Graceful labeling in a graph consisting chord with quadrilateral snake

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Abstract
We studied graphs which are quadrilateral with one chord, barycentric subdivision of quadrilateral with one chord, double quadrilateral snake and alternate double quadrilateral snake. We proved that Graph obtained by joining quadrilateral with one chord and double quadrilateral snake, Graph obtained by joining quadrilateral with one chord and alternate double quadrilateral snake and Graph obtained by joining barycentric subdivision of quadrilateral with one chord and quadrilateral snake are graceful.

Keywords
Graceful labeling, Quadrilateral snake, Barycentric subdivision, Double quadrilateral snake, Alternate double quadrilateral snake.

AMS Subject Classification
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1. Introduction
The concept of graceful labeling was introduced by Rosa \cite{5} in 1967 and for numbering in graph was defined by S.W. Golomb \cite{2}. Many researchers have studied gracefulness of graphs, refer Gallian survey \cite{1}. A good number of papers are found with variety of applications in coding theory, radar communication, cryptography etc. A depth details about applications of graph labeling is found in Bloom and Golomb \cite{2}. We accept all notations and terminology from Harary \cite{3}. We recall some definitions which are use in this paper.

A function f is called graceful labeling of a graph $G = (V,E)$ if $f : V \rightarrow \{0, 1, ..., q\}$ is injective and the induce function $f^* : E \rightarrow \{1, ..., q\}$ defined as $f^*(e) = |f(u) - f(v)|$ is bijective for every edge $e = (u,v) \in E(G)$. A graph G is called graceful graph if it admits a graceful labeling.

Cycle is a closed trail in which the "first vertex = last vertex"

The quadrilateral snake $Q_n$ is obtained from the path $P_n$ by replacing every edge of a path by cycle $C_4$.

A chord of a quadrilateral is an edge joining two non-adjacent vertices of quadrilateral.

The double quadrilateral snake $DQ_n$ consists of two quadrilateral snakes that have a common path.

An alternate double quadrilateral snake $ADQ_n$ is consist of two alternate double quadrilateral snakes that have common path.

Let $G = (V,E)$ be a graph. Let $e = uv$ be an edge of G, and $w$ is not a vertex of G. The edge $e$ is subdivided when it is replaced by edges $e' = uw$ and $e'' = wv$. 
Let $G = (V, E)$ be a graph if every edge of graph $G$ is subdivided, then the resulting graph is called barycentric subdivision of graph $G$. In other words barycentric subdivision is the graph obtained by inserting a vertex of degree 2 into every edge of original graph. The barycentric subdivision of any graph $G$ is denoted by $S(G)$. It is easy to observe that $|V S(G)| = |V(G)| + |E(G)|$ and $|E S(G)| = 2|E(G)|$.

In this paper we introduced gracefulness of (i) Graph obtained by joining quadrilateral with one chord and quadrilateral snake. For detail survey of the graph obtained by joining quadrilateral with one chord and quadrilateral snake $G$.

2. Main Results:

2.1 Theorem

The graph obtained by joining quadrilateral with one chord and double quadrilateral snake $G$ is graceful.

Proof:

Let $G = (V, E)$ be the graph, obtained by joining two graphs. Quadrilateral with one chord $G_1$ and double quadrilateral snake $G_2$ by a path $P_k$ of length $k$. Let $\{u_1, u_2, u_3, u_4\}$ be vertices of $G_1$ and $\{w_1, w_2, \ldots, w_k, w'_1, \ldots, w'_k\}$ be vertices of $G_2$ and $\{v_1, v_2, \ldots, v_k\}$ be the vertices of $P_k$ with $v_1 = u_4$ and $v_k = x_1$ and for $G_2$ join $x_1$ to $x_2$ (alternatively) to four new vertices $w_i$ to $w_i-1, w'_i$ and $w'_i-1$ by the edges $x_i w_i, w_i w_i+1, w_{i+1} x_i, x_i+1 w'_i, w'_i w'_i-1$ and $x_i w'_i, (i = 1, 2, \ldots)$. Here $|V(G)| = 5j + 5, |E(G)| = 7j + 6$

Case-1: $k$ is odd.

For vertices $f: v \rightarrow \{0, 1, \ldots, q\}$ where $q = 7j + (2l+4)$.

