1, \ldots, x_n = x\) from \(1\) to \(x\) of length \(n\). Then there exist \(g_i \in G\) and \(w_i \in \Omega\) for \(i \in \{1, 2, \ldots, n\}\) such that \(x_{i-1}, x_i \in e_i = \{g_n^k : 0 \leq k \leq t - 1\}\). Then \(x_i = x_{i-1}^k\) for some \(k_i \in \{1, 2, \ldots, t - 1\}\). Then \(x = x_n = 1, w_1^k_1 \ldots w_n^k_n\) implies \(x \in A^n\). Hence \(A^m \subseteq A^n\).

Conversely assume that \(A^m \subseteq A^n\). Let \(x, y \in G\) such that there exist a path from \(x\) to \(y\) of length \(m\). Then there exists \(w_1, w_2, \ldots, w_m \in \Omega\) and \(r_1, r_2, \ldots, r_m \in \{1, 2, \ldots, t - 1\}\) such that \(y = x w_1^r \ldots w_m^r\). Since \(A^m \subseteq A^n\), \(w_1^r \ldots w_m^r \in A^p\). This implies that there exists \(v_1, v_2, \ldots, v_n \in \Omega\) and \(k_1, k_2, \ldots, k_n \in \{1, 2, \ldots, t - 1\}\) such that \(w_1^r \ldots w_m^r = v_1^{k_1} v_2^{k_2} \ldots v_n^{k_n}\). Then \(y = x v_1^{k_1} v_2^{k_2} \ldots v_n^{k_n}\). Clearly \(x, x v_1^{k_1} v_2^{k_2} \ldots x v_1^{k_1} v_2^{k_2} \ldots v_n^{k_n} = y\) is a path from \(x\) to \(y\) of length \(n\). Hence \(H\) is \((m, n)\)-transitive.

References


