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A note on det-norm of circulant fuzzy matrices

M. Kavitha[1] and K. Gunasekaran[2]

Abstract

We discuss some operation on circulant fuzzy matrices. We also study circulant fuzzy det-norm matrices using the structure of Mn(F), the set of square circulant fuzzy det-norm matrices is introduced. We prove that circulant det-norm ordering is a partitions ordering on the set of all idempotent matrices in Mn(F). Finally we introduced relationship between these ordering with circulant det-norm ordering and also we investigate the concept of circulant fuzzy norm and partitions of Mn(F), properties of circulant fuzzy det-norm ordering.

Keywords

Fuzzy matrix, circulant fuzzy matrix, determinant of a circulant matrices.

[1,2] *Department of Mathematics, Government Arts College (Autonomous) [Affiliated to Bharathidasan University, Trichy], Kumbakonam-612002, Tamil Nadu, India.*

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1. Introduction

Fuzzy matrix was introduced by Thomason[ ] and he discussed about the convergence of powers of fuzzy matrix. In [8] presented some properties of the min-max composition of fuzzy matrices. S.maity[7] presented some important results on max-norm, square max-norm of fuzzy matrices. Perhaps first time, D.Diamond and P.Kloeden[1] introduced metric spaces of fuzzy sets. In 1994, Ragab.M.Z and Eman E.G[9] introduced the determinant and adjoint of a square matrix. M.Kon[2] introduced the concept of operation and ordering of fuzzy sets and fuzzy set-valued convex mapping. Bhowmik and pal [3] some results of circulant triangular fuzzy number matrices.

In this paper, we introduce the concept of circulant fuzzy det-norm ordering with circulant fuzzy matrices. The purpose of the introduction is to explain det-norm ordering with circulant fuzzy matrices and partitions of Mn(F). In section 2, Circulant fuzzy det-norm ordering with circulant fuzzy matrices is introduced in Mn(F). In section 3, properties of circulant det-norm ordering with circulant fuzzy matrices.

2. Preliminaries and Definitions

2.1 Circulant matrix

An $m \times n$ circulant matrix has the form

$$A=\begin{pmatrix} C_1 & C_2 & C_n-1 & \cdots & C_n \\ C_n & C_1 & C_2 & \cdots & C_{n-1} \\ C_{n-1} & C_n & C_1 & \cdots & C_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ C_2 & c_3 & C_4 & \cdots & C_1 \end{pmatrix}$$

2.2 Circulant fuzzy matrix

For any given $A_0,\ A_1,\ldots\ldots A_{n-1} \in f_{mx}n$ be the circulant fuzzy matrix $\tilde{A}=(c_j)_m \times n$ is defined by $(c_{ij})=\left(A_{j-1(modn)}\right)$ A circulant matrix is the form of

$$\tilde{A_1}=\begin{pmatrix} c_1 & c_2 & c_{n-1} & \cdots & c_n \\ C_n & c_1 & c_2 & \cdots & C_{n-1} \\ C_{n-1} & c_n & c_1 & \cdots & C_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_1 \end{pmatrix}$$

With entries in [0,1]

Definition 2.1. *An $m \times n$ matrix $A=(A_{jj})_{m\times n}$ whose components are in the unit interval [0,1] is called a fuzzy matrix.*

Definition 2.2. *The determinant $|A|$ of an nxn fuzzy matrix A is defined as follows*

$$|A| = \sum_{\alpha \in s_n} a_{1\alpha(1)} a_{2\alpha(2)} a_{3\alpha(3)} \cdots a_{n\alpha(n)}$$

Where S_n denotes the symmetric group of all permutation of the indices $(1,2,\ldots n)$.

