Logarithmic coefficients for starlike and convex functions of complex order defined by subordination

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Abstract
The aim of this paper is to find the bounds for the logarithmic coefficients \( \gamma \) of the general classes of starlike and convex functions of complex order, \( S_d(\Psi) \) and \( K_d(\Psi) \) respectively. Our results would generalize some of the previous paper like [1] E. A. Adegani et al., [3] Ali et al. etc.

Keywords
Starlike function and convex function of Complex order; subordination; logarithmic coefficients.

AMS Subject Classification
30C45.

1. Introduction
Suppose \( \mathcal{A} \) be the class containing functions which are of the form:

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

and are also analytic in the unit disk \( f(z) = \{ z : |z| < 1 \} \). Furthermore we assumes to be the subclass of \( \mathcal{A} \) which consists of all univalent functions in \( \Delta \), then the logarithmic coefficients \( \gamma \) of \( fS \), satisfies:

\[
\log \left( \frac{f(z)}{z} \right) = 2 \sum_{n=1}^{\infty} \gamma_n (f) z^n, z \in \Delta
\]

(1.2)

\( \gamma \) can be written as \( \gamma \). In the history of univalent function, these logarithmic coefficients play a significant role in various estimates. [2] Kayumov solved Brennan’s conjecture for conformal mappings using these logarithmic coefficients. Equation (1.2) can be written as

\[
2 \sum_{n=1}^{\infty} \gamma_n z^n = \left( \frac{t_2 z + t_3 z^2 + t_4 z^3 + \ldots}{z} \right) - \frac{1}{2} \left( \frac{t_2 z + t_3 z^2 + t_4 z^3 + \ldots}{z} \right)^2 + \frac{1}{2} \left( \frac{t_2 z + t_3 z^2 + t_4 z^3 + \ldots}{z} \right)^3 + \ldots
\]

Equating the coefficients of \( z^n \) for \( n = 1, 2, 3 \), on both sides of the above equation, we get:

\[
\begin{align*}
2\gamma_1 &= t_2 \\
2\gamma_2 &= t_3 - \frac{1}{2} t_2^2 \\
2\gamma_3 &= t_4 - t_2 t_3 + \frac{1}{3} t_2^3
\end{align*}
\]

(1.3)

Definition 1.1 Starlike function of complex order \( d \): For the function \( f(z) \in \mathcal{A} \) to be starlike of complex order \( d \) \( (d \in C \setminus \{0\}) \), it must follow the condition: \( \frac{f(z)}{z} \neq 0 \) \( (z \in \Delta) \) and

\[
\text{Re} \left\{ \frac{1 + \frac{1}{d} \left( \frac{z f'(z)}{f(z)} - 1 \right)}{z} \right\} > 0
\]

we denote this class by \( S^*_d(d) \).

Definition 1.2 Convex function of complex order \( d \): For the function \( f(z) \in \mathcal{A} \) to be convex of complex order \( d \) \( (d \in C \setminus \{0\}) \), it must follow the conditions given below:

\[
f''(z) \neq 0 \text{ and }
\text{Re} \left\{ \frac{1}{d} \left( \frac{z f''(z)}{f'(z)} \right) \right\} > 0, (z \Delta)
\]

We denote this class by \( K_d(d) \).

A function \( f(z) \in \mathcal{A} \) is close-to-convex of complex order \( d \) \( (d \in C \setminus \{0\}) \) if there exists a function \( g(z) \in K_d(d) (d \in C \setminus \{0\}) \) which satisfy the following condition:

\[
\text{Re} \left\{ \frac{1}{d} \left( \frac{f(z)}{f'(z)} - 1 \right) \right\} > 0, (z \in \Delta)
\]

We denote this class by \( \mathcal{C}_d \).

