On strict strong coloring of central graphs

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Abstract
A strict strong coloring of a graph $G$ is a proper coloring of $G$ in which every vertex of the graph is adjacent to every vertex of some color class. The minimum number of colors required for a strict strong coloring of $G$ is called the strict strong chromatic number of $G$ and is denoted by $\chi_{ss}(G)$. In this paper we discuss some results on strict strong chromatic number of central graphs.

Keywords
Proper coloring, strict strong coloring, strict strong chromatic number, central graphs.

AMS Subject Classification
05C15, 05C69.

1 Introduction
All graphs considered here are simple. For graph theoretic terminology we refer to [4]. Let $G = (V,E)$ be a graph. The degree of a vertex $v \in V$ in a graph $G$ is defined to be the number of edges incident with $v$ and is denoted by $\deg(v)$. A vertex of degree zero in $G$ is an isolated vertex and a vertex of degree one is a pendant vertex or a leaf. Any vertex which is adjacent to a pendant vertex is called a support vertex. The open neighborhood and closed neighborhood of $v$ is $N(v) = \{ u \in V : uv \in E \}$ and $N[v] = N(v) \cup \{v\}$ respectively. A subset $S$ of $V$ is called a dominating set (total dominating set) of $G$ if every vertex in $V - S$ (every vertex in $V$) is adjacent to a vertex in $S$. The domination number $\gamma$ (total domination number $\gamma_t$) is the minimum cardinality of a dominating set (total dominating set) in $G$. A dominating set $S$ of cardinality $\gamma(G)$ is called a $\gamma$-set.

A proper vertex coloring of $G$ is an assignment of colors to the vertices of $G$ in such a way that adjacent vertices are assigned distinct colors. The chromatic number $\chi(G)$ of a graph $G$ is the minimum number of colors required for a proper coloring of $G$. The concept of strong coloring was introduced by I.E. Zverovich [19]. It defines a dominance relation between graph vertices and color classes. A strict strong coloring of $G$ is a proper coloring of $G$ in which every vertex of the graph is adjacent to every vertex of some color class, that is, each vertex totally dominates every vertex of some other color class. The minimum number of colors required for a strict strong coloring of $G$ is called the strict strong chromatic number of $G$ and is denoted by $\chi_{ss}(G)$. Some basic results on strict strong colorings are given in [2, 10, 13, 17]. Given a graph $G$, we subdivide each edge of $G$ exactly once and join all the non-adjacent vertices of $G$. The graph obtained by this process is called central graph of $G$ denoted by $C(G)$. In this paper, we prove the results on strict strong chromatic number of central graphs on path, cycle, complete graph and complete bipartite graph.

2 Results
In this section, we prove the exact values for path, cycle, complete graph and complete bipartite graph.

Theorem 2.1. For a path $P_n$, 
$$\chi_{ss}(C(P_n)) = \left\{ \begin{array}{ll} \left\lfloor \frac{2n-1}{3} \right\rfloor + 1, & n \equiv 2 \pmod{3} \\ \left\lfloor \frac{2n-1}{3} \right\rfloor + 2, & \text{otherwise} \end{array} \right.$$ 

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of a path $P_n$ and let $c_{i,j}$ be the vertex which divides an edge $v_i v_j$. 

Case 2.2. $n \equiv 0 \pmod{3}$ 
Consider a coloring $\mathcal{C} = \{\{v_i\} : i \equiv 0 \pmod{3}\} \cup \{v_{n-1}\} \cup \{v_n\} \cup \{v_i \cup v_{i+1} : i = 1, 4, 7, \ldots, n-2\} \cup \{c_{i,j} : 1 \leq i \leq n-1 \text{ and } j = i+1\}$ of $C(P_n)$. Clearly the vertices $c_{i,j}$, where $i = 1, 4, 7, \ldots, n-4$ and $j = i+1$, totally
dominates the color class \(\{v_i \cup v_{i+1}\}\) and rest of the vertices totally dominates some color class \(\{v_i\} : i \equiv 0 \pmod{3}\). Hence \(\chi_{ss}(C(C_n)) = \left\lfloor \frac{2n-1}{3} \right\rfloor + 1\).

