Prime bi-interior Γ-ideals of TG-semiring

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Abstract
The notation of prime bi-interior ideal, semi prime bi-interior ideal, irreducible bi-interior ideal and strongly prime bi-interior ideal of TG-Semi ring (ternary gamma semi-ring) are introduced. We study properties of these ideals and relations between them and also characterize regular TG-Semi-ring and TG-Semi-ring using prime bi-interior ideals, irreducible and strongly irreducible bi-interior ideals in this article.

Keywords
TGS, bi-interior ideal, prime ideals, prime bi-interior ideal, strong prime bi-interior ideal and irreducible and strongly irreducible bi-interior ideals.

AMS Subject Classification
16W25, 16N60, 16U80.

1. Introduction

We introduced the notation of prime bi-interior ideals of TG Semi rings, in this paper. G. Srinivasa Rao et.al [29-36] studied ternary semi rings.

2. Preliminaries

Let \((R, +)\& (\Gamma, +)\), commutative semi-groups. Then we call \(R\) a \(T\) \(G\) -Semi ring (\(TGS\)), if there is a mapping \(R \times \Gamma \times R \rightarrow R\) (images of \((p, a, q, b, r)\) will be denoted by \(pagbr\)) \(\forall p, q, r, a, b, c \in \Gamma\) it satisfies the following axioms \(\forall p, q, r, s, t \in R\) and \(a, b, c, d \in \Gamma\)

1. \(pa(q+r)s = pagbs + parbs\)
2. \((p+q)arbs = parbs + qarbs\)

3. \(paqb(r+s) = paqbr + paqbs\)
4. \(pa(qbrcs)dt = paqbr(csdt) = (paqbr)csdt\).

ATGSR is said to be commutative TGS if \(paqbr = parbq = qarbp = qapbr = rapbq = raqbp, \forall p, q, r \in R\) and \(a, b \in \Gamma\). Suppose \(R\), a TGS. If for each \(p\) in \(R\), \(a, b \in \Gamma\) \(papbe = paebp = eabp = e\), then an element \(e \in R\) is called aunityelement or neutral element.

Definition 2.1. Suppose \(R\), a ternary \(\Gamma\)-semi-ring. \(\emptyset \neq S\) is said to be a ternary sub- \(\Gamma\)-semi-ring of \(R\), if \(S\) is an sub-semi-group with respect to + of \(R\) and \(\alpha\beta c \in S, \forall a, b, c \in S\) and \(\alpha, \beta \in \Gamma\).

Definition 2.2. The set \(\emptyset \neq I \subseteq R\), where \(R\) is a ternary \(\Gamma\)-semi-ring is said to be left (lateral, right)ternary \(\Gamma\)-ideal of \(R\), if

1. \(a, b \in I \Rightarrow a + b \in I\)
2. \(a, b \in R, i \in I, \alpha, \beta \in \Gamma \Rightarrow a\alpha\beta i \in I(a\alpha\beta b \in I, ia, a\beta b \in I)\).

An ideal \(I\) is said to be ternary \(\Gamma\)-ideal, if it is left, lateral and right \(\Gamma\)-ideal of \(R\).
Example 2.3. The set $Z = \{0, \pm 1, \pm 2, \pm 3, \ldots \}$ and $\Gamma$, the set of even natural numbers. Then with respect to usual addition and ternary multiplication, $Z$ is a ternary gamma semi-ring.

Example 2.4. The set $R = \{0, \pm i, \pm 2i, \pm 3i, \ldots \}$ and $\Gamma = N$. Then $R$ is a ternary gamma semi-ring with respect to usual addition and ternary multiplication.

Example 2.5. Let $Q = R$, the set of all rational numbers and $\Gamma$ the set of all real numbers. Define a mapping from $R \times \Gamma \times R \times \Gamma \to R$ by usual addition and ternary multiplication defined by $(p, a, b, r) = paqr \forall p, q, r \in R, a, b \in \Gamma$ then $R$ is a ternary gamma semi-ring.

Definition 2.6. Let $\phi \neq S \subseteq R$, where $R$ be a TGS. The set $S$ said to be a TG-sub-semi-ring of $R$ if $(S, +)$ is a sub-semi-group of $(R, +)$ and $S_{\Gamma S} \subseteq S$.

Definition 2.7. Let $R$ be a TGS and $\phi \neq S \subseteq R$. The set $S$ said to be a quasi-ideal $(QI)$ if $S$ is a TGsub-semi-ring (TGSSR) of $R$ and $S \cap (S \cap R \cap S \cap R \cap S) \cap R \subseteq S$.

