Density of an \(m\)-Bipolar Fuzzy Graph

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Abstract
In this article, density, balanced, strictly balanced, complement of an \(m\)-bipolar fuzzy graph (\(m\)-BPFG), self-complementary between \(m\)-BPFG and its complement are defined and corresponding properties are studied.

Keywords
\(m\)-BPFG, Density of an \(m\)-BPFG, Balanced, Self complementary.

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1. Introduction

Fuzzy sets are introduced for the parameters to solve problems related to vague and uncertain in real life situations are demonstrated by Zadeh [11] in 1965. The limitations of traditional model were overcome by the introduction of bipolar fuzzy concept in 1994 by Zhang [12, 13]. This was further improved by Chen et al. [3] to m-polar fuzzy set theory.

Free body diagrams using set of nodes connected by lines representing pairs are good problem solving tools in non-deterministic real life situations. Thus, Kaufmann [7] was first to introduced the idea of fuzzy graph from Zadeh fuzzy relation. Rosenfeld [8] gave the idea of fuzzy vertex, fuzzy edges and fuzzy cycle etc. Akram [1] studied the some properties of bipolar fuzzy graphs. Later Rashmanlou et al. [10] studied the categorical properties of bipolar fuzzy graphs. Ghorai and Pal [4-6] introduced the concept of m-polar fuzzy graphs and studied some properties on it. Ramprasad et al. [9] introduced the product m-polar fuzzy line and intersection graphs. Bera and pal [2] introduced the concept of m-polar interval-valued fuzzy graph and studied the algebraic properties of density, regularity and irregularity etc. on m-PIVFG.

This paper attempts to develop theory to analyze parameters combining concepts from mpolar fuzzy graph and bipolar fuzzy graph as a unique effort. The resultant graph is turned mBPFG and studied properties on it.

2. Preliminaries

All the vertices and edges of an \(m\)-polar fuzzy graph have \(m\) components and those components are fixed. But these components may be bipolar. Using this idea, \(m\)-BPFG has been introduced.

Before defining \(m\)-bipolar fuzzy graph, we assume the following:

For a given set \(V\), define an equivalence relation \(\leftrightarrow\) on \(V \times V - \{(k,k): k \in V\}\) as follows: \((k_1,k_1) \leftrightarrow (k_2,k_2) \leftrightarrow (k_1,k_2) \leftrightarrow (k_2,k_1)\) or \((k_1,k_1) = (k_2,k_2)\) or \(k_1 = k_2, k_1 = k_2\). The quotient set got in this way is denoted by \(V^2\).

**Definition 2.1.** An \(m\)-bipolar fuzzy set (\(m\)-BPFS) \(S\) on \(V\) is defined by

\[
S(s) = \{(p_1 \circ \psi^1_s(s), p_1 \circ \psi^2_s(s)), \ldots, [p_m \circ \psi^1_s(s), p_m \circ \psi^2_s(s)]\}
\]

for all \(s \in V\) or shortly

\[
S(s) = \left\{ \left[ p_j \circ \psi^1_s(s), p_j \circ \psi^2_s(s) \right]_{j=1}^m \right\} | s \in V
\]

where the functions \(p_j \circ \psi^1_s: V \to [0,1]\) and \(p_j \circ \psi^2_s: V \to [-1,0]\) denote the positive memberships and negative memberships of the element respectively.

