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Fixed-point of (α,β,Z) -contraction mapping under simulation functions in Banach space

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Abstract

Some fixed point results are discussed for the setting of a Banach space through the definition of a new contraction condition via a Cyclic (α,β,Z)-admissible mapping which is embedded in simulation function.

Keywords

Fixed point, Cyclic (α,β,Z)-admissible mapping, Banach Space, Simulation Functions.

AMS Subject Classification

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Contents

1. Introduction

Fixed point theorems have a great contribution in the applications of mathematical analysis, especially in differential and integral equations. Banach fixed point theorems is one of the most familiar fixed point theorems and its contraction principle is used as the most important tool in fixed point theory. The area of fixed point theorem is very contractive to the researches belonged to Physics, Biology, Engineering, Economics and Game theory. So, this Banach's contraction was extended and the existence of fixed and common fixed point theorems were proved by the researchers.

Recently various fixed point theorems were introduced in the setting of metric spaces by the using the concept of cyclic (α,β) admissible mappings[1]. The concept of admissible mappings is used to demonstrate the new results in many spaces. Several fixed point theorems were proved by the introduction of a new class of mapping which is called simulation functions [6]. Many results were showed from there obtained results in the literature.

Now we will give some basic definitions and results in Banach spaces before presenting our main results.

2. Preliminaries

Definition 2.1. *A norm on a linear space s is a mapping $\|.\| : S \to R^+$ which satisfies for each $x,y \in S$; $\quad \lambda \in R$*

(i) $\|x\| = 0 \Leftrightarrow x = 0$,

(ii) $\|\lambda x\| = |\lambda|\|x\|$,

(iii) $\|x+y\| \le \|x\| + \|y\|$.

A linear space with a norm is called a normed linear space.

Definition 2.2. *A normed space $(X,\|.\|)$ is said to be complete if every Cauchy sequence in X is convergent. A complete normed space is called a Banach space.*

Definition 2.3 ([1])**.** *Let X be a nonempty set, f be a self-mapping on X and $\alpha,\beta : X \to [0,+\infty)$ be two mappings. We say that f is a cyclic (α,β) -admissible mapping if $x \in X$ with $\alpha(x) \ge 1 \Rightarrow \beta(fx) \ge 1$ and $x \in X$ with $\beta(x) \ge 1 \Rightarrow \alpha(fx) \ge 1$.*

Definition 2.4 ([6])**.** *A function $\zeta : [0,\infty) \times [0,\infty) \to R$ is called a simulation function if ζ satisfies the following conditions:*

(i) $\zeta(0,0)=0$

(ii) $\zeta(t,s)<s-t$ *for all* $t,s>0$

(iii) *If* $\{t_n\}$ *and* $\{s_n\}$ *are sequences in* $(0,\infty)$ *such that*

$$\lim_{n\to\infty} t_n=\lim_{n\to\infty} s_n=l\in(l,\infty)>0,$$

then $\lim_{n\to\infty}\operatorname{Sup}\zeta(t_n,s_n)<0$.

Definition 2.5 ([3])**.** *A simulation function is a function* $\zeta:[0,\infty)\times[0,\infty)\to R$ *that satisfies thefollowing conditions:*

(i) $\zeta(t,s)<s-t$ *for all* $t,s>0$

(ii) *If* $\{t_n\}$ *and* $\{s_n\}$ *are sequences in* $(0,\infty)$ *such that*

$$\lim_{n\to\infty} t_n=\lim_{n\to\infty} s_n=l\in(l,\infty)>0,$$

then $\lim_{n\to\infty}\operatorname{Sup}\zeta(t_n,s_n)<0$.

It is clear that any simulation function is the same according to the definitions 2.4&2.5.

3. Main result

Lemma 3.1. *Let* $T:X\to X$ *be a cyclic* (α,β)*-admissible mapping. Assume that there exist* $x_0,x_1\in X$ *such that* $\alpha(x_0)\geq 1\Rightarrow\beta(x_1)\geq 1$ *and* $\beta(x_0)\geq 1\Rightarrow\alpha(x_1)\geq 1$. *Define a sequence* $\{x_n\}$ *by* $x_{n+1}=Tx_n$. *Then* $\alpha(x_n)\geq 1\Rightarrow\beta(x_m)\geq 1$ *and* $\beta(x_n)\geq 1\Rightarrow\alpha(x_m)\geq 1$ *for all* $m,n\in N$ *with* $n<m$.