Definition 2.3. *Let $M_n(F)$ be the set of all (nxn) fuzzy matrices over $F = [0,1]$ for every A in $M_n(F)$ define norm of A denoted by $\|A\| = \det[A]$ where $A = [a_{ij}]$.*

Definition 2.4. *A matrix A in $M_n(F)$ is called idempotent if $A^2 = A$ or $\|A^2\| = \det[A]$ where $A = [a_{ij}]$.*

Definition 2.5. *A fuzzy matrix A is said to be circulant fuzzy matrix if all the elements of A can be determined completed by its first row. Suppose the first row of A is $[A_1,\ A_2\ A_2, \dots A_n]$. Then any element A_{ij} of can be determined (throughout the element of the first row) as*

$$a_{ij} = a_i(n-i+j+1) \text{ with } A_l(n+k) = A_{1k}$$

A circulant fuzzy matrix is the form.

Definition 2.6. *A circulant matrix A in $M_n(AF)$ is called idemponent if $A^2 = A$ or $\|A\|^2 = \det[A]$ where $A = [a_{ij}]$.*

Example 2.7. *If,*

$$A = \begin{bmatrix} 0.3 & 0.7 & 0.5 \\ 0.5 & 0.3 & 0.7 \\ 0.7 & 0.5 & 0.3 \end{bmatrix}$$

then,

$$\begin{aligned} \|A\| &= 0.3\begin{bmatrix} 0.3 & 0.7 \\ 0.5 & 0.3 \end{bmatrix} + 0.7\begin{bmatrix} 0.5 & 0.7 \\ 0.7 & 0.3 \end{bmatrix} \\ &\quad + 0.5\begin{bmatrix} 0.5 & 0.3 \\ 0.7 & 0.5 \end{bmatrix} \\ &= 0.7 \\ A^2 &= \begin{bmatrix} 0.3 & 0.7 & 0.5 \\ 0.5 & 0.3 & 0.7 \\ 0.7 & 0.5 & 0.3 \end{bmatrix}\begin{bmatrix} 0.3 & 0.7 & 0.5 \\ 0.5 & 0.3 & 0.7 \\ 0.7 & 0.5 & 0.3 \end{bmatrix} \\ &= \begin{bmatrix} 0.5 & 0.5 & 0.7 \\ 0.7 & 0.5 & 0.5 \\ 0.5 & 0.7 & 0.5 \end{bmatrix} \\ &= 0.7 \end{aligned}$$

Therefore, $A^2 = A = 0.7$.

Definition 2.8. *For all A in $M_n(\ F)$ define*

$$\begin{aligned} A\{1\} =& \{x \in Mn(AF)/\|x\| > \|C\|\} \\ A\{2\} =& \{x \in Mn(AF)/\|x\| < \|C\|\} \\ A\{3\} =& \{x \in Mn(AF)/\|x\| = \|C\|\} \\ A\{4\} =& \{x \in Mn(AF)/CXC = C\} \\ A\{5\} =& \{x \in Mn(AF)/XCX = X\} \end{aligned}$$

Clearly $M_n(AF) = A\{1\} \cup A\{2\} \cup A\{3\}$. The set $A\{1\}$ is called as det-superior to A and $A\{2\}$ det inferior to A. Clearly $A\{3\}$ is det equivalent to A,$A\{4\}$ and $A\{5\}$ are known as the set of inner and outer inverses of A.

Example 2.9. *If $A = \begin{bmatrix} 0.3 & 0.7 & 0.8 \\ 0.8 & 0.3 & 0.7 \\ 0.7 & 0.8 & 0.3 \end{bmatrix}$ and $X = \begin{bmatrix} 0.5 & 0.9 & 0.2 \\ 0.2 & 0.5 & 0.9 \\ 0.9 & 0.2 & 0.5 \end{bmatrix}$ then*