**Definition 2.2.** Let \(S\) be an \(m\)-BPFS on a set \(V\). An \(m\)-bipolar fuzzy relation on a set \(S\) is an \(m\)-BPFS \(T\) of \(V \times V, T(s,t) =\)
\[
\{[p_j \circ \psi_R^n(s,t), p_j \circ \psi_L^n(s,t)] \mid [p_2 \circ \psi_R^n(s,t), p_2 \circ \psi_L^n(s,t)], \ldots, [p_m \circ \psi_R^n(s,t), p_m \circ \psi_L^n(s,t)]\}\text{ for all } s, t \in V \text{ or shortly }
\]
\[
T(s,t) = \left\{[p_j \circ \psi_R^n(s,t), p_j \circ \psi_L^n(s,t)]^{m}_{j=1} \mid s, t \in V \right\}
\]
such that \(p_j \circ \psi_R^n(s,t) \leq \min \{p_j \circ \psi_R^n(s,t), p_j \circ \psi_L^n(t)\}\) \(p_j \circ \psi_L^n(s,t) \geq \max \{p_j \circ \psi_R^n(s,t), p_j \circ \psi_L^n(t)\}\) for every \(j = 1, 2, \ldots, m\) and \(s, t \in V\).

**Definition 2.3.** An \(m\)-bipolar fuzzy graph (\(m\)-BPFG) of a graph \(G^* = (V, E)\) is a pair \(G = (V, S, T)\) where \(S = \left\{[p_j \circ \psi_R^n(s,t), p_j \circ \psi_L^n(s,t)]^{m}_{j=1}\right\}\), \(p_j \circ \psi_R^n : V \rightarrow [0, 1]\) and \(p_j \circ \psi_L^n : V \rightarrow [-1, 0]\) is an \(m\)-BPFS on \(V\), and \(T = \left\{[p_j \circ \psi_R^n(s,t), p_j \circ \psi_L^n(t)]^{m}_{j=1}\right\}, p_j \circ \psi_R^n : V^2 \rightarrow [0, 1]\) and \(p_j \circ \psi_L^n : V^2 \rightarrow [-1, 0]\) is an \(m\)-BPFS in \(V^2\) such that \(p_j \circ \psi_R^n(k,l) \leq \min \{p_j \circ \psi_R^n(k,l), p_j \circ \psi_L^n(l)\}\), \(p_j \circ \psi_L^n(k,l) \geq \max \{p_j \circ \psi_R^n(k,l), p_j \circ \psi_L^n(l)\}\) for all \((k,l) \in V^2, j = 1, 2, \ldots, m\) and \(p_j \circ \psi_R^n(k,l) = p_j \circ \psi_L^n(k,l) = 0\) for all \((k,l) \in V^2 - E\).

**Definition 2.4.** Let \(G = (V, S, T)\) be an \(m\)-BPFG of a graph \(G^* = (V, E)\). An \(m\)-BPFG \(N = (Q, C, D)\) is said to be an \(m\)-bipolar fuzzy subgraph of \(G\) induced by \(Q\) if \(Q \subseteq V\), \(C(s) = S(s)\) for all \(s \in Q\) and \(D(s,t) = T(s,t)\) for all \((s,t) \in Q^2\).

**Definition 2.5.** An \(m\)-BPFG \(G = (V, S, T)\) of a graph \(G^* = (V, E)\) is complete if for every \(s, t \in V\) and \(j = 1, 2, \ldots, m\) satisfying \(p_j \circ \psi_R^n(s,t) = \min \{p_j \circ \psi_R^n(s,t), p_j \circ \psi_L^n(t)\}\) \(p_j \circ \psi_L^n(s,t) = \max \{p_j \circ \psi_R^n(s,t), p_j \circ \psi_L^n(t)\}\).

**Definition 2.6.** An \(m\)-BPFG \(G = (V, S, T)\) of a graph \(G^* = (V, E)\) is strong if for every \(s, t \in V\) and \(j = 1, 2, \ldots, m\) satisfying \(p_j \circ \psi_R^n(s,t) = \min \{p_j \circ \psi_R^n(s,t), p_j \circ \psi_L^n(t)\}\) \(p_j \circ \psi_L^n(s,t) = \max \{p_j \circ \psi_R^n(s,t), p_j \circ \psi_L^n(t)\}\).

**3. Density of an \(m\)-BPFG**

In this section, complement, density, balanced, strictly balanced and self complementary of an \(m\)-BPFG are defined and studied some of its properties.