Proof. Since

$$\alpha(x_0)\geq 1\Rightarrow\beta(Tx_0)=\beta(x_1)\geq 1$$

and

$$\beta(x_0)\geq 1\Rightarrow\alpha(Tx_0)=\alpha(x_1)\geq 1.$$

By continuing the above process, we have

$$\alpha(x_n)\geq 1\Rightarrow\beta(Tx_n)=\beta(x_{n+1})\geq 1$$

and

$$\beta(x_n)\geq 1\Rightarrow\alpha(Tx_n)=\alpha(x_{n+1})\geq 1.$$

Since

$$\begin{aligned}\alpha(x_m)\geq 1&\Rightarrow\beta(Tx_m)=\beta(x_{m+1})\geq 1\\ \beta(x_m)\geq 1&\Rightarrow\alpha(Tx_m)=\alpha(x_{m+1})\geq 1\end{aligned}$$

for all $m,n\in N$ with $n<m$. Moreover since

$$\begin{aligned}\alpha(x_m)\geq 1&\Rightarrow\beta(x_{m+2})\geq 1\\ \beta(x_m)\geq 1&\Rightarrow\alpha(x_{m+2})\geq 1\end{aligned}$$

for all $m,n\in N$ with $n<m$. It is deduced that

$$\begin{aligned}\alpha(x_m)\geq 1&\Rightarrow\beta(x_{m+3})\geq 1\\ \beta(x_m)\geq 1&\Rightarrow\alpha(x_{m+3})\geq 1\end{aligned}$$

If the above process is continued, we have

$$\begin{aligned}\alpha(x_n)\geq 1&\Rightarrow\beta(x_m)\geq 1\\ \beta(x_n)\geq 1&\Rightarrow\alpha(x_m)\geq 1\end{aligned}$$

for all $m,n\in N$. ☐

Definition 3.2. *Let* $(X,\|\_\|)$ *be a Banach space,* $T:X\to X$ *be a mapping and* $\alpha,\beta:R\to[0,\infty)$ *be two functions. Then* T *is said to be a* (α,β,Z) *- contraction mapping if* T *satisfies the following conditions:*

(i) T *is a cyclic* (α,β) *-admissible.*

(ii) *There exists a simulation function* $\zeta\in Z$ *such that* $\alpha(x)\beta(y)\geq 1$

$\Rightarrow\zeta(\|Tx-Ty\|,L(x,y))\geq 0$ *holds for all* $x,y\in X$. *Where,*

$$L(x,y)=\max\left\{\|x-y\|,\|x-Tx\|,\|y-Ty\|,\frac{\|x-Tx\|\|y-Ty\|}{1+\|x-y\|},\frac{\|x-Tx\|\|y-Ty\|}{1+\|Tx-Ty\|}\right\}$$

Theorem 3.3. *Let* $(X,\|\cdot\|)$ *be a Banach space,* $T:X\to X$ *be a mapping and* $\alpha,\beta:X\times X\to[0,\infty)$ *be two functions. Suppose that the below conditions are followed:*

(i) T *is a* (α,β,Z) *- contraction mapping.*

(ii) *There exists anelement* $x_0\in X$ *such that* $\alpha(x_0)\geq 1$ *and* $\beta(x_0)\geq 1$

(iii) T *is continuous Then* T *has a fixed point* $u\in X$ *such that* $Tu=u$.

Then T *has a fixed point* $u\in X$ *such that* $Tu=u$.

Proof. Assume that there exists $x_0\in X$ such that $\alpha(x_0)\geq 1$ The proofis divided into the following steps:
Step 1. Define a sequence $\{x_n\}$ in X such that $x_{n+1}=Tx_n$ for all $n\in N\cup\{0\}$. If $x_n=x_{n+1}$ for all $n\in N\cup\{0\}$, then T has a fixed point and the proofis finished.