$$\begin{aligned} \|C\| &= 0.3\begin{bmatrix} 0.3 & 0.7 \\ 0.8 & 0.3 \end{bmatrix} + 0.7\begin{bmatrix} 0.8 & 0.7 \\ 0.7 & 0.3 \end{bmatrix} \\ &\quad + 0.8\begin{bmatrix} 0.8 & 0.3 \\ 0.7 & 0.8 \end{bmatrix} \\ \|C\| &= 0.3[0.3+0.7] + 0.7[0.3+0.7] + 0.8[0.8+0.3] \\ &= 0.3+0.7+0.8 \\ &= 0.8 \\ \|X\| &= 0.5\begin{bmatrix} 0.5 & 0.9 \\ 0.2 & 0.5 \end{bmatrix} + 0.9\begin{bmatrix} 0.2 & 0.9 \\ 0.9 & 0.5 \end{bmatrix} \\ &\quad + 0.2\begin{bmatrix} 0.2 & 0.5 \\ 0.9 & 0.2 \end{bmatrix} \\ &= 0.5[0.5+0.2] + 0.9[0.2+0.9] + 0.2[0.2+0.5] \\ &= 0.5+0.9+0.2 \\ &= 0.9 \end{aligned}$$

Therefore, $\|X\| > \|A\|$. If

$$A = \begin{bmatrix} 0.3 & 0.5 & 0.1 \\ 0.1 & 0.3 & 0.5 \\ 0.5 & 0.1 & 0.3 \end{bmatrix}$$

Then,

$$\begin{aligned} \|A\| &= 0.3[0.3+0.1] + 0.5[0.1+0.5] + 0.1[0.1+0.3] \\ &= 0.3+0.5+0.1 \\ &= 0.5 \end{aligned}$$

Therefore, $\|X\| < \|A\|$. If,

$$X = \begin{bmatrix} 0.8 & 0.5 & 0.3 \\ 0.3 & 0.8 & 0.5 \\ 0.5 & 0.3 & 0.8 \end{bmatrix}$$

Then,

$$\begin{aligned} \|X\| &= 0.8[0.8+0.3] + 0.5[0.3+0.5] + 0.3[0.3+0.5] \\ &= 0.8+0.5+0.3 \\ &= 0.8 \end{aligned}$$

Therefore, $\|X\| = \|A\|$.

Theorem 2.10. *For each A in the following results hold true;*

(i) If $X \in A\{i\}$ then X^T is also in $A\{i\}$ for $i = 1,2,3$ where X^T is the transpose of X

(ii) If $A_1 \in A\{1\}, A_2 \in A\{2\}, A_3 \in A\{3\}$ then $\|A_1 + A_2 + A_3\| = \det[A_3]$

(iii) $\|A_1A_2A_3\| = \det[A_2]$

(iv) $A^T \in A\{3\}$ *for all A in* $M_n(AF)$

Proof. (i) $\|X\| = \left\|X^T\right\|$, since $\|X\| = \det[A]$ for all X in $A\{3\}$

(ii) $\|A_1\| > \|A\|, \|A_2\| < \|A\|, \|A_3\| = \|A\|$, therefore ,

$$\begin{aligned}\|A_1+A_2+A_3\| &= \det[A_1+A_2+A_3]\\ &= \det[A_1]+\det[A_2]+\det[A_3]\\ &= \det[A_3]\\ &= A_3\end{aligned}$$

(iii)

$$\begin{aligned}\|A_1A_2A_3\| &= \det[A_1A_2A_3]\\ &= \det[A_1]+\det[A_2]+\det[A_3]\\ &= \det[A_2]\\ &= A_2\end{aligned}$$

(iv) $\|A\| = \left\|A^T\right\|$ or $\det[A] = \det\left[A^T\right]$
Therefore for all A in $M_n(AF), A^T \in A\{3\}$. □

Example 2.11. $\|A\| = \left\|A^T\right\|$ *for all X in* $A\{i\}$ *where* $i = 1,2,3$.
Case (i):

$$A\{1\}X = \begin{bmatrix} 0.5 & 0.4 & 0.7\\ 0.7 & 0.5 & 0.4\\ 0.4 & 0.7 & 0.5 \end{bmatrix}, \quad \|X\| = 0.7$$

$$X^T = \begin{bmatrix} 0.5 & 0.7 & 0.4\\ 0.4 & 0.5 & 0.7\\ 0.7 & 0.4 & 0.5 \end{bmatrix}, \quad X^T\| = 0.7$$

