Strongly perfect Plick and Lict graphs for some class of graphs

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Abstract
A graph $G$ is said to be strongly perfect if each of its induced subgraphs $H$ contains an independent set which meets all the cliques in $H$. In this paper, we develop results on strongly perfect graphs for plick and lict graphs of some class of graphs.

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Strongly perfect graph, plick graph, lict graph.

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1. Introduction
In this paper we utilize finite, simple and undirected graphs. Let $G$ be a graph. The vertex set of graph $G$ is denoted as $V(G)$ and its edge set is denoted as $E(G)$. We refer [4] for undefined terminologies used in this paper. An independent set [2] in a graph is a set of vertices no two of which are adjacent. A clique [4] of a graph is a maximal complete subgraph. The tadpole graph $T_{m,n}$ [3] is the graph obtained by joining a cycle $C_m$ to a path of length $n$, ($m$ indicates number of vertices in cycle $C_m$ and $n$ indicates number of vertices in path $P_n$). The friendship graph $F_x$ [3] is constructed by joining $x$ copies of the cycle $C_3$ with a common vertex. A wheel graph [3] $W_n$ is a graph with $n$ vertices formed by connecting a single vertex to all vertices of a cycle. The helm graph $H_y$ [3], where $y$ indicates the number of pendent edges, is the graph obtained from a wheel graph $W_n$ by adjoining a pendant edge at each vertex of the cycle. The plick graph $P(G)$ [5] of a graph $G$ is obtained from the line graph by adding a new vertex corresponding to each block of the original graph and joining this vertex to the vertices of the line graph which correspond to the edges of the block of the original graph. The lict graph $L_c(G)$ [6] of a graph $G$ is one whose vertex set is the union of the edges and the set of cutpoints of $G$ in which two vertices are adjacent if and only if the corresponding members of $G$ are adjacent or the corresponding members of $G$ are incident. The cutpoints and edges of a graph $G$ are called its members. A graph is called strongly perfect [8], if each of its induced subgraphs $H$ contains an independent set which meets all the cliques in $H$.

2. Preliminaries
In this section, we mention some standard results which will be used throughout this paper.

Theorem 2.1. [1] Every bipartite graph is strongly perfect.

Theorem 2.2. [1] Let $G$ be a graph with no induced $P_4$, then every maximal stable set (independent set) meets all the maximal cliques. Consequently, $G$ is strongly perfect.

Theorem 2.3. [7] If every odd cycle of length at least five in a graph $G$ has at least two chords, then $G$ is strongly perfect.
3. Results on plick graphs of some class of graphs to be strongly perfect

This section of paper develops results for plick graphs of some class of graphs to be strongly perfect.

**Theorem 3.1.** Plick graph of every path graph $P_n$ with $n \geq 2$ is strongly perfect.

**Proof.** Let $G$ be a path graph $P_n$, where $n$ is the number of vertices with $n \geq 2$. Let $P(G)$ denote the plick graph of $G$. Then plick graph $P(G)$ of path graph $P_n$ with $n \geq 2$ results into a bipartite graph. By Theorem 2.1, obtained plick graph is strongly perfect.

**Theorem 3.2.** If $G$ is a cycle graph $C_n$ with $n \geq 3$, then plick graph of $G$ is

$$P(G) = P(C_n) = \begin{cases} \text{strongly perfect} & \text{if } n = 3 \\ \text{not strongly perfect} & \text{if } n \text{ is odd} \\ \text{strongly perfect} & \text{if } n \text{ is even} \end{cases}$$

**Proof.** Let $G$ be a cycle graph $C_n$ with $n \geq 3$, where $n$ is the number of vertices in $C_n$. Let $P(G)$ be the plick graph of $G$ and $H$ be any induced subgraph of $P(G)$. We consider two cases,

**Case 1.** If $n = 3$.

In this case the plick graph $P(G)$ of a cycle graph $C_n$ with $n = 3$ is strongly perfect. The obtained plick graph $P(G)$ has no induced $P_4$. Thus by Theorem 2.2, plick graph $P(G)$ is strongly perfect.

