Fixed point theorems of integral type contraction in \(b\)-metric spaces

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**Abstract**
In this paper, we establish some new integral type contraction defined on \(b\)-metric spaces and prove some new fixed point theorems for these mappings. Our results are generalizations of previous research's.

**Keywords**
\(b\)-Metric Spaces, integral type contraction, Fixed Point Theorems.

**AMS Subject Classification**
47H10, 54H25.

**1. Introduction and Preliminaries**

Bakhtin [1] introduced the concept of \(b\)-metric space and used by Czerwik in [6]. Banach’s contraction mapping theorem says, A mapping \(T : X \to X\) where \((X,d)\) is a metric space, is said to be a contraction if there exists \(k \in [0,1)\) such that for all \(x,y \in X\)

\[
d(T(x),T(y)) \leq kd(x,y)
\]

If the metric space \((X,d)\) is complete the mapping satisfying (1.1) has a unique fixed point.

**Definition 1.1.** Let \(X\) be a non-empty set and let \(s \geq 1\) be a given real number. A function \(d : X \times X \to \mathbb{R}_+\) is called a \(b\)-metric provided that, for all \(p,q,r \in X\)

(i) \(d(p,q) = 0\) iff \(p = q\)

(ii) \(d(p,q) = d(q,p)\)

(iii) \(d(p,r) \leq sd(p,q) + d(q,r)\)

A pair \((X,d)\) is called a \(b\)-metric space.

**Definition 1.2.** Let \((X,d)\) be a \(b\)-metric space. Then a sequence \(\{p_n\}\) in \(X\) is called a Cauchy sequence if and only if for all \(\varepsilon > 0\) there exist \(n(\varepsilon) \in \mathbb{N}\) such that for each \(n,m \geq n(\varepsilon)\) we have \(d(p_n,p_m) < \varepsilon\).

**Definition 1.3.** Let \((X,d)\) be a \(b\)-metric space. Then a sequence \(\{p_n\}\) in \(X\) is called convergent sequence if and only if there exist \(x \in X\) such that for all \(n \geq n(\varepsilon)\) such that for all \(n \geq n(\varepsilon)\) we have \(d(p_n,x) < \varepsilon\).

**Definition 1.4.** The \(b\)-metric space is complete if every Cauchy sequence convergent.

**Definition 1.5.** Let \((X,d)\) be a metric space and \(T : X \to X\) be function, \(\zeta : [0,\infty) \to [0,\infty)\) be Lebesgue-integrable mapping, if there exist \(\alpha \in [0,1)\) such that for all \(p,q \in X\)

\[
\int_0^{d(Tp,Tq)} \zeta(t)dt \leq \alpha \int_0^{d(p,q)} \zeta(t)dt
\]

**2. Main Results**

**Theorem 2.1.** Let \((X,d)\) be a complete \(b\) metric space with constants \(s \geq 1\) and define the sequence \(\{x_n\}_{n=1} \subset X\) by the iteration \(x_n = T^nx_n\) and let \(T : X \to X\) be a function and \(\zeta : [0,\infty) \to [0,\infty)\) be Lebesgue-integrable mapping such
that
\[ \int_0^{d(Tp,Tq)} \zeta(t)dt \]
\[ \leq \alpha_1 \int_0^{d(p,q)} \zeta(t)dt + \alpha_2 \int_0^{d(Tp,p)} \zeta(t)dt \]
\[ + \alpha_3 \int_0^{d(Tq,p)} \zeta(t)dt + \alpha_4 \int_0^{d(Tp,Tq)} \zeta(t)dt \]
\[ + \alpha_5 \int_0^{d(Tq,p)} \zeta(t)dt + \alpha_6 \int_0^{d(Tq,Tp)} \zeta(t)dt \]
where \( \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 < 1 \), \( \forall p, q \in X \) then there exists \( p^* \in X \) such that \( p_n \to p^* \) and \( p^* \) is a unique fixed point.