Definition 2.8. Let $\phi \neq S \subseteq R$, where $R$ be a TGS. The set $S$ said to be a bi-ideal (BI) of $R$ if $S$ is a TGSSR of $R$ and $S_{\Gamma S} \subseteq S$.

Definition 2.9. Let $\phi \neq S \subseteq R$, where $R$ be a TGS. The set $S$ said to be an interior ideal (II) of $R$ if $S$ is a TGSSR of $R$ and $S_{\Gamma S} \subseteq S$.

Definition 2.10. Let $R$ be a TGS and $\phi \neq S \subseteq R$. The set $S$ said to be a rt. (medial, lt.) ideal of $R$ if $S$ is a TGSSR of $R$ and $S_{\Gamma S} \subseteq S, S_{\Gamma S} \subseteq S, S_{\Gamma S} \subseteq S$.

Theorem 3.7. A BII $P$ of a TGS $R$ is a prime bi-interior ideal (PBII) of $R$ if $P_{\Gamma P} P \subseteq P \Rightarrow P_{\Gamma P} P = P$.

Definition 3.3. A BII $P$ of $R$ is called a strongly prime bi-interior ideal (SPBI) of $R$ if for any bi-ideal $P_{\Gamma P} P \subseteq P \Rightarrow P_{\Gamma P} P = P$.

Definition 3.4. A bi-interior ideal (BII) $P$ of $R$ is said to be an irreducible bi-interior ideal (IBI) if $P_{\Gamma P} P \subseteq P \Rightarrow P_{\Gamma P} P = P$. Clearly every IBI is an IBI.

Note 3.6. (i) Every STPBII of a TGS $R$ is a PBII of $R$.

(ii) Every PBII $P$ of a TGS $R$ is a SPBI of $R$.

Theorem 3.8. If $P_{\Gamma P} P$ be bi-interior ideals of a TGS $R$ and $(P_{\Gamma P} P_{\Gamma P} P) \cap (P_{\Gamma P} P_{\Gamma P} P) = P_{\Gamma P} P_{\Gamma P} P$ then every bi-interior ideal of a TGS $R$ is a semi prime BII-ideal of $R$.

Proof. Let $P_{\Gamma P} P_{\Gamma P} P$ be bi-interior ideals of a TGS $R$ and $(P_{\Gamma P} P_{\Gamma P} P) \cap (P_{\Gamma P} P_{\Gamma P} P) = P_{\Gamma P} P_{\Gamma P} P$. Let $P$ be a bi-interior ideal of a TGS $R$. To prove, $P$ is a semi-prime ideal of $R$. Suppose $P_{\Gamma P} P_{\Gamma P} \subseteq P$. Then $P_{\Gamma P} P_{\Gamma P} \subseteq P_{\Gamma P} P_{\Gamma P} P \subseteq P_{\Gamma P} P_{\Gamma P} P$. Hence every bi-interior ideal of $R$ is semi prime.
**Definition 3.9.** Let $R$ be a TGS. An element $p \in R$ is said to be a regular element if there exists $x, y \in R$ and $a, b, c, d \in \Gamma$ such that $p = paxybpcydp$. Every element in TGS is a regular element then $R$ is known as Regular TGS.

**Theorem 3.10.** $R$ is a regular TG semi ring $\iff \Gamma_{\cap J \cap K} \subseteq I \cap J \cap K$ for any left ideal $K$, lateral ideal $J$ and right ideal $I$ of $R$.

**Proof.** Let $I, J, K$ be a rt. ideal, medial and a lt. ideal of a regular TGS $R$ respectively. Suppose $R$ be a regular TGS. Clearly $\Gamma_{\cap J \cap K} \subseteq I \cap J \cap K$. It enough to show $I \cap J \cap K \subseteq \Gamma_{\cap J \cap K}$. Let $p \in I \cap J \cap K$. Since $R$ is a regular TG semi ring, there exist $a, b, c, d \in \Gamma$ and $x, y \in R$ such that $p = paxybpcydp$. Since $paxybpcydp \in I$, and $paxybpcydp \in J$ $\Rightarrow$ paxybpcydp $\in \Gamma_{\cap J \cap K}$. Thus $p \in \Gamma_{\cap J \cap K}$. Hence $\Gamma_{\cap J \cap K} = I \cap J \cap K$. Against suppose that $\Gamma_{\cap J \cap K} = I \cap J \cap K$, for any left ideal $I$, for any lateral ideal or medial $J$ and right ideal $I$ of $R$. Let $p \in R$ and $I$ be the right ideal generated by $p$, $J$ be a lateral ideal generated by $p$ and $K$ be a left ideal generated by $p$. Implies $p \in I \cap J \cap K = \Gamma_{\cap J \cap K}$. Since $I$ is a right ideal generated by $p$, we have $p \in I$ implies $p = paxybpcydp$ and also since $J$ is a lateral ideal generated by $p$, we have $p \in J$ implies $p = pacybpcydp$. Consider $p = paxybpcydp \Rightarrow paxybpcydp$ $\Rightarrow$ paxybpcydp then $p$ is a regular element and thus $R$ is a regular TG semi ring. $\square$