**Definition 3.1.** Let \(G = (V, S, T)\) be an \(m\)-BPFG of \(G^* = (V, E)\). The complement of \(G\) is an \(m\)-BPFG \(\overline{G} = (V, S, T)\) of \(G^* = (V, V^2)\) such that \(\overline{S} = \overline{S}\) and \(\overline{T}\) is defined by
\[
p_j \circ \psi_R^n(s,t) = [p_j \circ \psi_R^n(s,t), p_j \circ \psi_L^n(s,t)]
p_j \circ \psi_L^n(s,t) = [p_j \circ \psi_R^n(s,t) \wedge p_j \circ \psi_L^n(t)] - p_j \circ \psi_L^n(s,t)
p_j \circ \psi_L^n(s,t) = [p_j \circ \psi_R^n(s,t) \lor p_j \circ \psi_L^n(t)] - p_j \circ \psi_R^n(s,t)
\]
for every \((s,t) \in V^2\) and \(j = 1, 2, \ldots, m\).

**Definition 3.2.** The density of an \(m\)-BPFG \(G = (V, S, T)\) of \(G^* = (V, E)\) is
\[
\chi(G) = \left\langle[p_j \circ \chi^n_1(G), p_j \circ \chi^n_2(G)]^{m}_{j=1}\right\rangle = \left\langle\frac{2\sum_{(s,t) \in E} p_j \circ \psi_R^n(s,t) \wedge p_j \circ \psi_L^n(t)}{\sum_{(s,t) \in E} [p_j \circ \psi_R^n(s,t) \lor p_j \circ \psi_L^n(t)]} \right\rangle
\]

**Definition 3.3.** An \(m\)-BPFG \(G = (V, S, T)\) of \(G^* = (V, E)\) is balanced if \(\chi(Q) \leq \chi(G)\) for all non-empty subgraphs \(Q\) of \(G\), i.e., for all \(j = 1, 2, \ldots, m\), we have \(p_j \circ \chi^n(Q) \leq p_j \circ \chi^n(G)\) and \(p_j \circ \chi^n(Q) \leq p_j \circ \chi^n(G)\).

**Definition 3.4.** An \(m\)-BPFG \(G = (V, S, T)\) of \(G^* = (V, E)\) is strictly balanced if \(\chi(G) = \chi(Q)\) for all non-empty subgraphs \(Q\) of \(G\).

**Example 3.5.** Consider a \(2\)-BPFG \(G = (V, S, T)\) of \(G^* = (V, E)\) as shown in the Figure 1.

![Figure 1](image)

**Figure 1.** Strictly balanced \(2\)-BPFG

For this \(2\)-BPFG, we have \(\chi(G) = \langle[0.5, 1.6], [1.5, 1.8]\rangle\) and the densities of a non-empty subgraphs of \(G\) are \(\chi(H_1 = (P, Q)) = \chi(H_2 = (R, Q)) = \chi(H_3 = (R, P)) = \langle[0.5, 1.6], [1.5, 1.8]\rangle\). Therefore \(G\) is a strictly balanced \(2\)-BPFG.

**Definition 3.6.** Let \(G_1 = (V_1, S_1, T_1)\) and \(G_2 = (V_2, S_2, T_2)\) be two \(m\)-BPFGs on crisp graphs \(G^*_1 = (V_1, E_1)\) and \(G^*_2 = (V_2, E_2)\) respectively. An isomorphism between \(G_1\) and \(G_2\) is a bijective mapping \(\phi\) from \(V_1\) to \(V_2\) such that for each \(j = 1, 2, \ldots, m\) \(p_j \circ \psi_R^n(s) = p_j \circ \psi_R^n(\phi(s))\) \(p_j \circ \psi_L^n(s) = p_j \circ \psi_L^n(\phi(s))\) \(\forall s \in V_1\).

