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Fixed point theorems of integral type contraction in b-metric spaces

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Abstract
In this paper, we establish some new integral type contraction defined on b-metric spaces and prove some new fixed point theorems for these mappings. Our results are generalizations of previous research's.

Keywords
b-Metric Spaces, integral type contraction, Fixed Point Theorems.

AMS Subject Classification
47H10, 54H25.

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1. Introduction and Preliminaries

Bakhtin [1] wasintroduced the concept of b-metric space and used by Czerwik in [6]. Banach's contraction mapping theorem says, A mapping $T : X \to X$ where (X,d) is a metric space, is said to be a contraction if there exists $k \in \{0,1\ )$ such that $\forall s,t \in X$

$$d(Tp,Tq) \leq kd(p,q) \tag{1.1}$$

If the metric space (X,d) is complete the mapping satisfying (1.1) has a unique fixed point.

Definition 1.1. *Let X be a non-empty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to R_+$ is called a b-metric provided that, for all $p,q,r \in X$*

(i) $d(p,q) = 0$ *iff* $p = q$

(ii) $d(p,q) = d(p,q)$

(iii) $d(p,r) \leq s[d(p,q) + d(q,r)]$

A pair (X,d) is called a b-metric space.

Definition 1.2. *Let (X,d) be a b-metric space. Then a sequence $\{p_n\}$ in X is called a Cauchy sequence if and only if for all $\varepsilon > 0$ there exist $n(\varepsilon) \in N$ such that for each $n,m \geq n(\varepsilon)$ we have $d(p_n,p_m) < \varepsilon$.*

Definition 1.3. *Let (X,d) be a b-metric space. Then a sequence $\{p_n\}$ in X is called convergent sequence if and only if there exist $x \in X$ such that for all there exists $n(\varepsilon) \in N$ such that for all $n \geq n(\varepsilon)$ we have $d(p_n,x) < \varepsilon$.*

Definition 1.4. *The b-metric space is complete if every Cauchy sequence convergent.*

Definition 1.5. *Let (X,d) be a metric space and $T : X \to X$ be function, $\zeta : [0,\infty) \to [0,\infty)$ be Lebesgue-integrable mapping, if there exist $\alpha \in [0,1)$ such that for all $p,q \in X$*

$$\int_0^{d(Tp,Tq)} \zeta(t)dt \leq \alpha \int_0^{d(p,q)} \zeta(t)dt$$

2. Main Results

Theorem 2.1. *Let (X,d) be a complete b metrics space with constants $s \geq 1$ and define the sequence $\{x_n\}_{n=1} \subset X$ by the iteration $x_n = Tx_{n-1} = T^n x_0$ and let $T : X \to X$ be a function and $\zeta : [0,\infty) \to [0,\infty)$ be Lebesgue-integrable mapping such* *that*

$$\int_0^{d(Tp,Tq)}\zeta(t)dt \le \alpha_1\int_0^{d(p,q)}\zeta(t)dt+\alpha_2\int_0^{d(p,Tp)}\zeta(t)dt + \alpha_3\int_0^{d(q,Tq)}\zeta(t)dt+\alpha_4\int_0^{d(p,Tq)}\zeta(t)dt + \alpha_5\int_0^{d(q,Tp)}\zeta(t)dt+\alpha_6\int_0^{[d(q,Tp)+d(p,Tq)]}\zeta(t)dt$$

where $\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+2\alpha_6<1, \forall p,q\in X$ *then there exists* $p^*\in X$ *such that* $p_n\to p^*$ *and* p^* *is a unique fixed point.*