**Proof.** Let \( p_0 \in X \) and \( \{p_n\}_{n=1}^\infty \) be a sequence in \( X \) defined as \( p_n = Tp_{n-1} = T^n p_0 \), \( n = 1, 2, 3, \ldots \).

\[ \int_0^{d(p_n,p_m)} \zeta(t)dt = \int_0^{d(p_{n-1},p_m)} \zeta(t)dt \]
\[ \leq \alpha_1 \int_0^{d(p_{n-1},p_m)} \zeta(t)dt + \alpha_2 \int_0^{d(p_{n-1},Tp_{n-1})} \zeta(t)dt \]
\[ + \alpha_3 \int_0^{d(p_{n-1},Tp_{n-1})} \zeta(t)dt + \alpha_4 \int_0^{d(p_{n-1},Tq)} \zeta(t)dt \]
\[ + \alpha_5 \int_0^{d(p_{n-1},Tq)} \zeta(t)dt + \alpha_6 \int_0^{d(Tq,Tp_{n-1})} \zeta(t)dt \]
\[ \leq \alpha_1 \int_0^{d(p_{n-1},p_m)} \zeta(t)dt + \alpha_2 \int_0^{d(p_{n-1},p_m)} \zeta(t)dt \]
\[ + \alpha_3 \int_0^{d(p_{n-1},p_m)} \zeta(t)dt + \alpha_4 \int_0^{d(p_{n-1},p_m)} \zeta(t)dt \]
\[ + \alpha_5 \int_0^{d(p_{n-1},p_m)} \zeta(t)dt + \alpha_6 \int_0^{d(Tq,Tp_{n-1})} \zeta(t)dt \]
\[ \leq (1 - \alpha_3 - \alpha_4 - \alpha_6) \int_0^{d(p_{n-1},p_m)} \zeta(t)dt \]
\[ \leq (\alpha_1 + \alpha_2 + \alpha_4 + \alpha_6) \int_0^{d(p_{n-1},p_m)} \zeta(t)dt \]
\[ \int_0^{d(p_{n-1},p_m)} \zeta(t)dt \leq \frac{\alpha_1 + \alpha_2 + \alpha_4 + \alpha_6}{1 - \alpha_3 - \alpha_4 - \alpha_6} \int_0^{d(p_{n-1},p_m)} \zeta(t)dt \]
\[ \int_0^{d(p_{n-1},p_m)} \zeta(t)dt \leq \eta \int_0^{d(p_{n-1},p_m)} \zeta(t)dt \]

where \( \eta = \frac{\alpha_1 + \alpha_2 + \alpha_4 + \alpha_6}{1 - \alpha_3 - \alpha_4 - \alpha_6} \).

\[ \int_0^{d(p_n,p_{n+1})} \zeta(t)dt \leq \eta \int_0^{d(p_{n-1},p_n)} \zeta(t)dt \]
\[ \leq \eta^2 \int_0^{d(p_{n-2},p_{n-1})} \zeta(t)dt \]

Continuing this process we get,
\[ \leq \eta^n \int_0^{d(p_0,p_1)} \zeta(t)dt \]

Now we show that \( \{p_n\}_{n=1}^\infty \) is a Cauchy sequence in \( X \). Let \( m, n > 0 \) with \( m > n \)
\[ \int_0^{d(p_n,p_m)} \zeta(t)dt \]
\[ \leq s \int_0^{d(p_n,p_{n+1})} \zeta(t)dt + s^2 \int_0^{d(p_{n+1},p_{n+2})} \zeta(t)dt \]
\[ + s^3 \int_0^{d(p_{n+2},p_{n+3})} \zeta(t)dt + \ldots \]
\[ \leq s \eta^n \int_0^{d(p_1,p_0)} \zeta(t)dt + s^2 \eta^{n+1} \int_0^{d(p_1,p_0)} \zeta(t)dt \]
\[ + \ldots + s^m \eta^{n+m-1} \int_0^{d(p_1,p_0)} \zeta(t)dt \]
\[ \leq s \eta^n \frac{1 - (s \eta)^{m-1}}{1 - s \eta} \int_0^{d(p_1,p_0)} \zeta(t)dt \]

Take \( m, n \to \infty \)
\[ \lim_{n \to \infty} d(p_n, p_m) = 0. \]

Hence \( \{p_n\}_{n=1}^\infty \) is a Cauchy sequence in \( X \). Since \( \{p_n\}_{n=1}^\infty \) is a Cauchy sequence \( \{p_n\} \) converges to \( p^* \in X \). Now we show that \( p^* \) is the unique fixed point of \( T \).