**Definition 3.7.** An \(m\)-BPFG is said to be self-complementary if \(G \cong \overline{G}\).

**Theorem 3.8.** Every complete \(m\)-BPFG is balanced.
Proof. Let $G = (V, S, T)$ be a complete $m$-BPFG and $Q$ be a non-empty subgraph of $G$. Then for every $j = 1, 2, \ldots, m$

$$[p_j \circ \chi^p(G), p_j \circ \chi^n(G)] = \left[ \frac{2 \sum_{s \in V} p_j \circ \psi^p_s(s, t)}{\sum_{s \in V} \chi^p_G(s) \vee \chi^n_G(s)} \right] = \left[ \frac{2 \sum_{s \in V} p_j \circ \psi^n_s(s, t)}{\sum_{s \in V} \chi^p_G(s) \vee \chi^n_G(s)} \right] = [2, 2]$$

where $V(Q)$ and $E(Q)$ represents the vertex set and edge set of $Q$. Thus $G = (V, S, T)$ is a balanced $m$-BPFG.

Corollary 3.9. Every strong $m$-BPFG is balanced.

Theorem 3.10. The complement of a strictly balanced $m$-BPFG is strictly balanced.

Proof. Follows from the definition.

Theorem 3.11. Let $G = (V, S, T)$ be a strictly balanced $m$-BPFG and let $\bar{G} = (V, \bar{S}, T)$ be its complement. Then $\chi(G) + \chi(\bar{G}) = [2, 2], [2, 2], \ldots, [2, 2]$.

Proof. Let $Q$ be any non-empty subgraph of $G$. Since $G$ is strictly balanced $m$-BPFG then $\chi(Q) = \chi(G)$ for every $Q \subseteq G$. Now, for all $(s, t) \in V^2$ and $j = 1, 2, \ldots, m$ we have

(i) $p_j \circ \psi^p_t(s, t) = \{p_j \circ \psi^n_t(s, t) \wedge p_j \circ \psi^p_t(t, t)\} - p_j \circ \psi^p_t(s, t)$

(ii) $p_j \circ \psi^p_t(s, t) = \{p_j \circ \psi^n_t(s, t) \vee p_j \circ \psi^p_t(t, t)\} - p_j \circ \psi^p_t(s, t)$

Dividing (i) by $\{p_j \circ \psi^n_t(s, t) \wedge p_j \circ \psi^p_t(t, t)\}$ on both sides, we get

$$\frac{p_j \circ \psi^p_t(s, t)}{p_j \circ \psi^n_t(s, t) \wedge p_j \circ \psi^p_t(t, t)} = \frac{1}{\{p_j \circ \psi^n_t(s, t) \wedge p_j \circ \psi^p_t(t, t)\}}$$

$$\sum_{s \in V} \frac{p_j \circ \psi^p_t(s, t)}{p_j \circ \psi^n_t(s, t) \wedge p_j \circ \psi^p_t(t, t)} = \frac{1}{\{p_j \circ \psi^n_t(s, t) \wedge p_j \circ \psi^p_t(t, t)\}}$$

$$2 \sum_{s \in V} \frac{p_j \circ \psi^p_t(s, t)}{p_j \circ \psi^n_t(s, t) \wedge p_j \circ \psi^p_t(t, t)} = 2 \frac{1}{\{p_j \circ \psi^n_t(s, t) \wedge p_j \circ \psi^p_t(t, t)\}}$$

i.e. $p_j \circ \chi^p(\bar{G}) = 2 - p_j \circ \chi^n(\bar{G})$. Similarly, $p_j \circ \chi^p(\bar{G}) = 2 - p_j \circ \chi^n(\bar{G})$. Therefore $\chi(G) + \chi(\bar{G}) = [2, 2], [2, 2], \ldots, [2, 2]$.

Corollary 3.12. Every balanced $m$-BPFG need not be complete.

Theorem 3.13. Let $G = (V, S, T)$ be a strong $m$-BPFG graph, then $\bar{G}$ is balanced.