Proof. Let $p_0\in X$ and $\{p_n\}_{n=1}^{\infty}$ be a sequence in X defined as $p_n=Tp_{n-1}=T^np_0, n=1,2,3,.$

$$\int_0^{d(p_n,p_{n+1})}\zeta(t)dt=\int_0^{d(Tp_{n-1},Tp_n)}\zeta(t)dt$$
$$\le\alpha_1\int_0^{d(p_{n-1},p_n)}\zeta(t)dt+\alpha_2\int_0^{d(p_{n-1},Tp_{n-1})}\zeta(t)dt + \alpha_3\int_0^{d(p_n,Tp_n)}\zeta(t)dt+\alpha_4\int_0^{d(p_{n-1},Tp_n)}\zeta(t)dt + \alpha_5\int_0^{d(p_n,Tp_{n-1})}\zeta(t)dt+\alpha_6\int_0^{[d(p_n,Tp_{n-1})+d(p_{n-1},Tp_n)]}\zeta(t)dt$$
$$\le\alpha_1\int_0^{d(p_{n-1},p_n)}\zeta(t)dt+\alpha_2\int_0^{d(p_{n-1},p_n)}\zeta(t)dt + \alpha_3\int_0^{d(p_n,p_{n+1})}\zeta(t)dt+\alpha_4\int_0^{d(p_{n-1},p_{n+1})}\zeta(t)dt + \alpha_5\int_0^{d(p_n,p_n)}\zeta(t)dt+\alpha_6\int_0^{[d(p_n,p_n)+d(p_{n-1},p_{n+1})]}\zeta(t)dt$$
$$\le\alpha_1\int_0^{d(p_{n-1},p_n)}\zeta(t)dt+\alpha_2\int_0^{d(p_{n-1},p_n)}\zeta(t)dt + \alpha_3\int_0^{d(p_n,p_{n+1})}\zeta(t)dt+s\alpha_4\int_0^{d(p_{n-1},p_n)}\zeta(t)dt + s\alpha_4\int_0^{d(p_n,p_{n+1})}\zeta(t)dt+s\alpha_6\int_0^{d(p_{n-1},p_n)}\zeta(t)dt + s\alpha_6\int_0^{d(p_n,p_{n+1})}\zeta(t)dt$$

$$(1-\alpha_3-s\alpha_4-s\alpha_6)\int_0^{d(p_n,p_{n+1})}\zeta(t)dt \le(\alpha_1+\alpha_2+s\alpha_4+s\alpha_6)\int_0^{d(p_{n-1},p_n)}\zeta(t)dt$$

$$\int_0^{d(p_n,p_{n+1})}\zeta(t)dt\le\frac{\alpha_1+\alpha_2+s\alpha_4+s\alpha_6}{1-\alpha_3-s\alpha_4-s\alpha_6}\int_0^{d(p_{n-1},p_n)}\zeta(t)dt$$
$$\int_0^{d(p_n,p_{n+1})}\zeta(t)dt\le\eta\int_0^{d(p_{n-1},p_n)}\zeta(t)dt$$

where $\eta=\frac{\alpha_1+\alpha_2+s\alpha_4+s\alpha_6}{1-\alpha_3-s\alpha_4-s\alpha_6}$

$$\int_0^{d(p_n,p_{n+1})}\zeta(t)dt\le\eta\int_0^{d(p_{n-1},p_n)}\zeta(t)dt \le\eta^2\int_0^{d(p_{n-2}p_{n-1})}\zeta(t)dt$$

Continuing this process we get,

$$\le\eta^n\int_0^{d(p_0,p_1)}\zeta(t)dt$$

Now we show that $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Let $m,n>0$ with $m>n$

$$\int_0^{d(p_n,p_m)}\zeta(t)dt \le s\int_0^{d(p_n,p_{n+1})}\zeta(t)dt+s^2\int_0^{d(p_{n+1},p_{n+2})}\zeta(t)dt + s^3\int_0^{d(p_{n+2},p_{n+3})}\zeta(t)dt+\cdots$$
$$\le s\eta^n\int_0^{d(p_1,p_0)}\zeta(t)dt+s^2\eta^{n+1}\int_0^{d(p_1,p_0)}\zeta(t)dt + \cdots+s^m\eta^{n+m-1}\int_0^{d(p_1,p_0)}\zeta(t)dt$$
$$\le s\eta^n[1+(s\eta)+(s\eta)^2+\cdots + (s\eta)^{m-1}]\int_0^{d(p_1,p_0)}\zeta(t)dt$$
$$\le s\eta^n\left[\frac{1-(s\eta)^{n-(m-1)}}{1-s\eta}\right]\int_0^{d(p_1,p_0)}\zeta(t)dt$$