Definition 1.3 Subordination: If \( f \) and \( g \) are two functions analytic in \( \Delta \), then the function \( f \) is subordinate to \( g \) in

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\( \Delta, i.e. f(z) \prec g(z) \), if there exists a Schwarz function \( \omega \), analytic in \( \Delta \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) such that \( f(z) = g(\omega(z)) (z^n \Delta) \). Articularly, if the function \( g \) is univalent in \( \Delta \), then \( f \prec g \) if the following conditions hold \( f(0) = 0 \) and \( f' (\Delta) \subseteq g' (\Delta) \).

Nasr and Aouf [4] introduced and studied the classes \( S_0'(b) \) and \( K_0(b) \). Ma and Minda [5] introduced and studied the class \( S^* (\phi) \) which consists of functions \( f \in S \) satisfying the following conditions

\[
\frac{zf'(z)}{f(z)} \prec \phi(z), (z \Delta).
\]

In this paper, we define a more general class of starlike function and convex function of complex order following Ma and Minda and find bounds for logarithmic coefficients for this class.

**Definitions 1.4**: Let \( S_0^d(\Psi) \) be a class consisting of all analytic function \( f \) where \( d \in (\mathbb{C} \setminus \{0\}) \) and \( \Psi(z) \) is an analytic function with positive real part on \( \Delta \) satisfying \( \Psi(0) = 1 \), \( \Psi'(0) > 0 \) and maps \( \Delta \) onto region starlike with respect to 1 and symmetric with respect to the real axis. Then \( S_0^d(\Psi) \) consists of all analytic functions \( f \) satisfying

\[
1 + \frac{1}{d} \left( \frac{zf'(z)}{f(z)} - 1 \right) \prec \Psi(z) \quad (1.4)
\]

The class \( K_0^d(\Psi) \) consists of the functions \( f \) which satisfies the following condition:

\[
1 + \frac{1}{d} \left( \frac{zf'(z)}{f(z)} - 1 \right) \prec \Psi(z) \quad (1.5)
\]

Furthermore, we let \( S^d(M,N,d) \) and \( K^d(M,N,d) \) \((d \neq 0, \text{complex})\) denote the class \( S_0^d(\Psi) \) and \( K_0^d(\Psi) \) respectively, where

\[
\Psi(z) = \frac{1 + Mz}{1 + Nz}, (-1 \leq N < M \leq 1).
\]

The class \( S^d(M,N,d) \) and the class \( S_0^d(\Psi) \), specialize to many well known classes of univalent functions for suitable choice of M, N, and d.

Recently many researchers have worked on the similar problems of logarithmic coefficients, such as the function \( k(z) = z(1 - e^{\theta})^{-2} \) has logarithmic coefficients \( \gamma_n = \frac{a_n}{n}, n \geq 1 \) for every \( \theta \). In [6] (Theorem 4), it has been proved that the logarithmic coefficients \( \gamma_n \) of every function \( f \in S \) satisfy:

\[
\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{\pi^2}{6},
\]

and the equality is attained for the Koebe function. Ali et al. [3] and P. Kumar et al. [7] in 2018 found the bounds for logarithmic coefficients \( \gamma_n \) for definite classes of close-to-convex functions. In 2019, E.A. Adegani, NakEun Cho and Mostafa Jafari [1] obtained bounds for logarithmic coefficients for certain subclasses of starlike and convex functions defined by subordination. But the problem for \( n \geq 3 \), for the logarithmic coefficients of univalent function is still a matter of concern.

On the basis of the results obtained in the previous paper, we have tried to obtain the bounds for the logarithmic coefficients \( \gamma_n \) of the general classes \( S_0^d(\Psi) \) and \( K_0^d(\Psi) \) in this paper.