\[ \int_0^{d(p^*,Tp^*)} \zeta(t)dt \leq s \int_0^{d(p^*,p_{n+1})+d(Tp^*,Tp_{n+1})} \zeta(t)dt \]
\[ \leq s \int_0^{d(Tp^*,Tp_{n+1})} \zeta(t)dt \]
\[ \leq s \int_0^{d(Tp^*,Tp_{n+1})} \left[ \alpha_1 d(p_n,p^*) + \alpha_2 d(p_n,Tp_n) + \alpha_3 d(Tp_n,Tp_{n+1}) + \alpha_4 d(p, Tp_{n+1}) + \alpha_5 d(p^*, Tp_{n+1}) + \alpha_6 d(Tp^*, Tp_{n+1}) \right] \zeta(t)dt \]
Another fixed point of $p$

$\Rightarrow$

$\leq$

$\leq$

$\int_0^{d(p',p^n)}\zeta(t)dt + s\alpha_1\int_0^{d(p^n,p^n)}\zeta(t)dt$

$+ s^2\alpha_2\int_0^{d(p^n,p^n)}\zeta(t)dt$

$+ s^2\alpha_2\int_0^{d(p^n,p^n)}\zeta(t)dt$

$+ s^2\alpha_2\int_0^{d(p^n,p^n)}\zeta(t)dt$

$+ s^2\alpha_2\int_0^{d(p^n,p^n)}\zeta(t)dt$

$+ s^2\alpha_2\int_0^{d(p^n,p^n)}\zeta(t)dt$

$(1-s\alpha_2-s^2\alpha_2-s^2\alpha_2)\int_0^{d(p',p^n)}\zeta(t)dt$

Now we show that $p^*$ is the fixed point of $T$. Assume that $p'$ is another fixed point of $T$. Then we have $T(p') = p'$.

$\int_0^{d(p',p')}\zeta(t)dt = \int_0^{d(p,p')}\zeta(t)dt$

$\leq\alpha_1\int_0^{d(p,p')}\zeta(t)dt + \alpha_2\int_0^{d(p,p')}\zeta(t)dt$

$+ \alpha_3\int_0^{d(p,p')}\zeta(t)dt + \alpha_4\int_0^{d(p,p')}\zeta(t)dt$

$+ \alpha_5\int_0^{d(p,p')}\zeta(t)dt + \alpha_6\int_0^{d(p,p')}\zeta(t)dt$

$\leq\alpha_1\int_0^{d(p,p')}\zeta(t)dt + \alpha_2\int_0^{d(p,p')}\zeta(t)dt$

$+ \alpha_3\int_0^{d(p,p')}\zeta(t)dt + \alpha_4\int_0^{d(p,p')}\zeta(t)dt$

$+ \alpha_5\int_0^{d(p,p')}\zeta(t)dt + \alpha_6\int_0^{d(p,p')}\zeta(t)dt$

$\int_0^{d(p,p')}\zeta(t)dt \leq (\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4)\int_0^{d(p,p')}\zeta(t)dt$

$\Rightarrow p^* = p'$. $T$ has a unique fixed point.

**Corollary 2.2.** Let $(X,d)$ be a complete $b$-metrics space with constants $s \geq 1$ and define the sequence $\{p_n\}_{n=1}^{\infty} \subset X$ by the iteration $p_0 = Tp_{n-1} = T^n \neq X_0$ and let $T : X \to X$ be a function and $\zeta : [0,\infty) \to [0,\infty)$ be Lebesgue-integrable mapping such that

$\int_0^{d(Tp,Tq)}\zeta(t)dt \leq \alpha_1\int_0^{d(p,q)}\zeta(t)dt + \alpha_2\int_0^{d(Tp,Tq)}\zeta(t)dt$

$+ \alpha_3\int_0^{d(Tq,Tq)}\zeta(t)dt + \alpha_4\int_0^{d(p,Tq)}\zeta(t)dt$

$\Rightarrow$ for $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 < 1, \forall p,q \in X$ then there exists $p^* \in X$ such that $p_n \to p^*$ and $p^*$ is a unique fixed point.

**Proof.** We take $\alpha_6 = 0$ in previous theorem we get the solution.

**References**


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ISSN(P):2319 – 3786
Malaya Journal of Matematik
ISSN(O):2321 – 5666
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