Take $m,n\to\infty$

$$\lim_{n\to\infty}d(p_n,p_m)=0.$$

Hence $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence $\{p_n\}$ converges to $p^*\in X$. Now we show that p^* is the unique fixed point of T.

$$\int_0^{d(p^*,Tp^*)}\zeta(t)dt\le s\int_0^{[d(p^*,p_{n+1})+d(p_{n+1},Tp^*)]}\zeta(t)dt \le s\int_0^{[d(p^*,p_{n+1})+d(Tp_n,Tp^*)]}\zeta(t)dt$$

$$\le s\int_0^{d(p^*,p_{n+1})}\zeta(t)dt + s\int_0^{\substack{[\alpha_1d(p_n,p^*)+\alpha_2d(p_n,Tp_n)\\ +\alpha_3d(p^*,Tp^*)+\alpha_4d(p_n,Tp^*)\\ +\alpha_5d(p^*,Tp_n)\\ +\alpha_6d(p^*,Tp_n)+d(p_n,Tp^*)]}}\zeta(t)dt$$

$$\begin{aligned}
&\leq s\int_0^{d(p^*,p_{n+1})}\zeta(t)dt+s\alpha_1\int_0^{d(p_n,x^*)}\zeta(t)dt\\
&+s^2\alpha_2\int_0^{d(p_n,p^*)}\zeta(t)dt+s^2\alpha_2\int_0^{d(p^*,p_{n+1})}\zeta(t)dt\\
&+s\alpha_3\int_0^{d(p^*,Tp^*)}\zeta(t)dt+s^2\alpha_4\int_0^{d(p^*,p_n)}\zeta(t)dt\\
&+s^2\alpha_4\int_0^{d(p^*,Tp^*)}\zeta(t)dt+s\alpha_5\int_0^{d(p^*,p_{n-1})}\zeta(t)dt\\
&+s\alpha_6\int_0^{d(p^*,p_{n-1})}\zeta(t)dt+s^2\alpha_6\int_0^{d(p_n,p^*)}\zeta(t)dt\\
&+s^2\alpha_6\int_0^{d(p^*,Tp^*)}\zeta(t)dt
\end{aligned}$$

$$\begin{aligned}
(1-s\alpha_3-s^2\alpha_4-s^2\alpha_6)&\int_0^{d(p^*,Tp^*)}\zeta(t)dt\\
&\leq\left(s+s^2\alpha_2\right)\int_0^{d(p^*,p_{n+1})}\zeta(t)dt\\
&+\left(s\alpha_1+s^2\alpha_2+s^2\alpha_4+s^2\alpha_6\right)\int_0^{d(p_n,p^*)}\zeta(t)dt\\
&+(s\alpha_5+s\alpha_6)\int_0^{d(p^*,p_{n+1})}\zeta(t)dt
\end{aligned}$$

$d(p^*,Tp^*)\leq 0$ asn $\to\infty$.
Now we show that p^* is the fixed point of T. Assume that p' is another fixed point of T. Then we have $Tp'=p'$.

$$\begin{aligned}
&\int_0^{d\left(p^*,p'\right)}\zeta(t)dt=\int_0^{d(Tp^*,Tpt)}\zeta(t)dt\\
&\leq\alpha_1\int_0^{d\left(p^*,p'\right)}\zeta(t)dt+\alpha_2\int_0^{d(p^*,Tp^*)}\zeta(t)dt\\
&\quad+\alpha_3\int_0^{d\left(p',Tp'\right)}\zeta(t)dt+\alpha_4\int_0^{d\left(p^*,Tp'\right)}\zeta(t)dt\\
&\quad+\alpha_5\int_0^{d(p\prime,Tp^*)}\zeta(t)dt+\alpha_6\int_0^{\left[d(p),Tp^*)+d\left(p^*,Tp'\right)\right]}\zeta(t)dt\\
&\leq\alpha_1\int_0^{d\left(p^*,p'\right)}\zeta(t)dt+\alpha_2\int_0^{d\left(p^*,p^*\right)}\zeta(t)dt\\
&\quad+\alpha_3\int_0^{d\left(p',p'\right)}\zeta(t)dt+\alpha_4\int_0^{d(p^*,pl)}\zeta(t)dt\\
&\quad+\alpha_5\int_0^{d\left(p',p^*\right)}\zeta(t)dt+\alpha_6\int_0^{\left[d\left(p',p^*\right)+d\left(p^*,p'\right)\right]}\zeta(t)dt
\end{aligned}$$