The lemmas will be used in our proofs as follows:

**Lemma 1.** [8] Let \( w \) be a Schwarz function such that \( w(z) = \sum_{n=1}^{\infty} w_n z^n \), then

\[
|w_1| \leq 1, |w_n| \leq 1 - |w_1|^2, n = 2, 3, \ldots
\]

**Lemma 2.** [9] Suppose \( \psi, \phi \in \mathbb{A} \) be convex in \( \Delta \), such that \( f(z) \prec \psi(z) \) and \( g(z) \prec \phi(z) \), then \( f(z) * g(z) \prec \psi(z) * \phi(z) \), where \( f, g \in \mathbb{A} \) and \( \ast \) represents convolution.

**Lemma 3.** [6, 10] Suppose \( I(z) = \sum_{n=1}^{\infty} a_n z^n \) and \( k(z) = \sum_{n=1}^{\infty} b_n z^n \) be analytic in \( \Delta \), and assume \( l \prec k \) where \( k \) is univalent in \( \Delta \).

Then \( \sum_{n=1}^{\infty} |a_n|^2 \leq \sum_{n=1}^{\infty} |b_n|^2, n = 1, 2, \ldots \)

**Lemma 4.** [6, 10] (Theorem 6.4(i)). Suppose \( j(z) = \sum_{n=1}^{\infty} j_n z^n \) and \( h(z) = \sum_{n=1}^{\infty} h_n z^n \) be analytic in \( \Delta \) and assuming \( j \prec h \) where \( h \) univalent in \( \Delta \), then

\[
\begin{align*}
|v_3 + p_1 v_1 v_2 + p_2 v_2^2| & \leq H(p_1, p_2),
\end{align*}
\]

where

\[
H(p_1, p_2) = \begin{cases}
1, & \text{if } (p_1, p_2) \in D_1, D_2 \cup \{(2, 1)\},
\frac{|p_1|}{|p_2|}, & \text{if } (p_1, p_2) \in \Omega, \quad \text{and} \\
\left( \frac{|p_1|}{|p_2|} + 1 \right)^{\frac{3}{2}}, & \text{if } (p_1, p_2) \in D_0 \cup D_9.
\end{cases}
\]

where the sets \( D_k, k = 1, 2, \ldots, 12 \) are given by

\[
D_1 = \{ (p_1, p_2) : |p_1| \leq \frac{1}{2}, |p_2| \leq 1 \},
\]

\[
D_2 = \{ (p_1, p_2) : \frac{1}{2} \leq |p_1| \leq 2, \frac{4}{27} ((|p_1| + 1)^3) - (|p_1| + 1) \leq |p_2| \leq 1 \},
\]

\[
D_3 = \{ (p_1, p_2) : |p_1| \leq \frac{1}{2}, |p_2| \leq -1 \},
\]

\[
D_4 = \{ (p_1, p_2) : |p_1| \geq \frac{1}{2}, |p_2| \leq -\frac{4}{3} (|p_1| + 1) \},
\]

\[
D_5 = \{ (p_1, p_2) : |p_1| \leq 2, |p_2| \geq 1 \}.
\]
D_6 = \{(p_1, p_2) : 2 \leq |p_1| \leq 4, |p_2| \geq \frac{1}{12} (p_1^2 + 8)\},
D_7 = \{(p_1, p_2) : |p_1| \geq 4, |p_2| \geq \frac{2}{3} (|p_1| - 1)\},
D_8 = \{(p_1, p_2) : \frac{2}{3} (|p_1| - 1) \leq |p_1| \leq 2, |p_2| \leq \frac{1}{12} (|p_1| + 1)\},
D_9 = \{(p_1, p_2) : |p_1| \geq 2, |p_2| \leq \frac{1}{12} (|p_1| + 1)\},
D_{10} = \{(p_1, p_2) : |p_1| \leq 4, |p_2| \leq \frac{1}{12} (p_1^2 + 8)\},
D_{11} = \{(p_1, p_2) : |p_1| \leq 4, |p_2| \leq \frac{1}{2} (p_1^2 + 4)\},
D_{12} = \{(p_1, p_2) : |p_1| \geq 4, |p_2| \leq \frac{2}{3} (|p_1| - 1)\}.