Proof. Suppose that $G$ is a strong $m$-BPFG. Then for all $(s, t) \in E$ and $j = 1, 2, \ldots, m$. We have

$$p_j \circ \psi^p_t(s, t) = \min \{p_j \circ \psi^n_t(s, t), p_j \circ \psi^p_t(t, t)\} = \max \{p_j \circ \psi^n_t(s, t), p_j \circ \psi^p_t(t, t)\}.$$

From $\bar{G}$, for all $(s, t) \in E$ and $j = 1, 2, \ldots, m$ we have

$$p_j \circ \psi^p_t(s, t) = \{p_j \circ \psi^n_t(s, t) \wedge p_j \circ \psi^p_t(t, t)\} - p_j \circ \psi^p_t(s, t)$$

$$p_j \circ \psi^p_t(s, t) = \{p_j \circ \psi^n_t(s, t) \vee p_j \circ \psi^p_t(t, t)\} - p_j \circ \psi^p_t(s, t).$$

As $G$ is strong, we get

$$\{p_j \circ \psi^p_t(s, t), p_j \circ \psi^p_t(s, t)\} = [0, 0],$$

$\forall (s, t) \in E$ and $p_j \circ \psi^p_t(s, t) = \{p_j \circ \psi^n_t(s, t) \wedge p_j \circ \psi^p_t(t, t)\}$ and $p_j \circ \psi^p_t(s, t) = \{p_j \circ \psi^n_t(s, t) \vee p_j \circ \psi^p_t(t, t)\}$ for all $(s, t) \in E$.

Hence, $\bar{G}$ is a strong $m$-BPFG and it implies that $\bar{G}$ is balanced.

Theorem 3.14. Let $G = (V, S, T)$ be a self complementary $m$-BPFG of $G^* = (V, E)$. Then for all $(s, t) \in E$ and $j = 1, 2, \ldots, m$ we have

$$\sum_{s \in V} p_j \circ \psi^p_t(s, t) = \frac{1}{2} \sum_{s \in V} \{p_j \circ \psi^n_t(s, t) \wedge p_j \circ \psi^p_t(t, t)\}$$

$$\sum_{s \in V} p_j \circ \psi^p_t(s, t) = \frac{1}{2} \sum_{s \in V} \{p_j \circ \psi^n_t(s, t) \vee p_j \circ \psi^p_t(t, t)\}.$$

Proof. Suppose that $G = (V, S, T)$ is a self complementary $m$-BPFG of $G^*$. Then there exists an isomorphism $\phi$ from $G$ to $\bar{G}$ such that $p_j \circ \psi^p_t(s, t) = p_j \circ \psi^p_t(\phi(s), \phi(t))$, $p_j \circ \psi^n_t(s, t) = p_j \circ \psi^n_t(\phi(s), \phi(t))$, $\forall s \in V$ and $p_j \circ \psi^p_t(s, t) = p_j \circ \psi^p_t(\phi(s), \phi(t))$, $p_j \circ \psi^n_t(s, t) = p_j \circ \psi^n_t(\phi(s), \phi(t))$, $\forall (s, t) \in V^2$. Let $(s, t) \in V^2$. Then for $j = 1, 2, \ldots, m$, we have

$$p_j \circ \psi^p_t(\phi(s), \phi(t)) = \{p_j \circ \psi^n_t(\phi(s), \phi(t)) \wedge p_j \circ \psi^p_t(\phi(s), \phi(t))\}$$

$$- p_j \circ \psi^p_t(\phi(s), \phi(t))$$

$$p_j \circ \psi^p_t(\phi(s), \phi(t)) = \{p_j \circ \psi^n_t(\phi(s), \phi(t)) \vee p_j \circ \psi^p_t(\phi(s), \phi(t))\}$$

$$- p_j \circ \psi^p_t(\phi(s), \phi(t))$$

i.e. $p_j \circ \psi^p_t(s, t) = \{p_j \circ \psi^n_t(\phi(s), \phi(t)) \wedge p_j \circ \psi^p_t(\phi(s), \phi(t))\}$