For vertices $f(u_1) = 7j + (2l+4)$, $f(u_2) = 0$, $f(u_3) = 7j + (2l+2)$, $f(u_4) = 1$

$f(v_1) = 1$, $f(v_2) = 7j + (2l+1)$, $f(v_3) = 2$, $f(v_4) = 7j + (2l)$

$f(v_2m-1) = m$, $f(v_2m) = 7j + (2l+2m))$

$f(v_{2l+1}) = 7j + (i+2)$

Case-2: $k$ is even.

For vertices $f: v \rightarrow \{0, 1, \ldots, q\}$ where $q = 7j + (2l+5)$.

For vertices $f(u_1) = 7j + (2l+5)$, $f(u_2) = 0$, $f(u_3) = 7j + (2l+3)$, $f(u_4) = 1$

$f(v_1) = 1$, $f(v_2) = 7j + (2l+2)$, $f(v_3) = 2$, $f(v_4) = 7j + (2l+1)$

$f(v_2m-1) = m$, $f(v_2m) = 7j + (2l+(3m))$

$f(v_{2l+1}) = 7j + (i+1)$

Here $f: V \rightarrow \{0, 1, \ldots, q\}$ is injective and the induce function $f^*: E \rightarrow \{1, \ldots, q\}$ is bijective. So graph $G$ is graceful.

2.3 Theorem

The graph obtained by joining quadrilateral with one chord and alternate double quadrilateral snake $G$ is graceful.

Proof:

Let $G = (V, E)$ be the graph, obtained by joining two graphs. Quadrilateral with one chord $G_1$ and alternate double quadrilateral snake $G_2$ by a path $P_k$ of length $k$. Let $\{u_1, u_2, u_3, u_4\}$ be vertices of $G_1$ and $\{w_1, w_2, \ldots, w_k, w'_1, \ldots, w'_k\}$ be vertices of $G_2$ and $\{v_1, v_2, \ldots, v_k\}$ be the vertices of $P_k$ with $v_1 = u_4$ and $v_k = x_1$ and for $G_2$ join $x_1$ to $x_2$ (alternatively) to four new vertices $w_i$ to $w_i+1, w'_i$ and $w'_{i+1}$ by the edges $x_i w_i, w_i w_i+1, w_{i+1} x_i, x_i+1 w'_i, w'_i w'_{i+1}$ and $x_i w'_i, (i = 1, 2, \ldots)$. Here $|V(G)| = 6j+4, |E(G)| = 8j+5$
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Case-1: k is odd.

\[ f : v \rightarrow \{0, 1, ..., q\} \text{ where } q = 8j + (2l+3). \]

For vertices:
\[
\begin{align*}
    f(u_1) &= 8j + (2l+3) \\
    f(u_2) &= 0 \\
    f(u_3) &= 8j + (2l+1) \\
    f(u_4) &= 1 \\
    f(v_1) &= 1 \\
    f(v_2) &= 8j + (2l) \\
    f(v_3) &= 2 \\
    f(v_4) &= 8j + (2l-1) \\
    f(v_{2m-1}) &= m \\
    f(v_{2m}) &= 8j + (2l+1-m) \\
    f(v_{2i}) &= 8j + (i+1) \\
    f(w_{2l-1}) &= 4l - (3-m) \\
    f(w_{2l}) &= 8j + ((4+m)-4l) \\
    f(w_{2l+1}) &= 8j + ((2+m)-4l) \\
    f(x_{2l-1}) &= 8j + ((5+m)-4l) \\
    f(x_{2l}) &= 4l + m
\end{align*}
\]

Hence \( f : V \rightarrow \{0, 1, ..., q\} \) is injective and the induce function \( f^* : E \rightarrow \{1, ..., q\} \) is bijective. So graph G is graceful.

2.5 Theorem
The graph obtained by joining barycentric subdivision of quadrilateral with one chord and quadrilateral snake is graceful.

Proof:
Let \( G = (V, E) \) be the graph, obtained by joining two graphs barycentric subdivision of quadrilateral with one chord \( G_1 \) and quadrilateral snake \( G_2 \) by a path \( P_k \) of length \( k \). Let \( \{u_1, u_2, u_3, u_4\} \) be vertices of quadrilateral with one chord and \( \{x_1, x_2, x_3, x_4, x_5\} \) are inserted vertices due to barycentric subdivision \( \{x_1, x_2, x_3, x_4, x_5\} \) are of \( G_2 \) and \( \{w_1, w_2, ..., w_k\} \) be vertices of quadrilateral snake \( G_2 \) and \( \{v_1, v_2, ..., v_k\} \) be the vertices of \( P_k \) with \( v_1 = u_4 \) and \( v_k = w_1 \) we consider the following two cases.

