**P_4-decomposition in Boolean function graph of B_3(G)**

S. Muthammal and S. Dhanalakshi

**Abstract**
For any graph G, let V(G) and E(G) denote the vertex set and edge set of G respectively. The Boolean function graph B(K_p, L(G), NINC) of G is a graph with vertex set V(G) ∪ E(G) and two vertices in B(K_p, L(G), NINC) are adjacent if and only if they correspond to two non adjacent edges of G or to a vertex and an edge not incident to it in G. For brevity, this graph is denoted by B_3(G). In this paper, P_4-decomposition in Boolean Function Graph B(K_p, L(G), NINC) of some standard graphs and corona graphs are obtained.

**Keywords**
Boolean Function graph, Edge Domination Number, Decomposition.

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**Article History**: Received 01 December 2020; Accepted 04 February 2021

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### 1. Introduction

Graphs discussed in this paper are undirected and simple graphs. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. A graph with p vertices and q edges is denoted by G(p, q). The corona G_1 ∘ G_2 of two graphs G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2, and then joining the ith vertex of G_1 to every vertex of in the ith copy of G_2. For any graph G, G ∘ K_1 is denoted by G⁺.

A decomposition of a graph G is a family of edge-disjoint subgraphs {G_1, G_2, …, G_k} such that E(G) = E(G_1) ∪ E(G_2) ∪ … ∪ E(G_k). If each G_i is isomorphic to H, for some subgraph H of G, then the decomposition is called a H-decomposition of G. In particular, a P_4-decomposition of a graph G is a partition of the edge set of G into paths of length 3. In this case, G is said to be P_4-decomposable. Several authors studied various types of decomposition by imposing conditions on G in the decomposition. Heinrich, Liu and Yu[3] proved that a connected 4-regular graph admits a P_4-decomposition if and only if |E(G)| ≡ 0 (mod 3). Sunil Kumar[10] proved that a complete r-partite graph is P_4-decomposable if and only if its size is a multiple of 3. P. Chithra Devi and J. Paulraj Joseph[1] gave a necessary and sufficient condition for the decomposition of the total graph of standard graphs and corona of graphs into paths on three edges. Janakiraman et al., introduced the concept of Boolean function graphs [4-6]. For any graph G, let V(G) and E(G) denote the vertex set and edge set of G respectively. The Boolean function graph B(K_p, L(G), NINC) of G is a graph with vertex set V(G) ∪ E(G) and two vertices in B(K_p, L(G), NINC) are adjacent if and only if they correspond to two non adjacent edges of G or to a vertex and an edge not incident to it in G. For brevity, this graph is denoted by B_3(G).

In this paper, P_4-decomposition in Boolean Function Graph B(K_p, L(G), NINC) of some standard graphs are obtained.

### 2. Prior Results

**Observation 2.1** ([6]). Let G be a graph with p vertices and q edges.

1. \(\overline{L(G)}\) is an induced subgraph of B_3(G) and the subgraph of B_3(G) induced by vertices of G in B_3(G) is totally disconnected.

2. If \(d_i = \deg_G(v_i), v_i \in V(G)\), then the number of edges in B_3(G) is \((q/2)(2p + q - 3) - 1/2\sum_{1 \leq i \leq p} d_i^2\).

3. The degree of a vertex of v in B_3(G) is \(q - \deg_G(v)\) and the degree of a vertex e of L(G) in B_3(G) is \(\deg_{L(G)}(e) + p - 2\).
4. Both $G$ and $B_3(G)$ are regular if and only if either $G$ is totally connected or $G$ is complete.

### 3. Main Results

In the following, $P_3$-decomposition of $B_3(P_n), B_3(C_n), B_3(K_{1,n})$ and corona graphs are found.

**Theorem 3.1.** Let $n \geq 3$

1. If $n$ is odd, then $B_3(P_n) - ((n + 1)/2)K_2$ is $P_4$-decomposable.

2. If $n$ is even, then $B_3(P_n) - ((n - 2)/2)K_2$ is $P_4$-decomposable.

**Proof.** Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $e_i = (v_i, v_{i+1}), i = 1, 2, \ldots, n - 1$ be the edges of $P_n$. Then $v_1,v_2,\ldots,v_n,e_1,e_2,\ldots,e_{n-1} \in V(B_3(P_n)) \cdot B_3(P_n)$ has $(2n - 1)$ vertices and $((3n^2 - 11n + 10)/2)$ edges. It is to be noted that, in all the sets the suffices in $y_i$ is integer modulo $n$ and $j$ in $e_j$ is integer modulo $n - 1, v_0 = v_n$ and $e_0 = e_{n-1}$.

