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# Some results in $b$-metric spaces

Nagaral Pandit Sanatammappa[1*], R. Krishnakumar[2] and K. Dinesh[3]

**Abstract**
In this paper, we establish some new contractive type condition for mappings defined on $b$-metric spaces and prove some new fixed point theorems for these mappings. Our results are generalizations of previous research's.

**Keywords**
$b$-Metric Spaces, Contractive Mapping, Fixed Point Theorems.

**AMS Subject Classification**
47H10, 54H25.

[1,2,3] *Department of Mathematics, Urumu Dhanalakshmi College (Affiliated to Bharathidasan University), Trichy-620019, Tamil Nadu, India.*
***Corresponding author**: [1] psn.mathsgt@gmail.com
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## Contents



## 1. Introduction and Preliminaries

Bakhtin [1] wasintroduced the concept of b-metric space and used by Czerwik in [6]. Banach's contraction mapping theorem says, A mapping $T: X \rightarrow X$ where $(X,d)$ is a metric space, is said to be a contraction if there exists $k \in \{0,1\ )$ such that $\forall s,t \in X$

$$d(Tp,Tq) \leq kd(p,q) \tag{1.1}$$

If the metric space $(X,d)$ is complete the mapping satisfying $(1.1)$ has a unique fixed point.

**Definition 1.1.** *Let $X$ be a non-empty set and let $s \geq 1$ be a given real number. A function $d: X \times X \rightarrow R_+$ is called a b-metric provided that, for all $p,q,r \in X$*

*(i)* $d(p,q)=0$ *iff* $p=q$

*(ii)* $d(p,q)=d(p,q)$

*(iii)* $d(p,r) \leq s[d(p,q)+d(q,r)]$

*A pair $(X,d)$ is called a b-metric space.*

**Definition 1.2.** *Let $(X,d)$ be a b-metric space. Then a sequence $\{p_n\}$ in $X$ is called a Cauchy sequence if and only if for all $\varepsilon > 0$ there exist $n(\varepsilon) \in N$ such that for each $n,m \geq n(\varepsilon)$ we have $d(p_n,p_m) < \varepsilon$.*

**Definition 1.3.** *Let $(X,d)$ be a b-metric space. Then a sequence $\{p_n\}$ in $X$ is called convergent sequence if and only if there exist $x \in X$ such that for all there exists $n(\varepsilon) \in N$ such that for all $n \geq n(\varepsilon)$ we have $d(p_n,x) < \varepsilon$.*

**Definition 1.4.** *The b-metric space is complete if every Cauchy sequence convergent.*

## 2. Main Results

**Theorem 2.1.** *Let $(X,d)$ be a complete b metrics space with constants $s \geq 1$ and define the sequence $\{p_n\}_{n=1}^{\infty} \subset X$ by the iteration $p_n = Tp_{n-1} = T^n p_0$ and let $T: X \rightarrow X$ be a mapping such that*

$$\begin{aligned}d(Tp,Tq) \leq{}& \alpha_1 d(p,q)+\alpha_2 d(p,Tp)+\alpha_3 d(q,Tq)+\alpha_4 d(p,Tp)\\&+\alpha_5 d(q,Tp)+\alpha_6[d(q,Tp)+d(p,Tq)]\end{aligned}$$

*where $\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+2\alpha_6 < 1, \forall p,q \in X$ then there exists $p^* \in X$ such that $p_n \rightarrow p^*$ and $p^*$ is a unique fixed point.*

*Proof.* Let $p_0 \in X$ and $\{p_n\}_{n=1}^{\infty}$ be a sequence in $X$ defined as $p_n = Tp_{n-1} = T^n p_0, n=1,2,3,.$

$$\begin{aligned}d(p_n,p_{n+1}) ={}& d(Tp_{n-1},Tp_n)\\ \leq{}& \alpha_1 d(p_{n-1},p_n)+\alpha_2 d(p_{n-1},Tp_{n-1})\\&+\alpha_3 d(x_n,Tp_n)+\alpha_4 d(p_{n-1},Tp_n)\\&+\alpha_5 d(p_n,Tp_{n-1})+\alpha_6[d(p_n,Tp_{n-1})\\&+d(p_{n-1},Tp_n)]\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_1 d(p_{n-1},p_n)+\alpha_2 d(p_{n-1},p_n)\\
&\quad+\alpha_3 d(p_n,p_{n+1})+\alpha_4 d(p_{n-1},p_{n+1})\\
&\quad+\alpha_5 d(p_n,p_n)+\alpha_6\left[d(p_n,p_n)+d(p_{n-1},p_{n+1})\right]\\
&\leq \alpha_1 d(p_{n-1},p_n)+\alpha_2 d(p_{n-1},p_n)\\
&\quad+\alpha_3 d(p_n,p_{n+1})+s\alpha_4 d(p_{n-1},p_n)\\
&\quad+s\alpha_4 d(p_n,p_{n+1})+s\alpha_6 d(p_{n-1},p_n)+s\alpha_6 d(p_n,p_{n+1})
\end{aligned}
$$

$$
\begin{aligned}
&(1-\alpha_3-s\alpha_4-s\alpha_6)d(p_n,p_{n+1})\\
&\qquad\leq(\alpha_1+\alpha_2+s\alpha_4+s\alpha_6)\,d(p_{n-1},p_n)
\end{aligned}
$$

$$
\begin{aligned}
d(p_n,p_{n+1})&\leq\frac{\alpha_1+\alpha_2+s\alpha_4+s\alpha_6}{1-\alpha_3-s\alpha_4-s\alpha_6}d(p_{n-1},p_n)\\
d(p_n,p_{n+1})&\leq\eta d(p_{n-1},p_n)
\end{aligned}
$$

where, $\eta=\frac{\alpha_1+\alpha_2+s\alpha_4+s\alpha_6}{1-\alpha_3-s\alpha_4-s\alpha_6}$

$$
\begin{aligned}
d(p_n,p_{n+1})&\leq\eta d(p_{n-1},p_n)\\
&\leq\eta^2 d(p_{n-2},p_{n-1})
\end{aligned}
$$