$$- p_j \circ \psi^p_t(\phi(s), \phi(t))$$

$$p_j \circ \psi^p_t(s, t) = \{p_j \circ \psi^n_t(\phi(s), \phi(t)) \vee p_j \circ \psi^p_t(\phi(s), \phi(t))\}$$

$$- p_j \circ \psi^p_t(\phi(s), \phi(t)).$$

Therefore,

$$\sum_{s \in V} p_j \circ \psi^p_t(\phi(s), \phi(t)) + \sum_{s \in V} p_j \circ \psi^p_t(s, t)$$

$$= \sum_{s \in V} \{p_j \circ \psi^n_t(\phi(s), \phi(t)) \wedge p_j \circ \psi^p_t(\phi(s), \phi(t))\}$$

$$= \sum_{s \in V} \{p_j \circ \psi^n_t(\phi(s), \phi(t)) \vee p_j \circ \psi^p_t(\phi(s), \phi(t))\}.$$
That is,
\[2 \sum_{s \neq t} p_j \circ \psi^p_s(s, t) = \sum_{s \neq t} \left\{ p_j \circ \psi^p_s(s) \land p_j \circ \psi^p_s(t) \right\},\]
\[\sum_{s \neq t} p_j \circ \psi^p_s(s, t) = \frac{1}{2} \sum_{s \neq t} \left\{ p_j \circ \psi^p_s(s) \lor p_j \circ \psi^p_s(t) \right\} \]

Similarly
\[\sum_{s \neq t} p_j \circ \psi^p_s(s, t) = \frac{1}{2} \sum_{s \neq t} \left( p_j \circ \psi^p_s(s) \lor p_j \circ \psi^p_s(t) \right).\]

**Theorem 3.15.** Let \( G_1 = (V_1, S_1, T_1) \) and \( G_2 = (V_2, S_2, T_2) \) be two isomorphic \( m \)-BPFGs. Then if \( G_2 \) is balanced then \( G_1 \) is balanced.

**Proof.** Since \( G_1 \) and \( G_2 \) are isomorphic, therefore there exists a bijective mapping \( \phi : V_1 \rightarrow V_2 \) such that \( p_j \circ \psi^p_{S_1}(s) = p_j \circ \psi^p_{S_2}(\phi(s)) \), \( p_j \circ \psi^p_{S_1}(t) = p_j \circ \psi^p_{S_2}(\phi(t)) \) for \( s \in V_1 \) and
\[ p_j \circ \psi^p_{T_1}(s, t) = p_j \circ \psi^p_{T_2}(\phi(s), \phi(t)), p_j \circ \psi^p_{T_1}(s, t) = p_j \circ \psi^p_{T_2}(\phi(s), \phi(t)) \]
\[ \forall (s, t) \in V_1^2, \forall j = 1, 2, \ldots, m. \]
Then,
\[ \sum_{s \in V_1} p_j \circ \psi^p_{T_1}(s, t) = \sum_{s \in V_2} p_j \circ \psi^p_{T_2}(\phi(s), \phi(t)) \]
\[ \sum_{s \in V_1} p_j \circ \psi^p_{T_1}(s, t) = \sum_{s \in V_2} p_j \circ \psi^p_{T_2}(\phi(s), \phi(t)) \]
Let \( N_1 \) and \( N_2 \) be two non-empty subgraphs of \( G_1 \) and \( G_2 \) respectively. Then, \( p_j \circ \psi^p_{S_1}(s) = p_j \circ \psi^p_{S_2}(\phi(s)) \), \( p_j \circ \psi^p_{T_1}(s, t) = p_j \circ \psi^p_{T_2}(\phi(s), \phi(t)) \), \( p_j \circ \psi^p_{T_1}(s, t) = p_j \circ \psi^p_{T_2}(\phi(s), \phi(t)) \), \( \forall s \in V_1(N_1), \forall j = 1, 2, \ldots, m, \) Here, \( V_1(N_1) \) are the vertices of \( N_1 \). Since \( G_2 \) is balanced, we have for \( j = 1, 2, \ldots, m \), \( p_j \circ \chi(N_2) \leq p_j \circ \chi(G_2) \).
4. Direct product of two m-bipolar fuzzy graphs