**Case 1.** $n$ is odd, $n \geq 7$.

Then the edge set of $B_3(P_n)$ can be decomposed into $((n^2 - 4n + 3)/2)P_4$ and $((n + 1)/2)K_2$.

The edge set of $((n^2 - 4n + 3)/2)P_4$ is given by the edge set $A^{(1)}$, $i = 1, 2, \ldots, 4$, where

$A^{(1)} = U_{i=1}^{n-2}/\{U_{i=1}^{n-2}A^{(1)}_{ji}\}$,

$A^{(2)}_{ji} = \{(v_i, e_{i+j+1}), (e_{i+j+1}, e_i), (e_j, v_{i+j+1})\}$

$A^{(2)} = \bigcup_{i=1}^{(n-2)/2} A^{(2)}_{ji}$

$A^{(3)} = U_{i=1}^{n-3}/A^{(3)}_i$,

$A^{(3)}_i = \{(e_{i+j+1}, v_i), (v_i, e_i + (n + 1)/2), (e_i + (n + 1)/2, v_{i+j+1})\}$

$A^{(4)} = \{(v_{i-1/2}, e_{i+(n-1)/2}), (e_{i+(n-1)/2}, v_i), (v_i, e_{i-(n-1)/2})\}$

Here,

$A^{(2)} \cong ((n-1)(n-5))/2P_4$, $A^{(2)} \cong ((n-1)/2)P_4$,

$A^{(3)} \cong ((n-3)/2)P_4$, $A^{(4)} \cong P_4$

Hence, $B_3(P_n) - ((n + 1)/2)K_2$ is $P_4$-decomposable.

**Case 2.** $n$ is even, $n \geq 6$.

Then the edge set of $B_3(P_n)$ can be decomposed into $((n^2 - 4n + 4)/2)P_4$ and $((n - 2)/2)K_2$. The edge set of $((n - 2)/2)K_2$ is given by the set $U_{i=1}^{n-4}/\{(v_n, e_{i+n-2})\}$.

The edge set of $((n^2 - 4n + 4)/2)P_4$ is given by the edge sets $A^{(5)}, A^{(6)}$ and $A^{(7)}$ where

$A^{(5)} = U_{j=1}^{n-4}/\{U_{j=1}^{n-4}A^{(5)}_{ji}\}$,

$A^{(6)} = U_{j=1}^{n-2}/\{U_{j=1}^{n-2}A^{(6)}_{ji}\}$,

$A^{(7)} = \{(v_{i+1}, v_i), (v_i, e_{i+(n/2)}), (e_{i+(n/2)}, v_{i+(n-2)/2})\}$

Here, $A^{(5)} \cong ((n-1)(n-4))/2P_4$, $A^{(6)} \cong ((n-2)/2)P_4$.

**Theorem 3.2.** For $n \geq 6$, the graph $B_3(C_n) - nK_2$ is $P_4$-decomposable.

**Proof.** Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $e_i = (v_i, v_{i+1}), i = 1, 2, \ldots, n - 1, e_n = (v_n, v_1)$ be the edges of $C_n$. Then $v_1,v_2,\ldots,v_n,e_1,e_2,\ldots,e_n \in V(B_3(C_n)) \cdot B_3(C_n)$ has $2n$ vertices and $2(3n^2 - 7n)$ edges. It is to be noted that, in all the sets the suffices in $y_i$ and $j$ in $e_j$ are integers modulo $n, v_0 = v_n$ and $e_0 = e_n$.

**Case 1.** $n$ is odd, $n \geq 7$.

Then the edge set of $B_3(C_n)$ can be decomposed into $1/2 ((n^2 - 3n)/2)P_4$ and $nK_2$.