Case 2.5. \(n \equiv 1 \pmod{3}\)

Consider a coloring \(C = \{v_i\} : i \equiv 0 \pmod{3}\) \(\bigcup \{v_i \cup v_{i+1}\} : i = 1, 4, 7, \ldots, n-3\)
\(\bigcup \{c_{i,j} : 1 \leq i \leq n-1 \text{ and } j = i+1\}\) of \(C(C_n)\). Clearly the vertices \(c_{i,j}\), where \(i = 1, 4, 7, \ldots, n-3\) and \(j = i+1\), totally dominates the color class \(\{v_i \cup v_{i+1}\}\) and rest of the vertices totally dominates some color class \(\{v_i\} : i \equiv 0 \pmod{3}\). Hence \(\chi_{ss}(C(C_n)) = \left\lfloor \frac{2n-1}{3} \right\rfloor + 1\).

Case 2.6. \(n \equiv 2 \pmod{3}\)

Consider a coloring \(C = \{v_i\} : i \equiv 0 \pmod{3}\) \(\bigcup \{v_i \cup v_{i+1}\} : i = 1, 4, 7, \ldots, n-1\)
\(\bigcup \{c_{i,j} : 1 \leq i \leq n-1 \text{ and } j = i+1\}\) of \(C(C_n)\). Clearly the vertices \(c_{i,j}\), where \(i = 1, 4, 7, \ldots, n-4\) and \(j = i+1\), totally dominates the color class \(\{v_i \cup v_{i+1}\}\) and rest of the vertices totally dominates some color class \(\{v_i\} : i \equiv 0 \pmod{3}\). Hence \(\chi_{ss}(C(C_n)) = \left\lfloor \frac{2n-1}{3} \right\rfloor + 1\).

Theorem 2.5. For a cycle \(C_n\),
\[\chi_{ss}(C(C_n)) = \begin{cases} \left\lfloor \frac{n}{3} \right\rfloor + 1, & \text{if } n \equiv 0 \pmod{3} \\ \left\lfloor \frac{n}{3} \right\rfloor + 2, & \text{otherwise} \end{cases}\]

Proof. Let \(v_1, v_2, \ldots, v_n\) be the vertices of a cycle \(C_n\), and let \(c_{i,j}\) be the vertex which divides an edge \(v_i v_j\).

Case 2.7. \(n \equiv 1 \pmod{3}\)

Consider a coloring \(C = \{v_i\} : i \equiv 0 \pmod{3}\) \(\bigcup \{v_i \cup v_{i+1}\} : i = 1, 4, 7, \ldots, n-3\)
\(\bigcup \{c_{i,j} : 1 \leq i \leq n-1 \text{ and } j = i+1\}\) of \(C(C_n)\). Clearly the vertices \(c_{i,j}\), where \(i = 1, 4, 7, \ldots, n-3\) and \(j = i+1\), totally dominates the color class \(\{v_i \cup v_{i+1}\}\) and rest of the vertices totally dominates some color class \(\{v_i\} : i \equiv 0 \pmod{3}\). Hence \(\chi_{ss}(C(C_n)) = \left\lfloor \frac{n}{3} \right\rfloor + 2\).

Case 2.8. \(n \equiv 2 \pmod{3}\)

Consider a coloring \(C = \{v_i\} : i \equiv 0 \pmod{3}\) \(\bigcup \{v_i \cup v_{i+1}\} : i = 1, 4, 7, \ldots, n-4\)
\(\bigcup \{c_{i,j} : 1 \leq i \leq n-1 \text{ and } j = i+1\}\) of \(C(C_n)\). Clearly the vertices \(c_{i,j}\), where \(i = 1, 4, 7, \ldots, n-4\) and \(j = i+1\), totally dominates the color class \(\{v_i \cup v_{i+1}\}\) and rest of the vertices totally dominates some color class \(\{v_i\} : i \equiv 0 \pmod{3}\). Hence \(\chi_{ss}(C(C_n)) = \left\lfloor \frac{n}{3} \right\rfloor + 2\).