**Theorem 3.11.** If $\Gamma_{\cap J \cap K} = P$, for all bi interior ideals $P$ of a TGS $R$, then $R$ is a regular and

$$P_1 \cap P_2 \cap P_3 = (P_1 \cap P_2 \cap P_3) \cap (P_2 \cap P_3 \cap P_1) \cap (P_3 \cap P_1 \cap P_2)$$

for any bi-interior ideals $P_1, P_2$ and $P_3$ of $R$.

**Proof.** Let $\Gamma_{\cap J \cap K} = P$, for all BIIs $P$ of a TGS $R$. Let $I$ be a rt. ideal, $J$ be a medial ideal and $L$ be a lt. ideal of $R$. Then $I \cap J \cap L$ is a BI of $R$. ... $(I \cap J \cap L) \subseteq I \cap J \cap L \subseteq \Gamma_{\cap J \cap K}$. Hence $I \cap J \cap L \subseteq \Gamma_{\cap J \cap K}$.

**Theorem 3.12.** If $P$ is a BI of $R$ and $p \in P$ such that $p \in P$ then $\exists$ an $\mathbb{Z}$ of $P \subseteq P$ and $j \in P$.

**Theorem 3.13.** Suppose $R$, a regular TGS and $\Gamma_{\cap J \cap K} = P$, for all BI $P$ of $R$. Then any BI $P$ of $R$ is STBIH $\iff P$ is STBII.

**Proof.** Given $R$ is a regular $\Gamma$ -semi ring and $\Gamma_{\cap J \cap K} = P$, for all BIIs $P$ of a TGS $R$. Suppose $P$ be a STBII of $R$. Now we show that $P$ is a STBII. Suppose that $(P_1 \cap P_2 \cap P_3) \cap (P_2 \cap P_3 \cap P_1) \cap (P_3 \cap P_1 \cap P_2) \subseteq P$ then $\Gamma_{\cap J \cap K} = P$. Then by Theorem [3.11], $P \cap P_1 \cap P_2 \cap P_3 = (P_1 \cap P_2 \cap P_3) \cap (P_2 \cap P_3 \cap P_1) \cap (P_3 \cap P_1 \cap P_2)$ for any bi-interior ideals $P_1, P_2$ and $P_3$ of $R$. $(P_1 \cap P_2 \cap P_3) \cap (P_2 \cap P_3 \cap P_1) \cap (P_3 \cap P_1 \cap P_2) \subseteq P \Rightarrow P \cap P_1 \cap P_2 \cap P_3 \subseteq P \Rightarrow P \subseteq P$ or $P \subseteq P$ or $P \subseteq P$. Thus $P$ is a strongly prime bi-interior ideal of $R$. Conversely assume $P$ is a STBII of $R$. Let $P_1, P_2$ and $P_3$ be BIIs of $R$. $P_1 \cap P_2 \cap P_3 \subseteq P \Rightarrow (P_1 \cap P_2 \cap P_3) \cap (P_2 \cap P_3 \cap P_1) \cap (P_3 \cap P_1 \cap P_2) = \{(P_1 \cap P_2 \cap P_3) \cap (P_2 \cap P_3 \cap P_1) \cap (P_3 \cap P_1 \cap P_2)\} \cup \{(P_1 \cap P_2 \cap P_3) \cap (P_2 \cap P_3 \cap P_1) \cap (P_3 \cap P_1 \cap P_2)\}$ and $(P_1 \cap P_2 \cap P_3) \cap (P_2 \cap P_3 \cap P_1) \cap (P_3 \cap P_1 \cap P_2) \subseteq P_1 \cap P_2 \cap P_3 \subseteq P$. Hence $P$ is a strongly prime bi-interior ideal of $R$. $\square$
\[(P_1 \cap P_2 \cap P_3)^3 \subseteq (P_2 \Gamma P_3 \Gamma P_1 \Gamma P_1 \Gamma P_1). (P_1 \cap P_2 \cap P_3)^3 \subseteq (P_3 \Gamma P_1 \Gamma P_2)\].