The edge set of $((n^2 - 3n)/2)P_4$ is given by the edge set $B^{(1)}$, where

$B^{(1)} = \bigcup_{j=1}^{n/3} \{U_{i=1}^{n-3}B^{(1)}_{ji}\}$,

$B^{(1)}_{ij} = \{(e_{2+j-1}, v_i), (v_i, e_{i+j}), (e_{i+j}, e_{i+j-1})\}$

Here, $B^{(1)} \cong (n-3)/2P_4$. Hence, $B_3(C_n) - nK_2$ is $P_4$-decomposable.

**Case 2.** $n$ is even, $n \geq 6$.

Then the edge set of $B_3(C_n)$ can be decomposed into $((n^2 - 3n)/2)P_4$ and $nK_2$.

The edge set of $((n^2 - 3n)/2)P_4$ is given by the set

$\bigcup_{i=1}^{n} \{(v_i, e_{i+n-2})\}$. 
The edge set of \((n^2 - 3n)/2\) is given by the edge sets \(B^{(2)}\) and \(B^{(3)}\), where
\[
B^{(2)} = \bigcup_{j=1}^{(n-4)/2} \left\{ \left( e_i, v_j \right), \left( e_i, v_{j+1} \right), \left( e_i, v_{j+2} \right) \right\},
\]
\[
B^{(2)}_i = \left\{ \left( e_i, v_{j+1} \right), \left( e_i, v_{j+2} \right), \left( e_i, v_{j+3} \right) \right\},
\]
\[
B^{(3)} = \bigcup_{i=1}^{n/2} \left\{ \left( e_i, v_{i+3} \right), \left( e_{i+1}, v_{i+3} \right), \left( e_{i+2}, v_{i+3} \right) \right\},
\]
\[
B^{(3)}_i = \left\{ \left( e_{i+1}, v_{i+3} \right), \left( e_{i+2}, v_{i+3} \right), \left( e_{i+3}, v_{i+3} \right) \right\}.
\]

Here, \(B^{(2)} \cong (n(n-1)/2)P_4\) and \(B^{(3)} \cong (n/2)P_4\). Therefore, \(B_3(C_n) - nK_2\) is decomposable.

**Theorem 3.3.** Let \(n \geq 4\).

1. If \(n \equiv 0(\mod 3)\), then \(B_3(K_{1,n}) - nP_3\) is \(P_4\)-decomposable.

2. If \(n \equiv 1(\mod 3)\), then \(B_3(K_{1,n})\) is \(P_4\)-decomposable.

3. If \(n \equiv 2(\mod 3)\), then \(B_3(K_{1,n}) - nK_2\) is \(P_4\)-decomposable.

**Proof.** Let \(V(K_{1,n}) = \{v, v_1, v_2, \ldots, v_n\}\), where \(v\) is the central vertex and \(e_i = (v, v_i), i = 1, 2, \ldots, n\) be the edges of \(K_{1,n}\). Then \(v, v_1, v_2, \ldots, v_{n-1}, e_1, e_2, \ldots, e_n \in V(B_3(K_{1,n})))\).

The edge set of \((n^2 - 3n)/3)\) is given by the edge set
\[
C^{(1)} = \bigcup_{j=1}^{n/3} \left\{ \left( e_i, v_j \right), \left( e_i, v_{j+1} \right), \left( e_i, v_{j+2} \right) \right\},
\]
\[
C^{(1)}_j = \left\{ \left( e_{i+3}, v_{j+3} \right), \left( e_{i+4}, v_{j+3} \right), \left( e_{i+5}, v_{j+3} \right) \right\}.
\]

Here, \(C^{(1)} \cong ((n(n-3)/3)P_4\) and \(C^{(3)} \cong n(P_4)\). Therefore, \(B_3(K_{1,n})\) is decomposable.

**Case 1.** \(n \equiv 0(\mod 3)\), \(n \geq 4\).

Then the edge set of \(B_3(K_{1,n})\) can be decomposed into
\[
\left( (n^2 - n)/3 \right) P_4.
\]

The edge set of \((n^2 - n)/3)\) is given by the edge set \(C^{(2)}\), where
\[
C^{(2)} = \bigcup_{j=1}^{(n-1)/3} \left\{ \left( e_i, v_j \right), \left( e_i, v_{j+1} \right), \left( e_i, v_{j+2} \right) \right\},
\]
\[
C^{(2)}_j = \left\{ \left( e_{i+2}, v_{j+2} \right), \left( e_{i+3}, v_{j+3} \right), \left( e_{i+4}, v_{j+4} \right) \right\}.
\]

Here, \(C^{(2)} \cong ((n(n-1)/3)P_4\) Therefore, \(B_3(K_{1,n})\) is decomposable.