Now, firstly to prove inequality (2.2), let us suppose that
\[ |\gamma| \leq \frac{d}{2n} |D_1|, n \in N \]
which gives the result:
\[ |\gamma| \leq \frac{d}{2n} |D_1|, n \in N \]

Again, to prove inequality (2.3), we define the analytic function
\[ h(z) = \left( \frac{f(z)}{z} \right)^\frac{1}{2} \]
which satisfy the following:
\[ \frac{zh'(z)}{h(z)} = \frac{1}{d} \left( \frac{zf'(z)}{f(z)} - 1 \right) \prec \phi(z), z \in \Delta \] (2.6)

Also we know that (see [12])
\[ E_0(z) = \log \left( \frac{1}{1 - z} \right) = \sum_{n=1}^{\infty} \frac{z^n}{n} \]
belongs to the class $K$, and for $f \in A$,
\[ f(z), E_0(z) = \int_{0}^{z} \frac{f(x)}{x} \, dx \] (2.7)

Then, by Lemma 2 and equation (2.6), we get
\[ \frac{zh'(z)}{h(z)} * E_0(z) \prec \phi(z) * E_0(z) \]

Using (2.7), the exceeding equation reduces to
\[ \frac{1}{d} \log \left( \frac{f(z)}{z} \right) \prec \frac{\varphi(x)}{x} \, dx \]

Also we know that (see [13]), the function $\int_{0}^{\infty} \frac{\varphi(x)}{x} \, dx$, is convex univalent. Using (1.2), the above relation becomes
\[ \frac{1}{d} \sum_{n=1}^{\infty} 2n \gamma_n z^n \prec \sum_{n=1}^{\infty} \frac{D_n z^n}{n} \]

Now by using Lemma 3, the above subordination yields
\[ \frac{4}{d^2} \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \sum_{n=1}^{\infty} \frac{|D_n|^2}{n^2} \]

This concludes inequality (2.3).
Assuming $k \to \infty$,
\[ \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{d^2}{4} \sum_{n=1}^{\infty} \frac{|D_n|^2}{n^2} \]
this gives inequality (2.4).
Lastly assume that $\Psi(z)$ is starlike with respect to 1 in $\Delta$, this implies $\phi(z)$ is starlike, therefore using Lemma 4(ii), we deduce
\[ \frac{2n}{d} |\gamma| \leq n |\phi'(a)| = n |D_1|, n \in N \]
This gives equation (2.5).
To get the sharp bounds, it is sufficient to consider the following:

\[
\frac{1}{d} \left[ \frac{z}{d} \left( \log \left( \frac{f'(z)}{f(z)} \right) \right) \right] = \frac{1}{d} \left[ \frac{zf'(z)}{f(z)} - 1 \right]
\]
and so these results are sharp in cases (i) and (ii), such that for any \( n \in \mathbb{N} \), there exists the function \( f_n \) given by \( 1 + \frac{1}{d} \left[ \frac{zf'(z)}{f(z)} - 1 \right] = \Psi(z^n) \), and the function \( f \) given by \( 1 + \frac{1}{d} \left[ \frac{zf'(z)}{f(z)} - 1 \right] = \Psi(z) \), respectively, hence proved.

**Corollary 1.** For \( 0 \leq a < 1 \), if \( f \in S_\alpha'(\alpha + (1 - \alpha)e^z) \). Then the logarithmic coefficients of \( f \), follows the conditions given below \(|\gamma_n| \leq \frac{d}{2} (1 - \alpha) \), \( n \in N \) and

\[
\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{d^2}{4} \sum_{n=1}^{\infty} (1 - \alpha)^2 \frac{1}{(n!)^2 n^2}
\]
The above conditions are sharp for function \( f_n \) satisfying:

\[
1 + \frac{1}{d} \left( \frac{zf_n'(z)}{f_n(z)} - 1 \right) = \alpha + (1 - \alpha) e^z, \quad n \in N
\]
and the function \( f \) given by:

\[
1 + \frac{1}{d} \left( \frac{zf'(z)}{f(z)} - 1 \right) = \alpha + (1 - \alpha) e^z.
\]