**Definition 4.1.** Let $G_1 = (V_1, S_1, T_1)$ and $G_2 = (V_2, S_2, T_2)$ be two $m$-BPFGs of $G_1^+ = (V_1, E_1)$ and $G_2^+ = (V_2, E_2)$ respectively such that $V_1 \cap V_2 = \emptyset$. Then the direct product of $G_1$ and $G_2$ is defined to be the $m$-BPFG $G_1 \times G_2 = (S_1 \times S_2, T_1 \times T_2)$ of the graph $G^+ = (V_1 \times V_2, E)$ where,

$$E = \{((s_1, t_1), (s_2, t_2)) \mid (s_1, t_1) \in E_1, (t_1, t_2) \in E_2\} \subseteq (V_1 \times V_2)^2$$

and for each $j = 1, 2, \ldots, m$,

(i) $p_j \circ \psi_{S_1 \times S_2}^p((s, t)) = \min\left\{p_j \circ \psi_{S_1}^p(s), p_j \circ \psi_{S_2}^p(t)\right\}$

(ii) $p_j \circ \psi_{T_1 \times T_2}^p((s, t)) = \max\left\{p_j \circ \psi_{S_1}^p(s), p_j \circ \psi_{T_2}^p(t)\right\}$

(iii) $p_j \circ \psi_{T_1 \times T_2}^p((p, q), (r, s)) = 0$

Also, for all $((p, q), (r, s)) \in \left((V_1 \times V_2)^2 - E\right)$, $j = 1, 2, \ldots, m$,

$$p_j \circ \psi_{(V_1 \times V_2)^2}^p((p, q), (r, s)) = 0$$

**Theorem 4.2.** The direct product $G_1 \times G_2$ of two $m$-BPFGs $G_1$ and $G_2$ is also an $m$-BPFG.

**Proof.** Let $((s_1, t_1), (s_2, t_2)) \in E$. Then $(s_1, s_2) \in E_1$ and $(t_1, t_2) \in E_2$. Hence for each $j = 1, 2, \ldots, m$; we have

$$p_j \circ \psi_{(V_1 \times V_2)^2}^p((s_1, t_1), (s_2, t_2)) = p_j \circ \psi_{(S_1 \times S_2)}^p((s_1, t_1), (s_2, t_2))$$

Similarly, $p_j \circ \psi_{(S_1 \times S_2)}^p((s_1, t_1), (s_2, t_2)) = 0$.

**Theorem 4.3.** Let $G_1 = (V_1, S_1, T_1)$ and $G_2 = (V_2, S_2, T_2)$ be two $m$-BPFGs of $G_1^+ = (V_1, E_1)$ and $G_2^+ = (V_2, E_2)$ respectively. Then $\chi(G_1) = \chi(G_2)$ if and only if $\chi(G_1) \leq \chi(G_1 \cap G_2)$ and $\chi(G_2) \leq \chi(G_1 \cap G_2)$.

**Proof.** Let $\chi(G_1) \leq \chi(G_1 \cap G_2)$ and $\chi(G_2) \leq \chi(G_1 \cap G_2)$. Then for $j = 1, 2, \ldots, m$,

$$p_j \circ \chi^p(G_1) \geq 2\sum_{s_1, s_2 \in V_1} p_j \circ \psi_{S_1}^p(s_1, s_2) \geq 2\sum_{s_1, s_2 \in V_1} p_j \circ \psi_{S_1}^p(s_1, s_2) \cap p_j \circ \psi_{S_1}^p(s_1, t_1) \cap p_j \circ \psi_{S_1}^p(s_2, t_2)$$

Similarly, $p_j \circ \chi^p(G_1) \geq \chi(G_1 \cap G_2)$.