Hence, We assume that $x_n\neq x_{n+1}$ for all $n\in N\cup\{0\}$. That is, $\|x_n-x_{n+1}\|\neq 0$ for all $n\in N\cup\{0\}$. Since T is a cyclic $(\alpha,\beta)-$ admissible mapping, we have $\alpha(x_0)\geq 1\Rightarrow\beta(x_1)=\beta(Tx_0)\geq 1\Rightarrow\alpha(x_2)=\alpha(Tx_1)\geq 1$ and $\beta(x_0)\geq 1\Rightarrow\alpha(x_1)=\alpha(Tx_0)\geq 1\Rightarrow\beta(x_2)=\beta(Tx_1)\geq 1$, then the above process is continued, we have $\alpha(x_n)\geq 1\Rightarrow\beta(x_n)\geq 1$ for all $n\in N\cup\{0\}$. Thus $\alpha(x_n)\beta(x_{n+1})\geq 1$ for all $n\in N\cup\{0\}$. Therefore, we get $\zeta(\|Tx_n-Tx_{n+1}\|,L(x_n,x_{n+1}))\geq 0(3.2)$ for all $n\in N\cup\{0\}$, where

$$\begin{aligned}&L(x_n,x_{n+1})\\ &=\max\left\{\|x_n-x_{n+1}\|,\|x_n-Tx_n\|,\|x_{n+1}-Tx_{n+1}\|,\right.\\ &\left.\frac{\|x_n-Tx_n\|\|x_{n+1}-Tx_{n+1}\|}{1+\|x_n-x_{n+1}\|},\frac{\|x_n-Tx_n\|\|x_{n+1}-Tx_{n+1}\|}{1+\|Tx_n-Tx_{n+1}\|}\right\}\\ &=\max\left\{\|x_n-x_{n+1}\|,\|x_n-x_{n+1}\|,\|x_{n+1}-x_{n+2}\|,\right.\\ &\left.\frac{\|x_n-x_{n+1}\|\|x_{n+1}-x_{n+2}\|}{1+\|x_n-x_{n+1}\|},\frac{\|x_n-x_{n+1}\|\|x_{n+1}-x_{n+2}\|}{1+\|x_{n+1}-x_{n+2}\|}\right\}\\ &=\max\{\|x_n-x_{n+1}\|,\|x_{n+1}-x_{n+2}\|\}\end{aligned}$$

It follows that

$$\zeta(\|x_{n+1}-x_{n+2}\|,\max\{\|x_n-x_{n+1}\|_n\|x_{n+1}-x_{n+2}\|\})\geq 0$$

Condition (ii) of definition 2.4 $\Rightarrow$
$0 \leq \zeta\left(\|x_{n+1}-x_{n+2}\|, \max\left\{\|x_n-x_{n+1}\|, \|x_{n+1}-x_{n+2}\|\right\}\right)$
$< \max\left\{\|x_n-x_{n+1}\|, \|x_{n+1}-x_{n+2}\|\right\} - \|x_{n+1}-x_{n+2}\|$.
Thus, it is concluded that $\|x_{n+1}-x_{n+2}\| < \max\{\|x_n-x_{n+1}\|$ $= \|x_{n+1}-x_{n+2}\|\}$ for all $n \geq 1$. The last inequality

$$\Rightarrow \|x_{n+1}-x_{n+2}\| < \|x_n-x_{n+1}\|$$

for all $n \geq 1$. It follows that the sequence $\{\|x_n-x_{n+1}\|$ is non-increasing. Therefore, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \|x_n-x_{n+1}\| = r.$$

Note that if $r \neq 0$; that is $r > 0$, then by condition (ii) of definition 2.4, we have

$$0 \leq \lim_{n\to\infty} \operatorname{Sup} \zeta\left(\|x_n-x_{n+1}\|, \|x_{n+1}-x_{n+2}\|\right) < 0$$

Which is a contradiction. $\Rightarrow r = 0$. i.e

$$\lim_{n\to\infty} \|x_n-x_{n+1}\| = 0 \tag{3.1}$$

Step 2. To prove: $\{x_n\}$ is a Cauchy sequence.
Suppose to the contrary; that is $\{x_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and two subsequences $\left\{x_{m(k)}\right\}$ and $\left\{x_{n(k)}\right\}$ of $\{x_n\}$ with $m(k) > n(k) > k$ and $m(k)$ is the smallest index in N such that $\left\|x_{m(k)}-x_{n(k)}\right\| \geq \varepsilon$. *So* $\left\|x_{n(k)}-x_{m(k)-1}\right\|$ $< \varepsilon$ Triangular inequality $\Rightarrow$

$$\begin{aligned}
\in &\leq \left\|x_{m(k)}-x_{n(k)}\right\| \\
&\leq \left\|x_{n(k)}-x_{m(k)-1}\right\| + \left\|x_{m(k)-1}-x_{m(k)}\right\| \\
&< \in + \left\|x_{m(k)-1}-x_{m(k)}\right\|
\end{aligned}$$