**Corollary 2.** Assuming \( \beta = 1 \), Class \( S_{\alpha}'(\Psi(z)) \) reduces to \( S'_{\alpha}'(\Psi(z)) \) defined by Ma and Minda [5]. For the function \( f \in S'_{\alpha}'(\Psi(z)) \), the results of Theorem 1 reduces to the logarithmic coefficients \( \gamma_n \) given by E.A. Adeagani et al. [3],(see Theorem 1). **Corollary 3.** Suppose the function \( f \in S_{\alpha}' \left( 1 + \frac{z}{1 - \alpha z^2} \right) \) and \( 0 \leq \alpha < 1 \). Then the logarithmic coefficients of \( f \) assures:

\[
|\gamma_n| \leq \frac{d}{2}, \quad n \in N
\]
The result is sharp for any \( n \in N \), there exists function \( f_n \) satisfying:

\[
1 + \frac{1}{d} \left( \frac{zf_n'(z)}{f_n(z)} - 1 \right) = 1 + \frac{z^n}{1 - \alpha z^{2n}}.
\]

**Corollary 4.** Suppose the function \( f \in S_{\alpha}' \left( z + \sqrt{1 + z^2} \right) \) then the logarithmic coefficients of \( f \) satisfies:

\[
|\gamma_n| \leq \frac{d}{2}, \quad n \in N.
\]
This result is sharp such that for any \( n \in N \), there exists function \( f_n \), satisfying:

\[
1 + \frac{1}{d} \left( \frac{zf_n'(z)}{f_n(z)} - 1 \right) = (z^n + \sqrt{1 + z^{2n}}).
\]

**Theorem 2.** Suppose the function \( f \in K_{d}(\Psi) \). Then the logarithmic coefficients of \( f \) satisfies the following conditions:

\[
|\gamma_1| \leq \frac{d |D_1|}{4}
\]

(2.8)

and this bound is sharp for \( |b_1| = 1 \).
Again, for $\gamma_2$, we apply Lemma 1 and obtain
\[
|\gamma_2| \leq d \left[ \frac{4|D_1| + (|4D_2 + dD_1^2| - 4|D_1|)|b_1|^2}{48} \right]
\]
\[
= \frac{d}{48} \left[ 4|D_1| + (|4D_2 + dD_1^2| - 4|D_1|) b_1^2 \right]
\]
\[
\leq \left\{ \begin{array}{ll}
\frac{4|D_1|}{48}, & \text{if } |4D_2 + dD_1^2| \leq 4|D_1| \\
\frac{4|D_1| + dD_1^2}{48}, & \text{if } |4D_2 + dD_1^2| > 4|D_1|
\end{array} \right.
\]
These bounds are sharp for $b_1 = 0$ and $|b_1| = 1$ respectively. At the last, for $\gamma_3$, using Lemma 5, we get
\[
2 |\gamma_3| \leq \frac{d|D_1|}{12} b_3 + \left( \frac{d|D_1| + dD_1^2}{2} \right) b_1 b_2 + \frac{(3-2d)dD_1^2}{2} b_3
\]
\[
\leq H(p_1; p_2). \frac{d|D_1|}{12}
\]
Where
\[
p_1 = \frac{d|D_1|}{2} \quad \text{and} \quad p_2 = \frac{(3-2d)dD_1^2}{2}
\]
Thus we get the result.

**Remark 1.** Assuming
\[
\Psi(z) = 1 + \frac{cz}{1-z} \quad (C(0,3))
\]
and $d = 1$, we obtain the result given by Ponnusamy et al. [14]

**Remark 2.** Let $d = 1$, in theorem 2, then we get the result obtained by E.A. Adegani et al. [2].

### References