Therefore $\|X\| = \left\|X^T\right\|$.
Case (ii):

$$A\ \{2\}, X = \begin{bmatrix} 0.2 & 0.4 & 0.5\\ 0.5 & 0.2 & 0.4\\ 0.4 & 0.5 & 0.2 \end{bmatrix}, \quad \|X\| = 0.5$$

$$X^T = \begin{bmatrix} 0.2 & 0.5 & 0.4\\ 0.4 & 0.2 & 0.5\\ 0.5 & 0.4 & 0.2 \end{bmatrix}, \quad \left\|X^T\right\| = 0.5$$

Therefore, $\|X\| = \left\|X^T\right\|$.
Case (iii):

$$A\{3\}, X = \begin{bmatrix} 0.3 & 0.6 & 0.5\\ 0.5 & 0.3 & 0.6\\ 0.6 & 0.5 & 0.3 \end{bmatrix}, \quad \|X\| = 0.6$$

$$X = \begin{bmatrix} 0.2 & 0.5 & 0.4\\ 0.4 & 0.2 & 0.5\\ 0.5 & 0.4 & 0.2 \end{bmatrix} \quad, \left\|X^T\right\| = 0.6$$

Therefore, $\|X\| = \left\|X^T\right\|$.

Example 2.12. *Let*

$$A = \begin{bmatrix} 0.3 & 0.7 & 0.1\\ 0.1 & 0.3 & 0.7\\ 0.7 & 0.1 & 0.3 \end{bmatrix}, A_1 = \begin{bmatrix} 0.3 & 0.1 & 0.5\\ 0.5 & 0.3 & 0.1\\ 0.1 & 0.5 & 0.3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.2 & 0.1 & 0.3\\ 0.3 & 0.2 & 0.1\\ 0.1 & 0.3 & 0.2 \end{bmatrix}, A_3 = \begin{bmatrix} 0.5 & 0.3 & 0.9\\ 0.9 & 0.5 & 0.3\\ 0.3 & 0.9 & 0.5 \end{bmatrix}$$

$$\begin{aligned}\|A\| &= 0.3[0.3+0.1]+0.7[0.1+0.7]+0.1[0.1+0.3]\\ &= 0.3+0.7+0.1\\ &= 0.7\\ \|A_1\| &= 0.3[0.3+0.1]+0.1[0.3+0.1]+0.5[0.5+0.1]\\ &= 0.3+0.1+0.5\\ &= 0.5\\ \|A_2\| &= 0.2[0.2+0.1]+0.1[0.2+0.1]+0.3[0.3+0.1]\\ &= 0.2+0.1+0.3\\ &= 0.3\\ \|A_3\| &= 0.5[0.5+0.3]+0.3[0.5+0.3]+0.9[0.9+0.3]\\ &= 0.5+0.3+0.9\\ &= 0.9\end{aligned}$$

$$A_1+A_2+A_3 = \begin{bmatrix} 0.5 & 0.3 & 0.9\\ 0.9 & 0.5 & 0.3\\ 0.3 & 0.9 & 0.5 \end{bmatrix}$$

$$\begin{aligned}\|A_1+A_2+A_3\| &= 0.5[0.5+0.3]+0.3[0.5+0.3]\\ &\quad +0.9[0.9+0.3]\\ &= 0.5+0.3+0.9\\ &= 0.9\\ \|A_1+A_2+A_3\| &= \det[A_1]+\det[A_2]+\det[A_3]\\ &= 0.5+0.3+0.9\\ &= 0.9 = \|A_3\|\end{aligned}$$

$$A_1\ A_2\ A_3 = \begin{bmatrix} 0.2 & 0.1 & 0.3\\ 0.3 & 0.2 & 0.1\\ 0.1 & 0.3 & 0.2 \end{bmatrix}$$

$$\begin{aligned}\|A_1\ A_2\ A_3\| &= 0.2[0.2+0.1]+0.1[0.2+0.1]\\ &\quad +0.3[0.3+0.1]\\ &= 0.2+0.1+0.3\\ &= 0.3\\ \|A_1\ A_2\ A_3\| &= \det[A_1]\det[A_2]\det[A_3]\\ &= (0.5)(0.3)(0.9)\\ &= 0.3 = \|A_2\|\end{aligned}$$

Therefore $\|A\| = \left\|A^T\right\| = 0.7$ *or* $det[A] = \det\left[A^T\right]$ *for all A in* $M_n(AF)$, $A^T \in A\{3\}$.

