Zumkeller labeling of some path related graphs

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Abstract

A positive integer \( n \) is said to be Zumkeller number if its positive factors can be partitioned into 2 disjoint parts with the equal sum, that is each part with sum \( \sigma(n)/2 \). Let \( G = (V(G), E(G)) \) be a graph. An one to one function \( f \) defined on \( V(G) \) to a subset of natural numbers is termed as Zumkeller labeling of \( G \) if the induced function \( f^*: E(G) \rightarrow \mathbb{N} \) defined as \( f^*(xy) = f(x)f(y) \) assigns a Zumkeller number for all \( xy \in E(G) \), \( x, y \in V(G) \). A graph \( G = (V(G), E(G)) \) admits Zumkeller labeling is called a Zumkeller graph. In this manuscript, we investigate Zumkeller labeling for several classes of path graph.

Keywords

Graph labeling, Zumkeller number.

AMS Subject Classification

05C78.

1 Introduction

The assignments of values typically represented by integers using several appropriate mathematical rules to the vertices and/or edges of the given graph is termed as graph labeling. Formally, in 1967 Alex Rosa [1] introduced the concept of graph labeling. There is an enormous literature regarding labeling of many familiar classes of graphs [8].

A natural number is said to be perfect if \( \sigma(n) = 2n \), where \( \sigma(n) \) represents the sum of positive factors. A generalization of concept of perfect number, termed as Zumkeller number. It has been explored by in 2003. Zumkeller observed a sequence of natural numbers in which the positive factors of all number has been partitioned into two disjoint parts whose sums are equal, i.e., each part with sum \( \sigma(n)/2 \). Formally Zumkeller numbers was proposed by Clark et. al. [7]. Later, Peng and Bhaskara Rao were studied in detail about Zumkeller number and half Zumkeller number [13]. B. J. Balamurugan et. al. introduced the concept of Zumkeller labeling in the year 2013 [2]. The idea of Zumkeller labeling of some simple graphs has been previously reported [3, 4, 5, 6, 11].

In this article, we investigated the existence of Zumkeller labeling in path related graphs such as splitting graph of path \( S(P_n) \), total graph of path \( T(P_n) \), shadow graph of path \( D_2(P_n) \), middle graph of path \( M(P_n) \) etc.

2 Preliminaries

Definition 2.1. The graph which is gained from \( G \) by the addition of new node \( v \) for every node \( v \) in \( G \) and join \( v \) to all nodes of \( G \) are neighbours to \( v \) are the splitting graph \( S'(G) \).

Definition 2.2. For a graph \( G \), the total graph \( T(G) \) has the vertex set \( V(G) \cup E(G) \) and two vertices are adjacent in \( T(G) \) whenever their corresponding elements are either incident or adjacent in \( G \).

Definition 2.3. The shadow graph \( D_2(G) \) the connected graph \( G \) is built by taking two copies of \( G \) namely \( G' \) and \( G'' \) join each vertex \( u \) in \( G \) to the neighbours of the corresponding vertex \( u \) in \( G'' \).

Definition 2.4. Middle graph \( M(G) \) of a graph \( G \) is the graph whose vertex set is \( V(G) \cup E(G) \) and in which two vertices are adjacent whenever either one is a vertex of \( G \) and other is an edge incident with it or they are adjacent edges of \( G \).
Definition 2.5. If a positive integer n can be subdivided into two disjoint subset of equal sum by the positive divisors of n, then n is called Zumkeller number.

Theorem 2.6. [7] If n is a Zumkeller number and p is a prime with (n, p) = 1, then np is Zumkeller for any positive integer l.

Theorem 2.7. [7] For any prime p ≠ 2 and positive integer k with p ≤ 2^{k+1} - 1 the number 2^k p is a Zumkeller number.

Theorem 2.8. [10] Let n = 2^k p^l be a positive integer. Then n is a Zumkeller number if and only if p ≤ 2^{k+1} - 1 and l is an odd number.

Definition 2.9. [12] Let G = (V(G), E(G)) be a graph. An injective map f : V(G) → N is said to be Zumkeller labeling of the graph G, if the induced function f∗ : E(G) → N defined as f∗(xy) = f(x)f(y) Zumkeller number for all xy ∈ E(G), x, y ∈ V(G).

3. Main Results

Theorem 3.1. Splitting graph of path S(P_n) a Zumkeller graph.