Taking $k \to \infty$ in the above inequality and using (3.1), we get

$$\lim_{k\to\infty} \left\|x_{n(k)}-x_{m(k)}\right\| = \in \tag{3.2}$$

Again, by triangular inequality $\Rightarrow$

$$\begin{aligned}
&\left\|x_{n(k)-1}-x_{m(k)-1}\right\| \\
&\leq \left\|x_{n(k)-1}-x_{n(k)}\right\| + \left\|x_{n(k)}-x_{m(k)}\right\| + \left\|x_{m(k)}-x_{m(k)-1}\right\| \\
&\leq 2\left\|x_{n(k)}-x_{n(k)-1}\right\| + \left\|x_{m(k)-1}-x_{n(k)-1}\right\| + 2\left\|x_{m(k)-1}-x_{m(k)}\right\|
\end{aligned}$$

Taking $k \to \infty$ in the above inequality and using (3.1) and (3.2), we get

$$\lim_{k\to\infty} \left\|x_{n(k)}-x_{m(k)}\right\| = \lim_{k\to\infty} \left\|x_{n(k)-1}-x_{m(k)-1}\right\| = \in \tag{3.3}$$

Since $\alpha(x_0) > 1$ and $\beta(x_0) > 0$, by lemma 3.1, we conclude that $\alpha\left(x_{n(k)-1}\right)\beta\left(x_{m(k)-1}\right) \geq 1$. Since T is a $(\alpha,\beta,Z)-$ contraction, we have

$$\zeta\left(\left\|Tx_{n(k)-1}-Tx_{m(k)-1}\right\|, L\left(x_{n(k)-1}, x_{m(k)-1}\right)\right) \geq 0,$$

for all $x, y \in X$, where

$$\begin{aligned}
&L\left(x_{n(k)-1}, x_{m(k)-1}\right) \\
&= \max\left\{\begin{array}{c}
\left\|x_{n(k)-1}-x_{m(k)-1}\right\|, \left\|x_{n(k)-1}-Tx_{n(k)-1}\right\|, \\
\left\|x_{m(k)-1}-Tx_{m(k)-1}\right\|, \\
\frac{\left\|x_{n(k)-1}-Tx_{n(k)-1}\right\|\left\|x_{m(k)-1}-Tx_{m(k)-1}\right\|}{1+\left\|x_{n(k)-1}-x_{m(k)-1}\right\|}, \\
\frac{\left\|x_{n(k)-1}-Tx_{n(k)-1}\right\|\left\|x_{m(k)-1}-Tx_{m(k)-1}\right\|}{1+\left\|Tx_{n(k)-1}-Tx_{m(k)-1}\right\|}
\end{array}\right\} \\
&= \max\left\{\begin{array}{c}
\left\|x_{n(k)-1}-x_{m(k)-1}\right\|, \left\|x_{n(k)-1}-Tx_{n(k)-1}\right\|, \\
\left\|x_{m(k)-1}-Tx_{m(k)-1}\right\|, \\
\frac{\left\|x_{n(k)-1}-x_{n(k)}\right\|\left\|x_{m(k)-1}-x_{m(k)}\right\|}{1+\left\|x_{n(k)-1}-x_{m(k)-1}\right\|}, \\
\frac{\left\|x_{n(k)-1}-x_{n(k)}\right\|\left\|x_{m(k)-1}-x_{m(k)}\right\|}{1+\left\|x_{n(k)}-x_{m(k)}\right\|}
\end{array}\right\} \\
&= \max\left\{\left\|x_{n(k)-1}-x_{m(k)-1}\right\|, \left\|x_{n(k)-1}-x_{n(k)}\right\|\right\} \\
&= \max\left\{\|x_n-x_{n+1}\|, \|x_{n+1}-x_{n+2}\|\right\}
\end{aligned}$$