Theorem 2.13. *(i) For all* $X \in A\{4\}$, $\|A\| \leq \|X\|$

(ii) For all $X \in A\{5\}$, $\|X\| \leq \|A$

For all X in $A\{4\} \cap A\{5\}$ *the matrices AX and XA are idempotent.*

Proof. If $X \in A\{4\}$, then $AXA = $ A therefore $\|AXA\| = \det[A]$ which implies

$$\det[A]\det[X]\det[A] = \det[A] = \|A\|$$
$$\|A\| \leq \|X\|$$

(i) If $X \in A\{5\}$, then $XAX = X$ therefore $\|xAX\| = \det[X]$, which implies

$$\det[X]\det[A]\det[X] = \det[X] = \|X\|$$
$$\|X\| \leq \|A\|$$

(ii) If X in $A\{4\} \cap A\{5\}$ then

$$AXA = A \quad (2.1)$$
$$XAX = X \quad (2.2)$$

$XAXA = XA \Rightarrow (XA)^2 = XA$ from equation (2.1)
$AXAX = AX \Rightarrow (AX)^2 = AX$ from equation (2.2)

That is AX and XA are idempotent □

Example 2.14. *(i)* *If $X \in A\{4\}$ then $\|AXA\| = \|A\| \leq \|X\|$*

$$A = \begin{bmatrix} 0.6 & 0.5 & 0.3 \\ 0.3 & 0.6 & 0.5 \\ 0.5 & 0.3 & 0.6 \end{bmatrix} \textit{ and } X = \begin{bmatrix} 0.8 & 0.5 & 0.3 \\ 0.3 & 0.8 & 0.5 \\ 0.5 & 0.3 & 0.8 \end{bmatrix}$$

$$\|AXA\| = \begin{bmatrix} 0.6 & 0.5 & 0.3 \\ 0.3 & 0.6 & 0.5 \\ 0.5 & 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} 0.8 & 0.5 & 0.3 \\ 0.3 & 0.8 & 0.5 \\ 0.5 & 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 0.6 & 0.5 & 0.3 \\ 0.3 & 0.6 & 0.5 \\ 0.5 & 0.3 & 0.6 \end{bmatrix}$$
$$=0.6$$
$$\|A\| \leq \|X\|$$

If $X \in A\{5\}$ then $\|XAX\| = \|X\| \leq \|A\|$

$$A = \begin{bmatrix} 0.3 & 0.7 & 0.5 \\ 0.5 & 0.3 & 0.7 \\ 0.7 & 0.5 & 0.3 \end{bmatrix} \textit{ and } X = \begin{bmatrix} 0.2 & 0.5 & 0.1 \\ 0.1 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.2 \end{bmatrix}$$

$$\|XAX\| = \begin{bmatrix} 0.2 & 0.5 & 0.1 \\ 0.1 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 & 0.5 \\ 0.5 & 0.3 & 0.7 \\ 0.7 & 0.5 & 0.3 \end{bmatrix} \begin{bmatrix} 0.2 & 0.5 & 0.1 \\ 0.1 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.2 \end{bmatrix}$$
$$=0.5$$
$$\|X\| \leq \|A$$

(ii) *If $X \in A\{5\}$ and $XAXA = XA \Rightarrow (XA)^2 = XA$*

$$XAXA = \begin{bmatrix} 0.2 & 0.5 & 0.1 \\ 0.1 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 & 0.5 \\ 0.5 & 0.3 & 0.7 \\ 0.7 & 0.5 & 0.3 \end{bmatrix} \begin{bmatrix} 0.2 & 0.5 & 0.1 \\ 0.1 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 & 0.5 \\ 0.5 & 0.3 & 0.7 \\ 0.7 & 0.5 & 0.3 \end{bmatrix}$$
$$= 0.5$$

$$XA = \begin{bmatrix} 0.2 & 0.5 & 0.1 \\ 0.1 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 & 0.5 \\ 0.5 & 0.3 & 0.7 \\ 0.7 & 0.5 & 0.3 \end{bmatrix}$$
$$= 0.5$$