**Case 2.** When $n > 3$. We consider following subcases,

**Subcase 2.1.** If $n$ is even with $n = 4$.

The plick graph $P(G)$ of a cycle graph $C_4$ produces a wheel graph $W_5$ and it has no induced $P_4$ in it. Thus by Theorem 2.2, plick graph of $C_4$ is strongly perfect.

**Subcase 2.2.** If $n$ is even with $n = 2k$, where $k = 3, 4, ...$

The plick graph $P(G)$ of a cycle graph $C_n$ when $n$ is even with $n = 2k$, where $k = 3, 4, ...$ is a wheel graph $W_n$ with odd number of vertices. Each of this plick graph has odd cycle of length at least five with at least two chords. Hence by Theorem 2.3, the plick graph $P(G)$ of a cycle graph $C_n$ is strongly perfect.

**Subcase 2.3.** If $n$ is odd with $n = 2k + 1$, where $k = 2, 3, ...$

The plick graph $P(G)$ of a cycle graph $C_n$ when $n$ is odd with $n = 2k + 1$, $k = 2, 3, ...$ is a wheel graph $W_n$ with even number of vertices. The obtained plick graph $P(G)$ has one induced subgraph $H$ as a cycle $C_n$, where $n = 5, 7, ...$. Let the vertex set of induced subgraph of plick graph $P(G)$ which is a cycle as shown in Figure 1 be $V(C_n) = \{v_1, v_2, ..., v_n\}$ and edge set be $E(C_n) = \{v_1v_2, v_2v_3, ..., v_nv_1\}$. Since the induced subgraph is a cycle $C_n$, it has clique as the complete graph $K_n$. Let $S = \{v_1, v_3, ..., v_{n-2}\}$ be the independent set of the induced subgraph, which is a cycle $C_n$, where $n = 5, 7, ...$. We find that the independent set $S$ meets all the cliques $\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_n, v_1\}$. But the independent set $S$ does not meet the clique $\{v_{n-1}, v_n\}$. Thus, definition of strongly perfect graph is not satisfied. Hence, plick graph $P(G)$ of a cycle graph $C_n$ is not strongly perfect.

**Theorem 3.3.** If $G$ is a tadpole graph $T_{m,n}$ with $m \geq 3$ and $n \geq 1$, then plick graph $P(G)$ of a graph $G$ is

$$P(G) = P(T_{m,n}) = \begin{cases} \text{strongly perfect} & \text{if } m = 3 \text{ and } n \geq 1 \\ \text{not strongly perfect} & \text{if } m = 3 \text{ and } n \leq 1 \end{cases}$$

**Proof.** Let $G = T_{m,n}$ with $m \geq 3$ and $n \geq 1$ be a tadpole graph.

Let $P(G)$ be the plick graph of $G$ and $H$ be any induced subgraph of $P(G)$.

Consider two cases as follows,

**Case 1.** If $m = 3$ and $n \geq 1$.

The plick graph $P(G)$ of a tadpole graph $T_{m,n}$ for $m = 3$ and $n \geq 1$ contains one complete graph $K_4$, a cycle $C_3$, and a tree.
Case 2. When \( m > 3 \) and \( n \geq 1 \).
This case consists of two subcases.

Subcase 2.1. If \( m \) is even and \( n \geq 1 \).
The plick graph \( P(G) \) of a tadpole graph \( T_{m,n} \), where \( m \) is even and \( n \geq 1 \) contains a wheel graph \( W_n \) with odd number of vertices, a cycle \( C_3 \) and a tree.

![Figure 3](image)

Figure 3. Tadpole graph \( T_{m,n} \) where \( m \) has even values, \( n \geq 1 \) and its plick graph \( P(T_{m,n}) \).

Let \( H \) be any induced subgraph of plick graph \( P(G) \) where \( G \) is a tadpole graph \( T_{m,n} \). It has an independent set \( S_1 \) which meets all cliques of \( H \). Now, consider the graph \( H - S_1 \). The independent set of \( H - S_1 \) is \( S_2 \) which meets all cliques of \( H - S_1 \). Continuation of this process results into a null graph. Since null graph is strongly perfect, hence each induced subgraph \( H \) of \( P(G) \) contains an independent set which meets all the cliques of \( H \). Thus, plick graph \( P(G) \) is strongly perfect.