**Theorem 4.4.** Let $G_1 = (V_1, S_1, T_1)$ and $G_2 = (V_2, S_2, T_2)$ be two balanced $m$-BPFGs. Then $G_1 \cap G_2$ is balanced if and only if $\chi(G_1) \geq \chi(G_1 \cap G_2)$.

**Proof.** Let $G_1 \cap G_2$ be balanced. Then $\chi(G_1) \leq \chi(G_1 \cap G_2)$ and $\chi(G_2) \leq \chi(G_1 \cap G_2)$. Hence, by Theorem 4.3, we have $\chi(G_1) = \chi(G_2) = \chi(G_1 \cap G_2)$.

Conversely, assume that $\chi(G_1) = \chi(G_2) = \chi(G_1 \cap G_2)$ and $N$ is a non-empty subgraph of $G_1 \cap G_2$. Then there exist two subgraphs $N_1$ and $N_2$ of $G_1$ and $G_2$ respectively. Let

$$\chi(G_1) = \chi(G_2) = \left\{\left[\begin{array}{c} p_j^1 \\ p_j^2 \\ \vdots \\ p_j^m \end{array}\right] \mid j = 1, 2, \ldots, m\right\}$$

Since $G_1$ and $G_2$ are balanced and $N_1$ and $N_2$ are non-empty subgraphs of $G_1$ and $G_2$, respectively, we have

$$\chi(G_1) = \chi(G_2) = \left\{\left[\begin{array}{c} p_j^1 \\ p_j^2 \\ \vdots \\ p_j^m \end{array}\right] \mid j = 1, 2, \ldots, m\right\}$$
where

\[ 0 \leq \left( \left[ \frac{p_j^p \ p_j^r}{k_j^p}, \frac{p_j^r}{k_j^r} \right] \right)_j^{m} \leq \langle [2, 2], [2, 2], \cdots, [2, 2] \rangle, \]

\[ \chi(N_1) = \left( \left[ \frac{p_j^p \ e_j^p}{d_j^p}, \frac{e_j^p}{d_j^p} \right] \right)_j^{m} \leq \left( \left[ \frac{p_j^p \ p_j^r}{k_j^p}, \frac{p_j^r}{k_j^r} \right] \right)_j^{m}, \]

\[ \chi(N_2) = \left( \left[ \frac{p_j^p \ e_j^p}{f_j^p}, \frac{e_j^p}{f_j^p} \right] \right)_j^{m} \leq \left( \left[ \frac{p_j^p \ p_j^r}{k_j^p}, \frac{p_j^r}{k_j^r} \right] \right)_j^{m}. \]

Thus \( c_j^p k_j^p + e_j^p k_j^p \leq d_j^p l_j^p + f_j^p l_j^p, c_j^n k_j^n + e_j^n k_j^n \leq d_j^n l_j^n + f_j^n l_j^n \) for all \( j = 1, 2, \cdots, m \). Hence,

\[ \chi(N) \leq \left( \left[ \frac{e_j^p + e_j^n}{d_j^p + f_j^n}, \frac{e_j^n + e_j^p}{d_j^n + f_j^p} \right] \right)_j^{m} \leq \left( \left[ \frac{p_j^p \ p_j^r}{k_j^p}, \frac{p_j^r}{k_j^r} \right] \right)_j^{m}, \]

\[ = \chi(G_1 \cap G_2). \]

Thus, \( \chi(N) \leq \chi(G_1 \cap G_2) \) for any subgraph \( N \) of \( G_1 \cap G_2 \). Therefore, \( G_1 \cap G_2 \) is balanced.

\[ \square \]

5. Conclusion

In this article, density and balanced m-BPFGs are defined. We studied the properties on selfcomplementary and density of an m-BPFG. We will extend our work to study the properties of morphism between two m-BPFGs and m-bipolar fuzzy line and intersection graphs.

References