Therefore, $(XA)^2 = XA = 0.5$
If $X \in A\{4\}$ and $AXAX = AX \Rightarrow (AX)^2 = AX$

$$AXAX = \begin{bmatrix} 0.6 & 0.5 & 0.3 \\ 0.3 & 0.6 & 0.5 \\ 0.5 & 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} 0.8 & 0.5 & 0.3 \\ 0.3 & 0.8 & 0.5 \\ 0.5 & 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 0.6 & 0.5 & 0.3 \\ 0.3 & 0.6 & 0.5 \\ 0.5 & 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} 0.8 & 0.5 & 0.3 \\ 0.3 & 0.8 & 0.5 \\ 0.5 & 0.3 & 0.8 \end{bmatrix}$$
$$=0.6$$

$$AX = \begin{bmatrix} 0.6 & 0.5 & 0.3 \\ 0.3 & 0.6 & 0.5 \\ 0.5 & 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} 0.8 & 0.5 & 0.3 \\ 0.3 & 0.8 & 0.5 \\ 0.5 & 0.3 & 0.8 \end{bmatrix}$$
$$=0.6$$

Therefore, $(AX)^2 = AX = 0.6$.
Therefore XA and AX are idempotent.

3. Properties of Det-Norm ordering with Fuzzy matrices

Definition 3.1. *The det-norm ordering $A \leq B$ in $M_n(F)$ is defined as $A \leq B <=> \|A\| \leq \|X\|$ or $A \leq B <=> \det[A] \leq \det[B]$.*

Example 3.2. $A = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.1 & 0.5 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0.9 & 0.6 & 0.3 \\ 0.3 & 0.9 & 0.6 \\ 0.6 & 0.3 & 0.9 \end{bmatrix}$, *$\|A\| = 0.5$ and $\|B\| = 0.9$. Therefore $A \leq B <= 2\| A\| \leq \|B$.*

Theorem 3.3. *The det-ordering is not a partial ordering.*

Proof. (i) $\mathrm{Det}[A] \leq \det[B]$ for all $A \in M_n(F)$ hence $A \leq B$. Therefore reflexivity is true.

(ii) $A \leq B \Rightarrow \|A\| \leq \|B\|, B \leq C \Rightarrow \|B\| \leq \|C\|$ which implies $\|A\| = \|B$. But $\|A\| = \|B\|$ does not imply $A = B$. Therefore anti-symmetry is not true.

(iii) $A \leq B, B \leq C \Rightarrow A \leq C$ for all $A,B,C \in M_n(F)$, for

$$A \leq B = \|A\| \leq \|B\|$$
$$B \leq C = \|B\| \leq \|C\|$$
$$A \leq C = \|A\| \leq \|C$$

Therefore transitivity condition satisfied. Then the det-ordering is not a partial ordering in $M_n(F)$. □

Example 3.4. $A = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.1 & 0.5 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}, B = \begin{bmatrix} 0.7 & 0.5 & 0.2 \\ 0.2 & 0.7 & 0.5 \\ 0.5 & 0.2 & 0.7 \end{bmatrix}$ *and* $C = \begin{bmatrix} 0.2 & 0.8 & 0.6 \\ 0.6 & 0.2 & 0.8 \\ 0.8 & 0.6 & 0.2 \end{bmatrix}$. *we get* $\|A\| = 0.5, \|B\| = 0.7$ *and* $\|C\| = 0.8$

(i) $\|A\| \leq \|A\|$ *for all A in* $M_n(F)$, $A \leq A$. *Therefore reflexivity condition satisfied.*

(ii) $A \leq B = \|A\| \leq \|B\| = 0.5 < 0.7,\ B \leq A = \|B\| \leq \|A\| = 0.7 \neq 0.5$. $A \leq B$ *and* $B \leq A$ *which implies* $\|A\| = \|B\|$. *But* $\|A\| = \|B\|$ *does not imply* $A = B$. *Therefore anti-symmetry is not true.*

(iii) $A \leq B = \|A\| \leq \|B\| = 0.5 \leq 0.7,\ B \leq C = \|B\| \leq \|C\| = 0.7 \leq 0.8$. $A \leq C$ *and* $\|B\| \leq \|C\| = 0.5 \leq 0.8$.