**Case 3.** \(n \equiv 2(\mod 3)\), \(n \geq 5\).

Then the edge set of \(B_3(K_{1,n})\) can be decomposed into
\[
\left( (n^2 - 2n)/3 \right) P_4
\]
and \(nK_2\). The edge set of \(nK_2\) is given by the set
\[
U_n = \left\{ \left( v_i, e_{i+1} \right) \right\}.
\]

The edge set \((n^2 - 2n)/3)P_4\) is given by the edge set \(C^{(3)}\), where
\[
C^{(3)} = \bigcup_{j=1}^{(n/2)} \left\{ \left( e_{i+2}, v_{i+2} \right), \left( e_{i+3}, v_{i+3} \right), \left( e_{i+4}, v_{i+4} \right) \right\},
\]
\[
C^{(3)}_j = \left\{ \left( e_{i+1}, v_{i+2} \right), \left( e_{i+2}, v_{i+3} \right), \left( e_{i+3}, v_{i+4} \right) \right\}.
\]

Here, \(C^{(3)} \cong (n(n-2)/3)P_4\) Therefore, \(B_3(K_{1,n}) - nK_2\) is decomposable.

**Theorem 4.3.** Let \(n \geq 6\).

1. If \(n\) is even, then \(B_3(P_n^+) - (3n-4)/2K_2\) is \(P_4\)-decomposable.

2. If \(n\) is odd, then \(B_3(P_n^+) - (3n-1)/2K_2\) is \(P_4\)-decomposable.

**Proof.** Let \(V(P_n^+ = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}\) where \(v_1, v_2, \ldots, v_n\) are the vertices of \(P_4\) and \(u_1, u_2, \ldots, u_n\) are the pendant vertices of \(P_n^+\) and \(e_i = (v_i, v_{i+1}) i = 1, 2, \ldots, n-1\) and \(f_i = (v_i, u_i), i = 1, 2, \ldots, n\) be the edges of \(P_n^+\).

Then \(v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n, e_1, e_2, \ldots, e_n-1, f_1, f_2, \ldots, f_n \in V(B_3(P_n^+)). B_3(P_n^+\) has \(4n-1\) vertices and \(6n^2 - 12n + 7\) edges.

In all the sets, suffix \(j\) in \(e_{ij}\) is integer modulo \(n-1\) and the suffix \(j\) in \(f_{ij}\) is integer modulo \(n\), \(u_0 = u_n, f_0 = v_n\) and \(v_0 = v_n - 1\).

**Case 1.** \(n\) is even, \(n \geq 6\).

Then the edge set of \(B_3(P_n^+)\) can be decomposed into
\[
\left( (4n^2 - 2n - 6)/2 \right) P_4
\]
and \((3n-4)/2)K_2\). The edge set of \((3n-4)/2)K_2\) is given by the set
\[
U_n = \left\{ \left( u_n, f_i \right) \right\} \cup \left\{ \left( v_{i+3}, e_{i+1} \right) \right\} \cup \left\{ \left( v_2, e_{n-1} \right) \right\}.
\]

The edge set of \((4n^2 - 9n + 6)/2)P_4\) is given by the edge set
sets $M^i, i = 1, 2, \ldots, 7$, where

\[
M^{(1)} = U_{j=1}^{(n-4)/2} \left\{ U_{i=1}^{n-1} M^i \right\},
\]

\[
M_j^i = \left\{ (e_{i+2j-1}, v_i), (v_i, e_{i+2j}), (e_{i+2j}, e_{i+j-1}) \right\},
\]

\[
M^{(2)} = U_{j=1}^{(n-2)/2} \left\{ U_{i=1}^{n-1} M^i \right\},
\]

\[
M_j^i = \left\{ (v_i, f_{i+j}), (f_{i+j}, f_i), (f_i, v_{i+j}) \right\},
\]

\[
M^{(3)} = \bigcup_{i=1}^{n/2} M^i,
\]