Subcase 2.2. If \( m \) is odd and \( n \geq 1 \).
The obtained plick graph \( P(G) \) of a tadpole graph \( T_{m,n} \) when \( m \) is odd and \( n \geq 1 \) contains a wheel graph \( W_n \) with even number of vertices, a cycle \( C_3 \) and a tree.
There exists one induced subgraph \( H \) of plick graph \( P(G) \) as a cycle \( C_m \) with \( m = 5,7,\ldots \).

![Figure 4](image)

Figure 4. Induced subgraph of plick graph \( P(T_{m,n}) \), where \( m \) has odd values and \( n \geq 1 \).

Let vertex set of the induced subgraph which is a cycle as shown in Figure 4 be \( V(C_m) = \{v_1,v_2,\ldots,v_m\} \) and edge set be \( E(C_m) = \{v_1v_2,v_2v_3,\ldots,v_mv_1\} \). Since the induced subgraph is a cycle \( C_m \), its clique is the complete graph \( K_2 \). Let \( S = \{v_1,v_3,\ldots,v_{m-2}\} \) be the independent set of this induced subgraph which is a cycle \( C_m \), where \( m = 5,7,\ldots \). From this it follows that the independent set \( S \) meets all the cliques \( \{v_1,v_2\}, \{v_2,v_3\},\ldots,\{v_{m-2},v_{m-1}\},\{v_m,v_1\} \). But the independent set \( S \) does not meet the clique \( \{v_{m-1},v_m\} \). Thus, definition of strongly perfect graph fails in this case. Hence, plick graph \( P(G) \) is not strongly perfect.

Theorem 3.4. If \( G \) is a friendship graph \( F_x \), where \( x \) is the number of copies of cycle \( C_3 \) with \( x \geq 2 \), then plick graph \( P(G) \) of a graph \( G \) is strongly perfect.

Proof. Let \( G \) be a friendship graph \( F_x \), where \( x \) is the number of copies of cycle \( C_3 \) with \( x \geq 2 \). Let \( P(G) \) be the plick graph of \( G \) and \( H \) be any induced subgraph of \( P(G) \).
The obtained plick graph \( P(G) \) of friendship graph \( F_x \) contains one complete graph \( K_{2x} \) with \( x \geq 2 \) and \( x \) copies of complete graph \( K_4 \).
Consider two cases as follows,

Case 1. When \( x = 2 \).
The obtained plick graph contains three copies of complete graph \( K_4 \). Let \( H \) be any induced subgraph of \( P(G) \). For induced subgraph \( H \) there exists an independent set \( S_1 \) which meets all the cliques of \( H \). Further consider the graph \( H - S_1 \), it has an independent set \( S_2 \) which meets all the cliques of \( H - S_1 \). Continuing this process results into a trivial graph. Since the trivial graph satisfies the definition of strongly perfect graph, it follows that each induced subgraph \( H \) of plick graph \( P(G) \) contains an independent set which meets all the cliques of \( H \). Hence, plick graph \( P(G) \) is strongly perfect.

Case 2. When \( x \geq 3 \).
The obtained plick graph contains one complete graph \( K_{2x} \), where \( x \geq 3 \) and \( x \) copies of complete graph \( K_4 \). This plick graph has an odd cycle of length five with at least two chords. From Theorem 2.3, plick graph \( P(G) \) is strongly perfect.

Theorem 3.5. Plick graph of every star graph \( K_{1,n} \), where \( n \geq 1 \) is strongly perfect.

Proof. Let \( G \) be a star graph \( K_{1,n} \), where \( n \geq 1 \).
Let \( P(G) \) be the plick graph of \( G \) and \( H \) be any induced subgraph of \( P(G) \).
The plick graph \( P(G) \) of a star graph \( K_{1,n} \) is obtained in the form of a complete graph \( K_n \), where \( n \geq 1 \) with one pendent edge at each vertex of \( K_n \), these pendent edges are \( K_2 \) in nature.
Consider the following two cases,

Case 1. When \( n = 1 \) and \( n = 2 \).
For $n = 1$ and $n = 2$ the plick graph $P(G)$ are path graph $P_2$ and $P_3$ respectively. Since path graph is a bipartite graph, by Theorem 2.1, plick graph $P(G)$ is strongly perfect.