Therefore transitivity is true. Thus the det-ordering is not a partial ordering in $M_n(F)$

Theorem 3.5. *If* $A \leq B$ *then*

(i) $A^T \leq B^T$

(ii) $AA^T \leq BB^T; B^TA \leq B^TB$

(iii) $A^TA \leq B^TB, AA^T \leq BB^T, A^n \leq B^n$ *for any positive integer n.*

Proof. (i)

$$\|A\| = \det\left[A^T\right], \|B\| = \det\left[B^T\right]$$
$$\|A\| \leq \det\left[A^T\right] \Rightarrow \|B\| \leq \det\left[B^T\right]$$
$$A \leq B \Rightarrow A^T \leq B^T$$

(ii)

$$\det\left[AB^T\right] \leq \det[A]\det\left[B^T\right] = \det[A]\det[B]^T$$
$$= \det[A] \text{ since } A \leq B$$
$$det[BB^T] = det[B^T]det[B]^T$$
$$= det[B]det[B]$$
$$= det[B]$$
$$A \leq B \Rightarrow \det[A] \leq \det[B]$$
$$\Rightarrow \det\left[AB^T\right] \leq \det\left[BB^T\right]$$

Similarly $A \leq B \Rightarrow B^TA \leq B^TB$.

(iii)

$$\det\left[A^TA\right] \leq \det\left[A^T\right]\det[A] = \det[A]\det[A] = \det[A]$$
$$\det\left[B^TB\right] \leq \det\left[B^T\right]\det[B] = \det[B]\det[B] = \det[B]$$
$$A \leq B \Rightarrow \det[A] \leq \det[B] \Rightarrow \det\left[A^TA\right] \leq \det\left[B^TB\right]$$
$$\Rightarrow A^TA \leq B^TB$$

Similarly $A \leq B \Rightarrow B^TA \leq B^TB$.

(iv)

$$\det\left[A^n\right] = \det[AA\ldots n \text{ times }] = \det[A]$$
$$\det[A] n \text{ times } = \det[A]$$
$$\det\left[B^n\right] = \det[B\ldots\ldots n \text{ times }] = \det[B]$$
$$\det[B] n \text{ times } = \det[B]$$
$$A \leq B \Rightarrow \det[A] \leq \det[B] \Rightarrow \det\left[A^n\right] \leq \det\left[B^n\right]$$
$$A^n \leq B^n \text{ for any positive integer } n.$$

□

Example 3.6. $A = \begin{bmatrix} 0.3 & 0.5 & 0.7 \\ 0.7 & 0.3 & 0.5 \\ 0.5 & 0.7 & 0.3 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0.1 & 0.9 & 0.4 \\ 0.4 & 0.1 & 0.9 \\ 0.9 & 0.4 & 0.1 \end{bmatrix}$, $\|A\| = 0.7$ *and* $\|B\| = 0.9$

$$A^T = \begin{bmatrix} 0.3 & 0.7 & 0.5 \\ 0.5 & 0.3 & 0.7 \\ 0.7 & 0.5 & 0.3 \end{bmatrix} = \left\|A^T\right\| = 0.7$$

$$B^T = \begin{bmatrix} 0.1 & 0.4 & 0.9 \\ 0.9 & 0.1 & 0.4 \\ 0.4 & 0.9 & 0.1 \end{bmatrix} \quad \|B\| = 0.9\ s$$