\[
M_j^i = \left\{ (v_i, v_n), (v_n, e_{i+(n-2)/2}), (v_{i+(n-2)/2}, v_{i+(n+2)/2}) \right\},
\]

\[
M^{(4)} = \left\{ (e_i, v_n), (v_n, e_{i+(n-2)/2}), (v_{i+(n-2)/2}, v_{i+(n+2)/2}) \right\},
\]

\[
M^{(5)} = \left\{ (v_i, v_{n-1}), (e_{n-1}, e_i), (e_i, v_1) \right\},
\]

\[
M^{(6)} = \bigcup_{i=1}^{n} M^i,
\]

\[
M_j^i = \left\{ (v_i, v_{n-i}), (v_{n-i}, e_i), (e_i, v_1) \right\},
\]

Here, $\langle M^{(1)} \rangle \cong ((n-1)(n-4))/3P_4$, $\langle M^{(2)} \rangle \cong (n(n-2)/2)P_4$, $\langle M^{(3)} \rangle \cong (n-2)/2P_4$ and $\langle M^{(4)} \rangle \cong (n-2)(n-1)/2P_4$, $\langle M^{(5)} \rangle \cong (n-1)/2P_4$, $\langle M^{(6)} \rangle \cong (n-1)/2P_4$. Hence, $B_3(P_n) - ((3n-2)/2)K_2$ is $P_4$-decomposable.

**Case 2.** $n$ is odd, $n \geq 9$.

Then the edge set of $B_3(P_n)$ can be decomposed into

\[(4n^2 - 9n + 5)/2 P_4 \]

and \(((3n-1)/2)K_2\). The edge set of \(((3n-1)/2)K_2\) is given by the set

\[
\left\{ U_{i=1}^{n-1} \left\{ (u_i, f_i) \right\} \right\} \bigcup \left\{ U_{i=1}^{n-5} \left\{ (v_{i+3}, e_{i+1}) \right\} \right\} \bigcup \left\{ (v_{i+n-2}), (v_1, e_{n-1}), (v_2, e_{n-1}) \right\}
\]

The edge set of \(((4n^2 - 9n + 5)/2) P_4\) is given by the edge sets $M^{(4)}, M^{(5)}, M^{(7)}$ as in Case 2 and the edge sets $M^{(8)}, M^{(9)}, M^{(10)}$ and $M^{(11)}$, where,

\[
M^{(8)} = U_{j=1}^{(n-5)/2} \left\{ U_{i=1}^{n-1} M^i \right\},
\]

\[
M_j^i = \left\{ (e_{i+2j-1}, v_i), (v_i, e_{i+2j}), (e_{i+2j}, e_{i+j-1}) \right\},
\]

\[
M^{(9)} = U_{j=1}^{(n-1)/2} \left\{ U_{i=1}^{n-1} M^i \right\},
\]

\[
M_j^i = \left\{ (v_i, f_{i+j}), (f_{i+j}, f_i), (f_i, v_{i+j}) \right\},
\]

\[
M^{(10)} = U_{j=1}^{n-3/2} M^i,
\]

\[
M_j^i = \left\{ (e_i, v_n), (v_n, e_{i+(n-2)/2}), (e_{i+(n-2)/2}, v_{i+(n+2)/2}) \right\},
\]

\[
M^{(11)} = U_{j=1}^{n-1/2} M^i,
\]

\[
M_j^i = \left\{ (v_i, e_{i+(n-4)}), (e_{i+(n-4)}, e_{i+(n-7)/2}), (e_{i+(n-7)/2}, v_{i+(n-2)/2}) \right\},
\]

Here, $\langle M^{(8)} \rangle \cong ((n-1)(n-5))/2P_4$, $\langle M^{(9)} \rangle \cong (n(n-1)/2)P_4$, $\langle M^{(10)} \rangle \cong (n(n-3)/2)P_4$ and $\langle M^{(11)} \rangle \cong (n-1)/2P_4$. Hence, $B_3(P_n) - ((3n-1)/2)K_2$ is $P_4$-decomposable.