Proof. The vertex set of Splitting graph of path S(P_n) is V = {vi, vi+1 : 1 ≤ i ≤ n} and the edge set of S(P_n) is E = {e_i = vi vi+1, e'_i = vi vi+1, e''_i = vi vi+1 : 1 ≤ i ≤ n - 1}. There are following cases:

Case 1 : n is odd
Define a one to one map f : V → N such that

\[
\begin{align*}
 f(v_i) &= \begin{cases} 
 2^{i+1} & i \equiv 1 \pmod{2} \\
 2^i & i \equiv 0 \pmod{2}
\end{cases} \quad 1 \leq i \leq n \\
 f(v'_i) &= \begin{cases} 
 2^{i+1} & i \equiv 1 \pmod{2} \\
 2^i & i \equiv 0 \pmod{2}
\end{cases} \quad 1 \leq i \leq n
\end{align*}
\]

where p is an odd prime less than 10 and q is any prime other than p. Now we define an induced function f∗ to f as follows

\[
\begin{align*}
 f^∗(e_i) &= f^∗(v_i v_{i+1}) = f(v_i)f(v_{i+1}) \quad 1 \leq i \leq n - 1 \\
 f^∗(e'_i) &= f^∗(v'_i v'_{i+1}) = f(v'_i)f(v'_{i+1}) \quad 1 \leq i \leq n - 1 \\
 f^∗(e''_i) &= f^∗(v''_i v''_{i+1}) = f(v''_i)f(v''_{i+1}) \quad 1 \leq i \leq n - 1
\end{align*}
\]

Then by definition of f we obtain

\[
\begin{align*}
 f^∗(v_i v_{i+1}) &= 2^{i+1} \cdot 2^{i+1} \cdot q = 2^{2i+1} \cdot pq \quad i \equiv 1 \pmod{2} \\
 f^∗(v_i v_{i+1}) &= 2^i \cdot 2^i \cdot q = 2^{2i} \cdot pq \quad i \equiv 0 \pmod{2}
\end{align*}
\]

Using Theorem 2.6 and Theorem 2.7 we get all edge labels are Zumkeller numbers.

Case 2 : n is even
Define a one to one map f : V → N such that

\[
\begin{align*}
 f(v_i) &= \begin{cases} 
 2^{i+1} & i \equiv 1 \pmod{2} \\
 2^i & i \equiv 0 \pmod{2}
\end{cases} \quad 1 \leq i \leq n \\
 f(v'_i) &= \begin{cases} 
 2^{i+1} & i \equiv 1 \pmod{2} \\
 2^i & i \equiv 0 \pmod{2}
\end{cases} \quad 1 \leq i \leq n
\end{align*}
\]

where p is an odd prime less than 10 and q is any prime other than p. Now we define an induced function f∗ to f as follows

\[
\begin{align*}
 f^∗(e_i) &= f^∗(v_i v_{i+1}) = f(v_i)f(v_{i+1}) \quad 1 \leq i \leq n - 1 \\
 f^∗(e'_i) &= f^∗(v'_i v'_{i+1}) = f(v'_i)f(v'_{i+1}) \quad 1 \leq i \leq n - 1 \\
 f^∗(e''_i) &= f^∗(v''_i v''_{i+1}) = f(v''_i)f(v''_{i+1}) \quad 1 \leq i \leq n - 1
\end{align*}
\]

Then by definition of f we obtain

\[
\begin{align*}
 f^∗(v_i v_{i+1}) &= 2^{i+1} \cdot 2^{i+1} \cdot q = 2^{2i+1} \cdot pq \quad i \equiv 1 \pmod{2} \\
 f^∗(v_i v_{i+1}) &= 2^i \cdot 2^i \cdot q = 2^{2i} \cdot pq \quad i \equiv 0 \pmod{2}
\end{align*}
\]

Using Theorem 2.6 and Theorem 2.7 we get all edge labels are Zumkeller numbers. By the above cases we get Splitting graph of path is a Zumkeller graph.

Example 3.2.

![Figure 1. S'(P_3) is an Zumkeller graph with p = 3 and q = 17](image-url)
Therefore Total graph of path $T(P_n)$ is a Zumkeller graph.

**Theorem 3.3.** Total graph of path $T(P_n)$ is a Zumkeller graph.

**Proof.** Let $V(T(P_n)) = \{v_i : 1 \leq i \leq n\}$ be the vertex set of the total graph of path $T(P_n)$. Let $E(T(P_n)) = \{v_iv_{i+1} : 1 \leq i < n\}$ and $E(T(P_n)) = \{v_iv_{i+1} : 1 \leq i < n\}$ be the edge set of the shadow graph of path $D_2(P_n)$.