by (3.1) and (3.3), we conclude that

$$\lim_{n\to\infty} L\left(x_{n(k)-1}, x_{m(k)-1}\right) = \in \tag{3.4}$$

Note that condition (ii) of definition 2.4 $\Rightarrow$

$$0 \leq \lim \operatorname{Sup} \zeta\left(\left\|Tx_{n(k)-1}-Tx_{m(k)-1}\right\|, L\left(x_{n(k)-1}, x_{m(k)-1}\right)\right) < 0$$

Which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence.
Step 3. Finally, To prove: T has a fixed point.
Since $\{x_n\}$ is a Cauchy sequence in the Banach space X, then there exists $u \in X$ such that $x_n \to u$. The continuity of T implies that $Tx_{2n} \to Tu$. Since $x_{2n+1} \to Tx_{2n}$ and $x_{2n+1} \to u$ by uniqueness of limit, we get $Tu = u$. So u is a fixed point of T.

Note that the continuity of the mapping T in Theorem 3.3 can be dropped if the Condition (iii) is replaced by a suitable one as in the following result. ☐

Theorem 3.4. *Let $(X, \|\cdot\|)$ be a Banach space, $T : X \to X$ be a mapping and $\alpha, \beta : X \times X \to [0,\infty)$ be two functions. Suppose the below conditions are followed:*

(i) T is a (α,β,Z) -contraction.

(ii) There exists an element $x_0 \in X$ such that $\alpha(x_0) > 1$ and $\beta(x_0) > 1$

(iii) If $\{x_n\}$ is a sequence in X converges to $x \in X$ with $\alpha(x_n) \geq 1$ (or $\beta(x_n) \geq 1$) for all $n \in N$ then $\alpha(x) \geq 1$ (or $\beta(x) \geq 1$) for all $n \in N$

Then T has a fixed point.

Proof. Following the same steps as in the proof of Theorem 3.3, we construct a sequence $\{x_n\}$ in X by defining $x_{2n+1} = Tx_{2n}$ for all $n \in N$ such that $x_n \to u \in X, \alpha(x_n) \geq 1, \beta(x_n) \geq 1$ for all $n \in N$. By Condition (iii), we have $\alpha(u) \geq 1$ and $\beta(u) \geq 1$ for all $n \in N$. So $\alpha(u)\beta(u) \geq 1$.
Claim: $Tu = u$.

Suppose not, that is ,$Tu \neq u$.
Therefore

$$\|Tu-u\| \neq 0 \text{ and } \lim_{n\to+\infty} \|x_{n+1}-Tu\| \neq 0 \tag{3.5}$$

Since T is a $(\alpha,\beta,Z)-$ contraction mapping, we have

$$\zeta\left(\|Tx_n-Tu\|_, L(x_n,u)\right) = \zeta\left(\|x_{n+1}-Tu\|_, L(x_n,u)\right) \geq 0, \tag{3.6}$$

for all $n \in N$. Now,

$$\begin{aligned} L(x_n,u) &= \max\{\|x_n-u\|, \|x_n-Tx_n\|, \|u-Tu\|, \\ &\frac{\|x_n-Tx_n\|\,\|x_{n+1}-Tx_{n+1}\|}{1+\|x_n-x_{n+1}\|}, \frac{\|x_n-Tx_n\|\,\|x_{n+1}-Tx_{n+1}\|}{1+\|Tx_n-Tx_{n+1}\|}\Big\} \\ &= \max\{\|x_n-u\|, \|x_n-x_{n+1}\|, \|u-Tu\|, \\ &\frac{\|x_n-x_{n+1}\|\,\|x_{n+1}-x_{n+2}\|}{1+\|x_n-x_{n+1}\|}, \frac{\|x_n-x_{n+1}\|\,\|x_{n+1}-x_{n+2}\|}{1+\|x_{n+1}-x_{n+2}\|}\Big\} \end{aligned}$$