Case-1: \( k \) is odd

\[ f : v \rightarrow \{0, 1, ..., q\} \text{ where } q = 4j + (9+2l). \]

For vertices:
\[
\begin{align*}
    f(u_1) &= 4j + (9+2l) \\
    f(u_2) &= 4j + (8+2l) \\
    f(u_3) &= 4j + (7+2l) \\
    f(u_4) &= 4j + (9+2l) \\
    f(v_1) &= 4j + (5+2l) \\
    f(v_2) &= 6 \\
    f(v_3) &= 4j + (4+2l) \\
    f(v_4) &= 7 \\
    f(v_{2m-1}) &= 4j + ((6-m)+2l) \\
    f(v_{2m}) &= 5+m \\
    f(v_{2i+1}) &= 5+i \\
    f(w_{2l-1}) &= 4l + (m+1) \\
    f(w_{2l}) &= 4l + (4l - (8+m)) \\
    f(w_{2l+1}) &= 4l + (4l - (9+m)) \\
    f(w_{2l+2}) &= 4l + (4l + (3+m)) \\
    f(w_{6l-3}) &= 4l + (6+m) \\
    f(w_{6l-2}) &= 4j + (4l - (6+m)) \\
    f(w_{6l-1}) &= 4j + (4l - (6+m)) \\
    f(x_1) &= 0 \\
    f(x_2) &= 1 \\
    f(x_3) &= 3 \\
    f(x_4) &= 4 \\
    f(x_5) &= 5
\end{align*}
\]

2.6 Illustration
Graceful labeling of the graph obtained by joining barycentric subdivision of quadrilateral with one chord and quadrilateral snake.
**figure 3:** The graph obtained joining by 3 copies of quadrilateral snake and barycentric subdivision of quadrilateral with one chord with p = 19 and q = 23 is graceful labeling.

Case-2: k is even
\[
f : v \rightarrow \{0, 1, \ldots, q\} \text{ where } q = 4j + (10+2l).
\]

For vertices
\[
\begin{align*}
f(u_1) &= 4j + (10+2l) \\
f(u_2) &= 4j + (9+2l) \\
f(u_3) &= 4j + (8+2l) \\
f(u_4) &= 4j + (6+2l)
\end{align*}
\]
\[
\begin{align*}
f(v_1) &= 4j + (6+2l) \\
f(v_2) &= 4j + (5+2l) \\
f(v_3) &= 4j + (4+2l) \\
f(v_4) &= 4j + (2+2l)
\end{align*}
\]
\[
\begin{align*}
f(v_{2m-1}) &= 4j + ((7-m)+2l) \\
f(v_{2m}) &= 4j + (6+l)
\end{align*}
\]
\[
\begin{align*}
f(w_{6l-5}) &= 4j + ((10+m) - 4l) \\
f(w_{6l-4}) &= 4l + (2+m) \\
f(w_{6l-3}) &= 4j + ((7+m) - 4l) \\
f(w_{6l-2}) &= 4l + (3+m) \\
f(w_{6l-1}) &= 4j + (6+l) \\
f(w_{6l}) &= 4l + (5+m)
\end{align*}
\]
\[
\begin{align*}
f(x_1) &= 0 \\
f(x_2) &= 1 \\
f(x_3) &= 3 \\
f(x_4) &= 4
\end{align*}
\]

Hence \( f : V \rightarrow \{0, 1, \ldots, q\} \) is injective and the induce function \( f^* : E \rightarrow \{1, \ldots, q\} \) is bijective. So graph G is graceful.

**2.7 Concluding Remark**

Present work contributes some new results. We discussed gracefulness of graph obtained by joining (barycentric subdivision of quadrilateral with one chord and quadrilateral snake, quadrilateral with one chord and double quadrilateral snake, quadrilateral with one chord and alternate double quadrilateral snake). The labeling pattern is demonstrated by means of illustrations which provide better understanding to derived results.

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**References**