We have two cases:

- **Case 1:** $n \equiv 1 \pmod{2}$

  Define a one to one map $f : V \rightarrow \mathbb{N}$ such that
  
  \[
  f(v_i) = \begin{cases} 
    2^{i+1}, & i \equiv 1 \pmod{2} \\
    2^p, & i \equiv 0 \pmod{2}
  \end{cases} \quad 1 \leq i \leq n
  \]

  where $p, q, r$ are distinct odd primes less than 10.

  An induced function $f^* : E \rightarrow \mathbb{N}$ such that
  
  \[
  f^*(v_iv_{i+1}) = f(v_i)f(v_{i+1}) \quad 1 \leq i < n
  \]

  Now we can show that the labels on the edges of Total graph of path $T(P_n)$ Zumkeller numbers through the following cases.

  - $f^*(v_iv_{i+1}) = 2^i \cdot 2^{i+1} = 2^{i+1} \quad 1 \leq i < n
  \]
  - $f^*(v_iv_{i+1}) = 2^{i+1} \cdot 2^i = 2^{i+1} \quad 1 \leq i < n
  \]
  - $f^*(v_iv_{i+1}) = 2^i \cdot 2^i = 2^{i+1} \quad 1 \leq i < n
  \]
  - $f^*(v_iv_{i+1}) = 2^i \cdot 2^{i+1} = 2^{i+1} \quad 1 \leq i < n
  \]

  Consuming theorem 2.6 and Theorem 2.7 we get all edge labels are Zumkeller numbers.

  Therefore Total graph of path $T(P_n)$ is a Zumkeller graph.

  **Example 3.4.**
Using Theorem 2.8 all edge labels are Zumkeller numbers. Thus Middle graph of path $D_2(P_n)$ admits Zumkeller labeling.

**Example 3.6.**

![Figure 4. $D_2(P_5)$ is an Zumkeller graph with $p = 19$ and $k = 4$.](image)

![Figure 5. $D_2(P_6)$ is an Zumkeller graph with $p = 19$ and $k = 6$.](image)

**Theorem 3.7.** Middle graph of path $M(P_n)$ admits Zumkeller labeling.

**Proof.** Let $V(M(P_n)) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n-1\}$ and $E(M(P_n)) = \{v_iu_i : 1 \leq i \leq n-1\} \cup \{v_iu_{i-1} : 1 \leq i \leq n\} \cup \{u_iu_{i+1} : 1 \leq i \leq n-2\}$ be the vertex set, edge set respectively of middle graph of path $M(P_n)$.

Define a one to one map $f : V \to \mathbb{N}$ such that

$$f(u_i) = r_i^2 \quad 1 \leq i \leq n-1$$

$$f(v_i) = \begin{cases} 2^{k+\frac{r-1}{2}}p & i \equiv 1 \pmod{2} \\ 2^{k+\frac{r+1}{2}}q & i \equiv 0 \pmod{2} \end{cases} \quad 1 \leq i \leq n$$

where, $p$ and $q$ are distinct odd prime numbers and $p,q \leq 2^{k+1}-1$ ($k \geq 2, k \in \mathbb{N}$) and $r$ is any prime number other than $p$ and $q$.

An induced function $f^* : E \to \mathbb{N}$ such that

$$f^*(v_iu_i) = f(v_i)f(u_i) \quad 1 \leq i \leq n-1$$

$$f^*(v_iu_{i-1}) = f(v_i)f(u_{i-1}) \quad 2 \leq i \leq n$$

$$f^*(u_iu_{i+1}) = f(u_i)f(u_{i+1}) \quad 1 \leq i \leq n-2$$

Now we calculate the edge labels

$$f^*(v_iu_i) = \begin{cases} 2^{k+\frac{r-1}{2}}pr & i \equiv 1 \pmod{2} \\ 2^{k+\frac{r+1}{2}}qr & i \equiv 0 \pmod{2} \end{cases}$$

$$f^*(v_iu_{i-1}) = \begin{cases} 2^{k+\frac{r+1}{2}}pr & i \equiv 1 \pmod{2} \\ 2^{k+\frac{r-1}{2}}qr & i \equiv 0 \pmod{2} \end{cases}$$

$$f^*(u_iu_{i+1}) = 2^{2k+1}pq$$

Using Theorem 2.6 and Theorem 2.7 we get all the edge labels are Zumkeller numbers.

Thus the graph Middle graph of path $M(P_n)$ admits Zumkeller labeling.

**Example 3.8.**

![Figure 6. $M(P_3)$ is an Zumkeller graph with $p = 7, q = 5, r = 41$ and $k = 4$.](image)

**References**


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