Case 2. When $n = 3$.

For $n = 3$ the plick graph $P(G)$ obtained contains a complete graph $K_3$ and one pendent edge $K_2$ at each vertex of $K_3$.

Let $H$ be any induced subgraph of plick graph $P(G)$. For induced subgraph $H$ we find an independent set $S_1$ which meets all the cliques of $H$. Further, consider graph $H - S_1$, for this graph we have $S_2$ as an independent set in $H - S_1$ such that $S_2$ meets all the cliques of $H - S_1$. Continuation of this process leads into a trivial graph. Since trivial graph satisfies the definition of strongly perfect graph, it follows that each induced subgraph $H$ of plick graph $P(G)$ contains an independent set which meets all the cliques of $H$. Hence, plick graph $P(G)$ is strongly perfect.

Case 3. When $n \geq 4$.

For $n \geq 4$ the obtained plick graph $P(G)$ contains one complete graph $K_n$, where $n \geq 4$ with one pendent edge at each vertex of $K_n$, these pendent edges are $K_2$ in nature. In this case the plick graph obtained contains an odd cycle of length at least five with at least two chords. Thus, by Theorem 2.3, plick graph $P(G)$ is strongly perfect.

**Theorem 3.6.** If $G$ is a helm graph $H_y$ with $y \geq 3$, then plick graph $P(G)$ of a graph $G$ is

$$P(G) = P(H_y) = \begin{cases} 
\text{strongly perfect} & \text{if } y = 3 \\
\text{not strongly perfect} & \text{if } y > 3.
\end{cases}$$

**Proof.** Let $G = H_y$ with $n \geq 3$ be a helm graph.

Let $P(G)$ be the plick graph of $G$ and $H$ be any induced subgraph of $P(G)$.

We discuss the following two cases,

**Case 1.** If $y = 3$. The plick graph $P(G)$ of a helm graph $H_3$ contains an odd cycle of length at least five with at least two chords. By Theorem 2.3 plick graph $P(G)$ of $H_3$ is strongly perfect.

**Case 2.** If $y > 3$.

Let $y = 4$. The helm graph considered is $H_4$ and its plick graph is $P(H_4)$.

4. Results on lict graphs of some class of graphs to be strongly perfect

In this section, we prove lict graphs of some class of graphs to be strongly perfect.

**Theorem 4.1.** Lict graph of any path graph $P_n$ with $n \geq 3$ is strongly perfect.

**Proof.** Let $G$ be the path graph $P_n$ with $n \geq 3$.

Let $L(G)$ denote the lict graph of a graph $G$ and $H$ be any induced subgraph of lict graph $L(G)$. The obtained lict graph $L(G)$ of path graph $P_n$ with $n \geq 3$ contains cycles of length 3 which are $(n - 2)$ in number, where $n$ is the number of vertices in $P_n$.

For induced subgraph $H$ we find an independent set $S_1$ which meets all the cliques of $H$. Consider the graph $H - S_1$, It has an independent set $S_2$ which meets all the cliques of $H - S_1$. Continuation of this process leads to a trivial graph, for trivial graph the definition of strongly perfect graph holds.
Theorem 4.2. If \( G \) is a tadpole graph \( T_{m,n} \) with \( m \geq 3 \) and \( n \geq 1 \), then lict graph \( L_c(G) \) of a graph \( G \) is

\[
L_c(G) = L_c(T_{m,n}) = \begin{cases} 
\text{strongly perfect} & \text{if } m = 3 \text{ and } n \geq 1, \\
\text{strongly perfect} & \text{if } m \text{ is even and } n \geq 1, \\
\text{not strongly perfect} & \text{if } m \text{ is odd and } n \geq 1.
\end{cases}
\]

Proof. Let \( G = T_{m,n} \) with \( m \geq 3 \) and \( n \geq 1 \) be a tadpole graph. Let \( L_c(G) \) be the lict graph of \( G \) and \( H \) be any induced subgraph of \( L_c(G) \).

Consider two cases as follows.

Case 1. If \( m = 3 \) and \( n \geq 1 \).