$$\int_0^{d\left(p^*,p'\right)}\zeta(t)dt\leq(\alpha_1+\alpha_4+\alpha_5+2\alpha_6)\int_0^{d\left(p^*,p'\right)}\zeta(t)dt$$

$\Rightarrow p^*=p'$. $\therefore T$ has a unique fixed point. □

Corollary 2.2. *Let (X,d) be a complete b metrics space with constants $s\geq 1$ and define the sequence $\{p_n\}_{n=1}^{\infty}\subset X$ by the iteration $p_n=Tp_{n-1}=T^nx_0$ and let $T:X\to X$ be a function and $\zeta:[0,\infty)\to[0,\infty)$ be Lebesgue-integrable mapping such that*

$$\begin{aligned}
\int_0^{d(Tp,Tq)}\zeta(t)dt\leq\alpha_1\int_0^{d(p,q)}\zeta(t)dt&+\alpha_2\int_0^{d(p,Tq)}\zeta(t)dt\\
&+\alpha_3\int_0^{d(p,Tq)}\zeta(t)dt+\alpha_4\int_0^{d(p,Tq)}\zeta(t)dt\\
&+\alpha_5\int_0^{d(q,Tp)}\zeta(t)dt
\end{aligned}$$

where $\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+2\alpha_6<1,\forall p,q\in X$ then there exists $p^*\in X$ such that $p_n\to p^*$ and p^* is a unique fixed point.

Proof. We take $\alpha_6=0$ in previous theorem we get the solution.

□

References

[1] Aydiet. al., A fixed point theorem for set valued quasi-contractions in b-metric spaces, *Fixed point theory and Applications*, 88(2012).

[2] Bakhtin I.A, The contraction mapping principle in quasimetric spaces, *Funct. Anal. UnianowskGos. Ped. Inst.* 30(1989), 26-37.

[3] Berinde, V., *Iterative Approximation of Fixed points*, Springer, (2006).

[4] Branciari A: A fixed point theorem for mappings satisfying a general contractive condition of integral type. *International Journal of Mathematics and Mathematical Sciences,* 29(9)(2002), 531–536.

[5] Boriceanu, M., Fixed Point theory for multivalued generalized contraction on a set with two b-metrics, studia Univ Babes, Bolya: *Math. LIV,* (3)(2009), 1-14.

[6] Dinesh.K, Krishnakumar. R, Nagaral Pandit Sanatammappa, Some new rational expressions in complete metric space, *Journal of the Mharaja Sayajirao university of Baroada,* 2021.

[7] Dinesh.K, Krishnakumar. R, Nagaral Pandit Sanatammappa, Existence of fixed point theorems of contractive mappings in complete generalized metric spaces, *Malaya Journal of Mathematik*, 9(1)(2021).

[8] Chatterjea, S.K. Fixed point theorems, *C. R. Acad. Bulgare Sci.*, 25(1972), 727-730.

[9] Chaudhary, B.S., Unique fixed point theorem for weakly c-contractive mappings, Kathmandu Univ. *J. Sci, Eng. And Tech*, 5(1)(2009), 6-13.

[10] Czerwik, S., Contraction mappings in b-metric space, *Acta Math. Inf. Univ. Ostraviensis,* 1(1993), 5-11.

[11] Czerwik, S., Non-linear set-valued contraction mappings in b-metric spaces. *AttiSem Math Fig Univ Modena*, 46(2)(1998), 263-276.

[12] Kannan,R., some results on fixed points, *Bull.Calcutta Math.Soc.,* 60(1968), 71-76.

[13] Mehmet Kir, Hiikmikiziltunc; On some well known fixed point theorems in b-metric spaces, *Turkish Journal of Analysis and Number theory*, 1(1)(2013), 13-16.

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