Theorem 2.9. For a complete graph \(K_n\), \(\chi_{ss}(C(K_n)) = 2(n-1)\).

Proof. Let \(v_1, v_2, \ldots, v_n\) be the vertices of a cycle \(C_n\) and let \(c_{i,j}\) be the vertex which divides an edge \(v_i v_j\). Consider a coloring \(C = \{v_i\} : 1 \leq i \leq n \} \bigcup \{v_{n-1} \cup v_n\}
\(\bigcup \{c_{i,j} : 1 \leq i \leq n-2 \text{ and } j = i+1\}\) of \(C(K_n)\). Each vertex \(v_i, 1 \leq i \leq n-2\), totally dominates the color class \(\{c_{i,j} : 1 \leq i \leq n-2 \text{ and } j = i+1\}\) and vice versa. Further the vertices \(v_{n-1}\) and \(v_n\) totally dominates the color class \(\{v_{n-1} \cup v_n\}\) and the vertex \(c_{n-1,n}\) totally dominates the color class \(\{v_{n-1} \cup v_n\}\). Hence \(\chi_{ss}(C(K_n)) = 2(n-1)\)
Theorem 2.10. For a complete bipartite graph $K_{m,n}$,
\[
\chi_{ss}(C(K_{m,n})) = \begin{cases} 
m+n, & \text{if } m \neq 2\text{ and } n \neq 2 \\
m+n+1, & \text{otherwise} 
\end{cases}
\]

Proof. Let $v_{n1}, v_{n2}, \ldots, v_{nk}, v_{m1}, v_{m2}, \ldots, v_{mk}, k \geq 1$ be the vertices of $K_{m,n}$ and let $c_{i,j}$ be the vertex which divides the edge $v_{mi}v_{nj}, 1 \leq i, j \leq k$.

Case 2.11. $m = n = 1$.

In this case it is easy to observe from figure 5 that $\chi_{ss}(C(K_{1,1})) = 2 = m+n$.

Figure 3. Strict strong chromatic number of central graph on complete graph $K_5$ is 8.

Figure 4. Strict strong chromatic number of central graph on complete bipartite graph $K_{1,1}$ is 2.

Case 2.12. $m = 2$ or $n = 2$.

Consider a coloring $\mathcal{C} = \{v_{mi} \cup v_{nj}\} \cup \{v_{mi}\} : 2 \leq i \leq k\}$
$\cup \{v_{nj} : 2 \leq j \leq k\} \cup \{c_{i,j} : i = m1, m2, n1 \leq j \leq n(k-1)\}\$
$\cup \{c_{m2, n1}\}$. Clearly each vertex $v_{mi}, v_{nj}$ and $c_{i,j}, i \neq m1$ and $j \neq n1$, totally dominates some color class $\{v_{mi}\} : 2 \leq i \leq k\}$. Further the vertex $c_{m1,n1}$ totally dominates the color class $\{v_{mi} \cup v_{nj}\}$ and the vertex $v_{m2}$ totally dominate the color class $\{c_{m2, n1}\}$. Hence $\chi_{ss}(C(K_{2,2})) = m+n+1$. Similarly, we can prove that $\chi_{ss}(C(K_{m,2})) = m+n+1$.

Case 2.13. $m \geq 3$ and $n \geq 3$.

Consider a coloring $\mathcal{C} = \{v_{mi} \cup v_{nj}\} \cup \{v_{mi}\} : 2 \leq i \leq k\}$
$\cup \{v_{nj} : 2 \leq j \leq k\} \cup \{c_{i,j} : m1 \leq i \leq mk, n1 \leq j \leq nk\}$. Clearly each vertex $v_{mi}, v_{nj}$ and $c_{i,j}$, where $i \neq m1$ and $j \neq n1$ totally dominates some color class $\{v_{mi}\} : 2 \leq i \leq k\}$ or some color class $\{v_{nj}\} : 2 \leq j \leq k\}$. Further the vertex $c_{m1,n1}$ totally dominate the color class $\{v_{mi} \cup v_{nj}\}$. Hence $\chi_{ss}(C(K_{m,n})) = m+n$.

References


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