The lict graph \( L_c(G) \) of tadpole graph \( T_{m,n} \) for \( m = 3 \) and \( n \geq 1 \) contains a cycle \( C_3 \), one complete graph \( K_4 \) and \( (n-1) \) number of cycles \( C_3 \) corresponding to path \( P_n \), where \( n \) is the number of vertices in \( P_n \).

In this case the induced subgraph \( H \) contains an independent set \( S_1 \) such that \( S_1 \) meets all the cliques of \( H \). Consider the graph \( H - S_1 \). There exists an independent set \( S_2 \) in \( H - S_1 \) such that \( S_2 \) meets all the cliques of \( H - S_1 \). Continuation of this process results into a trivial graph which is strongly perfect. Thus, each induced subgraph of lict graph \( L_c(G) \) contains an independent set which meets all the cliques of \( H \). Hence, lict graph \( L_c(G) \) is strongly perfect.

Case 2. When \( m > 3 \) and \( n \geq 1 \).

This case consists of two subcases.

Subcase 2.1. If \( m \) is even and \( n \geq 1 \).

The lict graph \( L_c(G) \) of tadpole graph \( T_{m,n} \), where \( m \) is even and \( n \geq 1 \) is made up of one even cycle \( C_m \) \((m = 4, 6, ...)\), one complete graph \( K_4 \) and \((n-1) \) number of cycles \( C_3 \) corresponding to path \( P_n \), where \( n \) is number of vertices in path \( P_n \).

Consider an induced subgraph \( H \), there exists an independent set \( S_1 \) in \( H \) which meets all the cliques of \( H \). Consider graph \( H - S_1 \). For this graph there exists an independent set \( S_2 \) which meets all the cliques of \( H - S_1 \). Continuation of this process results into a trivial graph which is strongly perfect. Thus, each induced subgraph of lict graph \( L_c(G) \) contains an independent set which meets all the cliques of \( H \). Hence, lict graph \( L_c(G) \) is strongly perfect.

Subcase 2.2. If \( m \) is odd and \( n \geq 1 \).

The obtained lict graph \( L_c(G) \) consists of an odd cycle \( C_m \) with \( m = 5, 7, ..., \) one complete graph \( K_4 \) and \((n-1) \) number of cycles \( C_3 \) corresponding to path \( P_n \), where \( n \) is number of vertices in \( P_n \). In lict graph \( L_c(G) \) one induced subgraph \( H \) is a cycle \( C_m \) with \( m = 5, 7, ..., \).

Let the vertex set of this induced subgraph be \( V(C_m) = \{v_1, v_2, ..., v_m\} \) and edge set be \( E(C_m) = \{v_1v_2, v_2v_3, ..., v_mv_1\} \).

Since the induced subgraph is a cycle \( C_m \), its clique is the complete graph \( K_2 \). Let \( S = \{v_1, v_3, ..., v_{m-1}\} \) be the independent set of this induced subgraph which is a cycle \( C_m \) with...
Consider the following two cases, 

**Case 1.** If $y = 3$. 

The lict graph $L_c(G)$ of a helm graph $H_3$ contains an odd cycle of length at least five with at least two chords. Thus by Theorem 2.3 lict graph $L_c(G)$ of $H_3$ is strongly perfect.

**Case 2.** If $y > 3$. 

This case consists of two subcases,

**Subcase 2.1.** When $y$ is even. 

Let $y = 4$. The helm graph considered is $H_4$ and its lict graph is $L_c(H_4)$.

**Subcase 2.2.** When $y$ is odd. 

Let $y = 5$. The helm graph considered is $H_5$ and its lict graph is $L_c(H_5)$.
Consider one of the induced subgraph of lict graph $L_c(H_5)$ as shown in the Figure 18, which is an odd cycle of length 5. It has an independent set $S = \{e_1, e_3\}$ which does not meet the clique $\{e_4, e_5\}$. Thus definition of strongly perfect graph is not satisfied. Hence lict graph $L_c(H_5)$ is not strongly perfect. For the remaining values of $y$ (i.e., $y = 7, 9, ...$) of helm graph the same argument holds. Thus respective lict graphs of helm graphs are not strongly perfect. Hence for $y > 3$, lict graphs $L_c(G)$ of respective helm graphs are not strongly perfect.

\[\square\]

**References**