Therefore
\[(P_1 \cap P_2 \cap P_3)^3 = (P_1 \Gamma P_2 \Gamma P_3 \Gamma P_1 \Gamma P_1 \Gamma P_1) \cap (P_2 \Gamma P_3 \Gamma P_1 \Gamma P_1 \Gamma P_1) \subseteq P\).

Since \(P\) is a semi-prime bi-interior ideal of \(R\), \(P_1 \cap P_2 \cap P_3 \subseteq P\) and also since \(P\) is a strongly irreducible bi-interior ideal, we have \(P_1 \subseteq P\) or \(P_2 \subseteq P\) or \(P_3 \subseteq P\). Hence \(P\) is a strongly prime bi-interior ideal of \(R\).

Theorem 3.19. Let \(R\) be a TG-semi ring. Prove the following:

1. The family of BI ideals of \(R\) is totally ordered set with respect to set inclusion \(\Leftarrow\)
2. Every BII of \(R\) is strongly irreducible \(\Leftrightarrow\) (3) Every BII of \(R\) is irreducible.

Proof. Given \(R\) is a TG-semi ring. Suppose, the set of BI ideals of \(R\) is a totally ordered set with respect to \(\subseteq\). Now we show that each BII of \(R\) is strongly irreducible. Let \(P\) be any BI ideal of \(R\). It is enough to show \(P\) is a STIBI ideal of \(R\). Let \(P_1, P_2\) and \(P_3\) be BI ideals of \(R\) such that \(P_1 \cap P_2 \cap P_3 \subseteq P\). From the hypothesis, we have either \(P_1 \subseteq P_2 \cap P_3 \subseteq P_1\) or \(P_2 \subseteq P_1 \cap P_3 \subseteq P_2\) or \(P_3 \subseteq P_1 \cap P_2 \subseteq P_3\). Hence \(P_1 \subseteq P_2 \cap P_3 \subseteq P\) or \(P_2 \subseteq P_1 \cap P_3 \subseteq P\) or \(P_3 \subseteq P_1 \cap P_2 \subseteq P\). Thus \(P\) is a STIBI ideal of \(R\). Hence \((1) \Rightarrow (2)\).

\((2) \Rightarrow (3)\): Let, every BII of \(R\) be STII. To show every BI ideal of \(R\) is irreducible. Let \(B\) be any BII of \(R \ni B_1 \cap B_2 \cap B_3 = B\). By BIIs \(B_1\), \(B_2\) and \(B_3\) of \(R\). Hence from our assumption \((2)\), we have \(B_1 \subseteq B \) or \(B_2 \subseteq B \) or \(B_3 \subseteq B \). As \(B_2 \subseteq B_1 \) and \(B_1 \subseteq B_2 \) and \(B_2 \subseteq B_3\), we have \(B_1 = B\) or \(B_2 = B\) or \(B_3 = B\). Therefore \(B\) is an IBII of \(R\).

\((2) \Rightarrow (1)\): Let each BII of \(R\) is an IBII. Let \(P_1, P_2\) and \(P_3\) be any BIIs of \(R\). Then by the remark, \(P_1 \cap P_2 \cap P_3\) is also a BI of \(R\). Hence \(P_1 \cap P_2 \cap P_3 = P_1 \cap P_2 \cap P_3 \Rightarrow P_1 = P_1 \cap P_2 \cap P_3 \) or \(P_2 = P_1 \cap P_2 \cap P_3 \) or \(P_3 = P_1 \cap P_2 \cap P_3 \) by our assumption. \(P_1 \subseteq P_2 \subseteq P_3 \) or \(P_2 \subseteq P_1 \subseteq P_3 \) or \(P_3 \subseteq P_1 \subseteq P_2 \). The collection of all BIIs of \(R\) is a totally ordered set under \(\subseteq\).

Therefore given conditions are equivalent.

4. Conclusion

Generalization of ideals of algebraic structures and ordered algebraic structure plays a very remarkable role and also necessary for further advance studies and applications of various algebraic structures. We introduced the notion of prime bi-interior ideal, semi prime bi-interior ideal, irreducible bi-interior ideal and strongly prime bi-interior ideal of TGSR and explained axioms and relations between them and also characterized regular TGSR and TGSR using PBI ideals.

References