**Theorem 3.5.** For $n \geq 6$, the graph $B_3(C_n^+) - nK_2$ is $P_4$-decomposable.

**Proof.** Let $V(C_n^+) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$, where $v_1, v_2, \ldots, v_n$ are the vertices of $C_n$ and $u_1, u_2, \ldots, u_n$ are the pendant vertices of $C_n^+$ and $e_i = (v_i, v_{i+1})$, $i = 1, 2, \ldots, n-1$, $e_n = (v_n, v_1)$ and $f_i = (v_i, u_i)$, $i = 1, 2, \ldots, n$ be the edges of $C_n^+$. Then $v_1, v_2, v_n, u_1, u_2, \ldots, u_n$, $e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n \in V(B_3(C_n^+) \cdot B_3(C_n^+))$ has 4n vertices and $(6n^2 - 8n)$ edges.

It can be shown that in all the sets, all the edges are integers modulo $n$. $f_0 = f_n = v_n$ and $e_0 = e_n, u_0 = u_n$.

**Case 1.** $n$ is even, $n \geq 6$.

Then the edge set of $B_3(C_n^+)$ can be decomposed into

\[(2n^2 - 3n) P_4 \]

and $nK_2$. The edge set of $nK_2$ is given by the set $\{U_{i=1}^n \{ (u_i, f_{i+1}) \} \}$. The edge sets of $(2n^2 - 3n) P_4$ is given by the edge sets $N(i)$ $i = 1, 2, \ldots, 6$, where

\[
N^{(1)} = \bigcup_{i=1}^{(n-4)/2} \{ U_{i=1}^{n-i} N^{(1)} \},
\]

\[
N_j^{(1)} = \{ (e_{i+j-1}, v_i), (v_i, e_{i+j}), (e_{i+j}, e_{i+j-1}) \},
\]

\[
N^{(2)} = \bigcup_{i=1}^{n/2} N^{(2)} ,
\]

\[
N_j^{(2)} = \{ (v_i, e_{i+(n-3)}), (e_{i+(n-3)}, e_i + (n-6)/2), (v_{i+(n-6)/2}, v_{i+n/2}) \},
\]

\[
N^{(3)} = \bigcup_{i=1}^{(n-2)/2} \{ U_{i=1}^{n-i} N^{(3)} \},
\]

\[
N_j^{(3)} = \{ (v_i, f_{i+j}), (f_{i+j}, f_i), (f_i, v_{i+j}) \},
\]

\[
N^{(4)} = \bigcup_{i=1}^{n/2} N^{(4)} ,
\]

\[
N_j^{(4)} = \{ (v_i, f_{i+n/2}), (f_{i+n/2}, f_i), (f_i, v_{i+n/2}) \},
\]

\[
N^{(5)} = \bigcup_{i=1}^{n} \{ U_{i=1}^{n-i} N^{(5)} \},
\]

\[
N_j^{(5)} = \{ (e_i, f_{i+j}), (f_{i+j}, f_i), (f_i, v_{i+j}) \},
\]

\[
N^{(6)} = \bigcup_{i=1}^{n} N^{(6)} ,
\]

\[
N_j^{(6)} = \{ (e_{i+(n-2)/2}), (v_{i+(n-2)/2}), (e_{i+(n-2)/2} + (n-6)/2), (v_{i+(n-6)/2}) \}. 
\]

Here, $\langle N^{(1)} \rangle \cong (n(n-4))/2P_4$, $\langle N^{(2)} \rangle \cong (n/2)P_4$, $\langle N^{(3)} \rangle \cong (n(n-2))/2P_4$ and $\langle N^{(4)} \rangle \cong (n/2)P_4$, $\langle N^{(5)} \rangle \cong (n(n-2))/2P_4$, $\langle N^{(6)} \rangle \cong (n/2)P_4$. Hence, $B_3(C_n^+) - nK_2$ is $P_4$-decomposable.
Case 2. \(n\) is odd, \(n \geq 5\).