Now, $\|A\| = \det\left[A^T\right] = 0.7, \|B\| = \det\left[B^T\right] = 0.9$.
Therefore $\|A\| \leq \|B\| \Rightarrow \det\left[A^T\right] \leq \det\left[B^T\right] \Rightarrow 0.7 \leq 0.9$.
i.e, $A \leq B \Rightarrow A^T \leq B^T$

$$\left\|AB^T\right\| = \begin{bmatrix} 0.3 & 0.5 & 0.7 \\ 0.7 & 0.3 & 0.5 \\ 0.5 & 0.7 & 0.3 \end{bmatrix}\begin{bmatrix} 0.1 & 0.4 & 0.9 \\ 0.9 & 0.1 & 0.4 \\ 0.4 & 0.9 & 0.1 \end{bmatrix}$$
$$= \begin{bmatrix} 0.5 & 0.7 & 0.4 \\ 0.4 & 0.5 & 0.7 \\ 0.7 & 0.4 & 0.5 \end{bmatrix}$$
$$\left\|AB^T\right\| = 0.7$$

$$\left\|A^T\right\|\left\|B^T\right\| = [0.7][0.9] = 0.7$$
$$\left\|BB^T\right\| = \begin{bmatrix} 0.1 & 0.9 & 0.4 \\ 0.4 & 0.1 & 0.9 \\ 0.9 & 0.4 & 0.1 \end{bmatrix}\begin{bmatrix} 0.1 & 0.4 & 0.9 \\ 0.9 & 0.1 & 0.4 \\ 0.4 & 0.9 & 0.1 \end{bmatrix}$$
$$= \begin{bmatrix} 0.9 & 0.4 & 0.4 \\ 0.4 & 0.9 & 0.4 \\ 0.4 & 0.4 & 0.9 \end{bmatrix}$$
$$\left\|BB^T\right\| = 0.9$$

$$\|A^TA\| = \begin{bmatrix} 0.3 & 0.7 & 0.5 \\ 0.5 & 0.3 & 0.7 \\ 0.7 & 0.5 & 0.3 \end{bmatrix} \begin{bmatrix} 0.3 & 0.5 & 0.7 \\ 0.7 & 0.3 & 0.5 \\ 0.5 & 0.7 & 0.3 \end{bmatrix}$$
$$= \begin{bmatrix} 0.7 & 0.5 & 0.5 \\ 0.5 & 0.7 & 0.5 \\ 0.5 & 0.5 & 0.7 \end{bmatrix}$$
$$\|A^TA\| = 0.7$$
$$\| A^T\| \, \|A\| = [0.7][0.7] = 0.7$$
$$\| B^TB\| = \begin{bmatrix} 0.1 & 0.4 & 0.9 \\ 0.9 & 0.1 & 0.4 \\ 0.4 & 0.9 & 0.1 \end{bmatrix} \begin{bmatrix} 0.1 & 0.9 & 0.4 \\ 0.4 & 0.1 & 0.9 \\ 0.9 & 0.4 & 0.1 \end{bmatrix}$$
$$= \begin{bmatrix} 0.9 & 0.4 & 0.4 \\ 0.4 & 0.9 & 0.4 \\ 0.4 & 0.4 & 0.9 \end{bmatrix}$$
$$\|B^TB\| = 0.9$$
$$A \leq B \Rightarrow \|A\| \leq \|B\| = \|A^TA\| \leq \|B^TB\| = 0.4 \leq 0.6$$
$$\Rightarrow A^TA \leq B^TB$$
$$\|AA^T\| = \begin{bmatrix} 0.3 & 0.5 & 0.7 \\ 0.7 & 0.3 & 0.5 \\ 0.5 & 0.7 & 0.3 \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 & 0.5 \\ 0.5 & 0.3 & 0.7 \\ 0.7 & 0.5 & 0.3 \end{bmatrix}$$
$$= \begin{bmatrix} 0.7 & 0.5 & 0.5 \\ 0.5 & 0.7 & 0.5 \\ 0.5 & 0.5 & 0.7 \end{bmatrix} = 0.7$$
$$\| A\| \, \|A^T\| = [0.7][0.7] = 0.7$$

Therefore $A \leq B \Rightarrow AA^T \leq BB^T$.

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