Pythagorean triangles and addition of nonagonal, triangular numbers

D. Ramprasad

Abstract
Oblong numbers as figurate numbers, which were first studied by the Pythagoreans are studied in terms of special Pythagorean Triangles. The two consecutive sides and their perimeters of Pythagorean triangles are investigated. In this study, the perimeter of Pythagorean triangles is obtained as addition of nonagonal and triangular numbers.

Keywords
Nonagonal numbers, Triangle numbers, Pythagorean Triangles, Diophantine equation.

AMS Subject Classification
11D09.

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Article History: Received 01 January 2021; Accepted 22 February 2021

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1. Introduction

2. Method of Analysis
The primitive solutions of the Pythagorean Equation,
\[
X^2 + Y^2 = Z^2
\]  
(2.1)
is given by [5]
\[
X = m^2 - n^2, Y = 2mn, Z = m^2 + n^2
\]  
(2.2)
for some integers \( m, n \) of opposite parity such that \( m > n > 0 \) and \( (m, n) = 1 \).

2.1 Perimeter is an addition of nonagonal and triangular numbers
Definition 2.1. A natural number \( P \) is called addition of nonagonal and triangular numbers if it can be written in the form
\[
\frac{(7w^2 - 5w)}{2} + \frac{(w^2 + w)}{2} = 2\left(2w^2 - w\right), w \in \mathbb{N}.
\]

If the perimeter of the Pythagorean triangle \((X, Y, Z)\) is addition of nonagonal and triangular numbers \( W \), then
\[
X + Y + Z = 2\left(2w^2 - w\right) = P
\]  
(2.3)
From the equations (2.2) & (2.3) \( 2m^2 + 2mn = 2\left(2w^2 - w\right), w \in \mathbb{N} \)
\[
m(m + n) = w(2w - 1)
\]  
(2.4)
2.2 Hypotenuse and one leg are consecutive

In such cases,

\[ m = n + 1. \]  

(2.5)

This gives equation (2.4) as \( (n + 1)(2n + 1) = w(2w - 1) \). Take,

\[ w = n + 1. \]  

(2.6)

Equations (2.2), (2.5) & (2.6) give solution of equations (2.1) in correspondence with equations (2.3) and (2.4) i.e., \( X = 2n + 1; Y = 2n(n + 1); Z = 2n(n + 1) + 1; \)

First ten such special Pythagorean triangles \((X, Y, Z)\) are given in the Table 1 below:

<table>
<thead>
<tr>
<th>S. No.</th>
<th>( n )</th>
<th>( w )</th>
<th>( P )</th>
<th>( X )</th>
<th>( Y )</th>
<th>( Z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>12</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>30</td>
<td>5</td>
<td>12</td>
<td>13</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
<td>56</td>
<td>7</td>
<td>24</td>
<td>25</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>5</td>
<td>90</td>
<td>9</td>
<td>40</td>
<td>41</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>6</td>
<td>132</td>
<td>11</td>
<td>60</td>
<td>61</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>7</td>
<td>182</td>
<td>13</td>
<td>84</td>
<td>85</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>8</td>
<td>240</td>
<td>15</td>
<td>112</td>
<td>113</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>9</td>
<td>306</td>
<td>17</td>
<td>144</td>
<td>145</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>10</td>
<td>380</td>
<td>19</td>
<td>180</td>
<td>181</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>11</td>
<td>462</td>
<td>21</td>
<td>220</td>
<td>221</td>
</tr>
</tbody>
</table>

Table 2. Verification of \( X^2 + Y^2 = Z^2 \) and \( X + Y + Z = 2w(2w - 1) \)

<table>
<thead>
<tr>
<th>S.No</th>
<th>( X^2 )</th>
<th>( Y^2 )</th>
<th>( X^2 + Y^2 )</th>
<th>( Z^2 )</th>
<th>( X + Y + Z ) =2 ( w(2w -1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9</td>
<td>16</td>
<td>25</td>
<td>25</td>
<td>12 = 2.23</td>
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<tr>
<td>2</td>
<td>25</td>
<td>144</td>
<td>169</td>
<td>169</td>
<td>30 = 2.35</td>
</tr>
<tr>
<td>3</td>
<td>49</td>
<td>576</td>
<td>625</td>
<td>625</td>
<td>56 = 2.47</td>
</tr>
<tr>
<td>4</td>
<td>81</td>
<td>1600</td>
<td>1681</td>
<td>1681</td>
<td>90 = 2.59</td>
</tr>
<tr>
<td>5</td>
<td>121</td>
<td>3600</td>
<td>3721</td>
<td>3721</td>
<td>132 = 2.611</td>
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<tr>
<td>6</td>
<td>169</td>
<td>7056</td>
<td>7225</td>
<td>7225</td>
<td>182 = 2.713</td>
</tr>
<tr>
<td>7</td>
<td>225</td>
<td>12544</td>
<td>12769</td>
<td>12769</td>
<td>240 = 2.815</td>
</tr>
<tr>
<td>8</td>
<td>289</td>
<td>20736</td>
<td>21025</td>
<td>21025</td>
<td>306 = 2.917</td>
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<tr>
<td>9</td>
<td>361</td>
<td>32400</td>
<td>32761</td>
<td>32761</td>
<td>380 = 2.1019</td>
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<tr>
<td>10</td>
<td>441</td>
<td>48400</td>
<td>48841</td>
<td>48841</td>
<td>462 = 2.1121</td>
</tr>
</tbody>
</table>

3. Observations and conclusion

1. \( (X + Y - Z)^2 = (Y + Z - 2X + 1) \)

2. \( (X + Z - Y)^2 = (Y + Z + 2X + 1) \)

3. \( Y + Z = X^2 \)

4. \( (2X - Y + Z)^2 = X^2 + 2(X + Y + Z) + 2(X + Z) \)

References

[1] M.A.Gopalan and P.Vijalakshmi, Special Pythagorean triangles generated through the integral solutions of the equation \( y^2 = (k^2 + 1)x^2 + 1 \), Antarctica J. Math., 7(5)(2010), 503-507.