Then the edge set of \(B_3(C_n^+)\) can be decomposed into

\[
(2n^2 - 3n) P_4
\]

and \(nK_2\). The edge set of \(nK_2\) is given by the set

\[
U_i^{n} = \{ (u_i, f_{i+1}) \}.
\]

The edge set of \((2n^2 - 3n) P_4\) is given by the edge sets \(N^5\), \(N^6\) as in Case 1 and the sets \(N^7\) and \(N^8\), where

\[
N^7 = \bigcup_{j=1}^{(n-3)/2} \left\{ \bigcup_{i=1}^{n} N^7_j \right\},
\]

\[
N^7_j = \{ (e_{i+2j-1}, v_i), (v_i, e_i + 2j), (e_{i+2j}, e_{i+j-1}) \},
\]

\[
N^8 = \bigcup_{j=1}^{(n-1)/2} \left\{ \bigcup_{i=1}^{n} N^8_j \right\},
\]

\[
N^8_j = \{ (v_i, u_i), (f_{i+j}, f_i), (f_i, v_{i+2j}) \}.
\]

Here, \(\langle N^7 \rangle \approx ((n(n-3)/2)P_4 \) and \(\langle N^8 \rangle \approx ((n(n-1)/2)P_4\).

Hence, \(B_3(C_n^+) - nK_2 \) is P_4-decomposable.

\[\Box\]

**Theorem 3.6.** For \(n \geq 3\) and \(n \equiv 0(\text{mod } 3)\), the graph

\[B_3 \left( K_{1,n}^+ \right) - nK_2\]

is P_4-decomposable.

**Proof.** Let

\[V \left( K_{1,n}^+ \right) = \{ v, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n \},\]

where \(v\) is the central vertex and \(\{ v_1, v_2, \ldots, v_n \} \approx K_{1,n} \), and \(u_1, u_2, \ldots, u_n \) are the pendant vertices of \(K_{1,n}^+\) and \(e_i = (v, v_i)\), \(i = 1, 2, \ldots, n\) and \(f = (v, u), f_i = (v_i, u)\), \(i = 1, 2, \ldots, n\) be the edges of \(K_{1,n}^+\) Then \(v, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n, e_1, e_2,\ldots, e_n, f, f_1, f_2, \ldots, f_n \in V \left( B_3 \left( K_{1,n}^+ \right) \right) B_3 \left( K_{1,n}^+ \right) \) has \((4n+1)\) vertices and \((11n^2 - n)/2\) edges. In all the sets, suffix in \(v_i, u_i \) and \(f_i \) is integers modulo \(n\), \(f_0 = f_n, v_0 = v_n \) and \(e_0 = e_n, u_0 = u_n\).

**Case 1.** \(n\) is even, \(n \geq 6\).

Then the edge set of \(B_3 \left( K_{1,n}^+ \right) \) can be decomposed into \((11n^2 - 3n)/6\) \(P_4\) and \(nK_2\). The edge set of \(nK_2\) is given by the set \(U_i^{n} = \{ (v, f_i) \}\). The edge set of \((11n^2 - 3n)/6\) \(P_4\) is given by the edge sets \(Q^{(1)}, \ldots, Q^{(6)}\), where

\[
Q^{(1)} = \bigcup_{j=1}^{n(n-3)/3} \left\{ \bigcup_{i=1}^{n} Q^{(1)}_{ji} \right\},
\]

\[
Q^{(2)}_{ji} = \{ (e_{i+j}, v_i), (v_i, e_{i+j+1}), (e_{i+j+1}, v_{i+3j}) \},
\]

\[
Q^{(2)} = \bigcup_{j=1}^{n(n-2)/2} \left\{ \bigcup_{i=1}^{n} Q^{(2)}_{ji} \right\},
\]

\[
Q^{(3)}_{ji} = \{ (v_i, f_{i+j}), (f_{i+j}, f_i), (f_i, v_{i+j}) \},
\]

\[
Q^{(2)} = \bigcup_{j=1}^{n(n-2)/2} \left\{ \bigcup_{i=1}^{n} Q^{(2)}_{ji} \right\},
\]

\[
Q^{(3)} = \bigcup_{j=1}^{n(n-2)/2} \left\{ \bigcup_{i=1}^{n} Q^{(3)}_{ji} \right\},
\]

\[
Q^{(3)} = \bigcup_{j=1}^{n(n-2)/2} \left\{ \bigcup_{i=1}^{n} Q^{(3)}_{ji} \right\},
\]

\[
Q^{(4)} = \bigcup_{j=1}^{(n-1)} \left\{ \bigcup_{i=1}^{n} Q^{(4)}_{ji} \right\},
\]

\[
Q^{(5)} = \{ (v_i, f_{i+2j}), (f_{i+2j}, f_i), (f_i, v_{i+2j}) \},
\]

\[
Q^{(5)} = \bigcup_{j=1}^{n(n-2)/2} \left\{ \bigcup_{i=1}^{n} Q^{(5)}_{ji} \right\},
\]

\[
Q^{(6)} = \bigcup_{j=1}^{n(n-2)/2} \left\{ \bigcup_{i=1}^{n} Q^{(6)}_{ji} \right\},
\]