Continuing this process we get,

$$\leq\eta^n d(p_0,p_1)$$

Now we show that $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $X$. Let $m,n>0$ with $m>n$

$$
\begin{aligned}
d(p_n,p_m)&\leq sd(p_n,p_{n+1})+s^2d(p_{n+1},p_{n+2})\\
&\quad+s^3d(p_{n+2},p_{n+3})+\cdots\\
&\leq s\eta^n d(p_1,p_0)+s^2\eta^{n+1}d(p_1,p_0)\\
&\quad+\cdots+s^m\eta^{n+m-1}d(p_1,p_0)\\
&\leq s\eta^n d(p_1,p_0)\left[1+(s\eta)+(s\eta)^2\right.\\
&\quad\left.+\cdots+(s\eta)^{m-1}\right]\\
&\leq s\eta^n d(p_1,p_0)\left[\frac{1-(s\eta)^{n-(m-1)}}{1-s\eta}\right]
\end{aligned}
$$

Take $m,n\to\infty$

$$\lim_{n\to\infty}d(p_n,p_m)=0$$

Hence $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $X$.
Since $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence $\{p_n\}$ converges to $p^*\in X$.
Now we show that $x^*$ is the unique fixed point of $T$.

$$
\begin{aligned}
d(p^*,Tp^*)&\leq s\left[d(p^*,p_{n+1})+d(p_{n+1},Tp^*)\right]\\
&\leq s\left[d(p^*,p_{n+1})+d(Tp_n,Tp^*)\right]\\
&\leq sd(p^*,p_{n+1})+s\left[\alpha_1 d(p_n,p^*)+\alpha_2 d(p_n,Tp_n)\right.\\
&\quad+\alpha_3 d(p^*,Tp^*)+\alpha_4 d(p_n,Tp^*)+\alpha_5 d(p^*,Tp_n)\\
&\quad\left.+\alpha_6\left[d(p^*,Tp_n)+d(p_n,Tp^*)\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq sd(p^*,p_{n+1})+s\alpha_1 d(p_n,p^*)+s^2\alpha_2 d(p_n,p^*)\\
&\quad+s^2\alpha_2 d(p^*,p_{n+1})+s\alpha_3 d(p^*,Tp^*)+s^2\alpha_4 d(p^*,p_n)\\
&\quad+s^2\alpha_4 d(p^*,Tp^*)+s\alpha_5 d(p^*,p_{n-1})+s\alpha_6 d(p^*,p_{n-1})\\
&\quad+s^2\alpha_6 d(p_n,p^*)+s^2\alpha_6 d(p^*,Tp^*)\\
&\quad\left(1-s\alpha_3-s^2\alpha_4-s^2\alpha_6\right)d(p^*,Tp^*)\\
&\leq\left(s+s^2\alpha_2\right)d(p^*,p_{n+1})+\left(s\alpha_1+s^2\alpha_2+s^2\alpha_4+s^2\alpha_6\right)\\
&\qquad d(p_n,p^*)+(s\alpha_5+s\alpha_6)\,d(p^*,p_{n+1})
\end{aligned}
$$

$d(p^*,Tp^*)\leq 0$ as $n\to\infty$. Now we show that $p^*$ is the fixed point of $T$. Assume that $p$ ' is another fixed point of $T$. Then we have $Tp'=p'$.

$$
\begin{aligned}
d\left(p^*,p'\right)&=d\left(Tp^*,Tp'\right)\\
&\leq\alpha_1 d\left(p^*,p'\right)+\alpha_2 d(p^*,Tp^*)+\alpha_3 d\left(p',Tp'\right)\\
&\quad+\alpha_4 d\left(p^*,Tp'\right)+\alpha_5 d\left(p',Tp^*\right)\\
&\quad+\alpha_6\left[d\left(p',Tp^*\right)+d\left(p^*,Tp'\right)\right]\\
&\leq\alpha_1 d\left(p^*,p'\right)+\alpha_2 d(p^*,p^*)+\alpha_3 d\left(p',p'\right)\\
&\quad+\alpha_4 d\left(p^*,p'\right)+\alpha_5 d\left(p',p^*\right)\\
&\quad+\alpha_6\left[d\left(p',p^*\right)+d\left(p^*,p'\right)\right]\\
d\left(p^*,p'\right)&\leq(\alpha_1+\alpha_4+\alpha_5+2\alpha_6)\,d\left(p^*,p'\right)
\end{aligned}
$$

$\Rightarrow p^*=p'$. $\therefore T$ has a unique fixed point. □

**Corollary 2.2.** *Let $(X,d)$ be a complete b metrics space with constants $s\geq 1$ and define the sequence $\{p_n\}_{n=1}^{\infty}\subset X$ by the iteration $p_n=Tp_{n-1}=T^np_0$ and let $T:X\to X$ be a mapping such that*

$$
\begin{aligned}
d(Tp,Tq)\leq&\alpha_1 d(p,q)+\alpha_2 d(p,Tp)+\alpha_3 d(q,Tq)\\
&+\alpha_4 d(p,Tq)+\alpha_5 d(q,Tp)
\end{aligned}
$$

*where $\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5<1,\forall p,q\in X$ then there exists $p^*\in X$ such that $p_n\to p^*$ and $p^*$ is a unique fixed point.*

*Proof.* We take $\alpha_6=0$ in previous theorem we get the solution □

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