Here, \(\langle Q^{(1)} \rangle \approx (n(n-3)/3)P_4, \langle Q^{(2)} \rangle \approx (n(n-2)/2)P_4, \langle Q^{(3)} \rangle \approx (n(n-2)/2)P_4, \langle Q^{(4)} \rangle \approx (n(n-1)/2)P_4, \langle Q^{(5)} \rangle \approx nP_4, \langle Q^{(6)} \rangle \approx nP_4\).

Hence, \(B_3 \left( K_{1,n}^+ \right) - nK_2 \) is P_4-decomposable.

**Case 2.** \(n\) is odd, \(n \geq 9\).

Then the edge set of \(B_3 \left( K_{1,n}^+ \right) \) can be decomposed into \((11n^2 - 3n)/6\) \(P_4\) and \(nK_2\). The edge set of \(nK_2\) is given by the set \(\bigcup_{i=1}^{n} \{ (v, f_i) \}\). The edge set of \((11n^2 - 3n)/6\) \(P_4\) is given by the sets \(Q^{(4)}, Q^{(5)}, Q^{(6)}\) as in Case 1 and the sets \(Q^{(7)}\) and \(Q^{(8)}\), where

\[
Q^{(7)} = \bigcup_{j=1}^{n(n-3)/3} \left\{ \bigcup_{i=1}^{n} Q^{(7)}_{ji} \right\},
\]

\[
Q^{(7)}_{ji} = \{ (e_{i+j}, v_i), (v_i, e_{i+j+1}), (e_{i+j+1}, v_{i+2j-2}) \},
\]

\[
Q^{(8)} = \bigcup_{j=1}^{n(n-2)/2} \left\{ \bigcup_{i=1}^{n} Q^{(8)}_{ji} \right\},
\]

\[
Q^{(8)}_{ji} = \{ (v_i, f_{i+j}), (f_{i+j}, f_i), (f_i, v_{i+2j}) \},
\]

Here, \(\langle Q^{(7)} \rangle \approx (n(n-3)/3)P_4, \langle Q^{(8)} \rangle \approx (n(n-1)/2)P_4\).

Hence, \(B_3 \left( K_{1,n}^+ \right) - nK_2 \) is P_4-decomposable.

\[\Box\]

**Theorem 3.7.** For \(n \geq 4\) and \(n \equiv 1(\text{mod } 3)\), the graph

\[B_3 \left( K_{1,n}^+ \right) - 2nK_2\]

is \(P_4\)-decomposable.

**Proof.** Then the edge set of \(B_3 \left( K_{1,n}^+ \right) \) can be decomposed into \((11n^2 - 5n)/6\) \(P_4\) and \(2nK_2\). The edge set of \(2nK_2\)
is given by the set $U^{n}_{j=1} \{(v,f_i),(u_i,e_i)\}$. The edge set of $(11n^2 - 7n)/6$ $P_4$ is given by the edge sets $Q^{(2)}, Q^{(3)}, Q^{(4)}, Q^{(5)}$ as in Theorem 3.6 and the set $Q^{(9)}$ where,

$$Q^{(9)}=U^{n}_{j=1}Q^{(9)}_{ji},$$

$$Q^{(9)}_{ji} = \{(e_{i+j},v_i), (v_i,e_{i+j+1}), (e_{i+j+1},v_{i+3j})\}.$$

Here, $Q^{(9)} \cong (n(n-1)/3)P_4$. Hence, $B_3\left(K_{1,n}^+\right) - 3nK_2$ is $P_4$- decomposable.

4. Conclusion

In this paper, $P_4$- Decomposition of Boolean Function Graph $B(KP, L(G), NINC)$ of path, cycle, stars and corona graphs are obtained.

References