Letting $n\to\infty$ in the above equation, we obtain,

$$\lim_{n\to+\infty} L(x_n,u) = \|u-Tu\| \neq 0 \tag{3.7}$$

By using (3.5),(3.4) and (3.6), then condition (ii) of Definition 2.4

$$\Rightarrow 0 \leq \lim\mathrm{Sup}\,\zeta\left(\|Tx_n-Tu\|_z L(x_n,u)\right) < 0$$

Which is a contradiction. So $Tu=u$. Thus u is a fixed point of T. Now, an example is introduced to show that if T is satisfied all hypothesis of Theorem 3.3 or Theorem 3.4 then the fixed point of T is not necessary to be unique. □

Example

Example 3.5. *Let $X=[0,1]$. Define $d: X\times X\to R$ by $d(x,y)=|x-y|\cdot$ Also, define the mapping $T: X\times X$ by $Tx=x^2$. Define the function $\alpha,\beta: X\to R$ by*

$$\alpha(x)=\beta(x)=\begin{cases} 1, \text{ if } x=0 \\ 0, \text{ otherwise} \end{cases}.$$

Define $\zeta:[0,+\infty)\times[0,+\infty)\to R$ by $\zeta(t,s)=\frac{s}{s+1}-t$. Then we have the following:

1. *T is continuous and there exists $x_0\in X$ such that $\alpha(x)\geq 1$ and $\beta(x)\geq 1$*
2. *T is cyclic $(\alpha,\beta)-$ admissible mapping*
3. *For any $x,y\in X$ with $\alpha(x)\beta(y)\geq 1$ then*

$$\zeta(\|Tx-Ty\|,L(x,y))\geq 0,$$

where

$$L(x,y) = \max\left\{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \frac{\|x-Tx\|\|y-Ty\|}{1+\|x-y\|}, \frac{\|x-Tx\|\|y-Ty\|}{1+\|Tx-Ty\|}\right\}$$

4. *If $\{x_n\}$ is a sequence in X converges to $x\in X$ with $\alpha(x_n)\geq 1$ for all $n\in N$, then $\alpha(x)\geq 1$.*

Proof. The Proof of (1) is clear.
To prove (2), let $x\in X$, if $\alpha(x)\geq 1$ then $x=0$. So, $T(x)=T(0)=0$ and $\beta(Tx)=\beta(0)=1\geq 1$.
If $\beta(x)\geq 1$, then $x=0$. So $T(x)=T(0)=0$ and $\alpha(Tx)=\alpha(0)=1\geq 1$. So T is cyclic (α,β) -admissible mapping.
To prove (3), let $x,y\in X$, with $\alpha(x)\beta(y)\geq 1$. Then $x=y=0$. So $T(x)=T(y)=0$. Therefore, we have

$$\begin{aligned} L(x,y) &= \max\left\{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \frac{\|x-Tx\|\|y-Ty\|}{1+\|x-y\|}, \frac{\|x-Tx\|\|y-Ty\|}{1+\|Tx-Ty\|}\right\} \\ &= \max\left\{\|0-0\|, \|0-0\|, \|0-0\|, \frac{\|0-0\|\|0-0\|}{1+\|0-0\|}, \frac{\|0-0\|\|0-0\|}{1+\|0-0\|}\right\} \\ &= 0 \end{aligned}$$

So $\zeta(\|Tx-Ty\|,L(x,y))=\zeta(0,0)=\frac{0}{0+1}-0\geq 0$.
To prove (4), Let $\{x_n\}$ is a sequence in X such that $x_n\to x$ with $\beta(x_n)\geq 1$. Then $x_n=0$ for all $n\in N$, So $x=0$. Hence $\beta(x)=\beta(0)=1\geq 1$. Note that T satisfies all the conditions of Theorem 3.3 and Theorem 3.4. Here 0,1 are fixed point of T. So, the fixed point of T is not unique. □

Corollary 3.6. *Let $(X,\|\cdot\|)$ be a Banach space, $T: X\to X$ be a mapping and $\alpha: X\times X\to[0,+\infty)$ be a function. Suppose that below conditions are followed:*

(i) There exists $\zeta\in Z$ such that, if $x,y\in X$ with $\alpha(x)\beta(y)\geq 1$ then $\zeta(\|Tx-Ty\|_z L(x,y))\geq 0$, where

$$L(x,y) = \max\left\{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \frac{\|x-Tx\|\|y-Ty\|}{1+\|x-y\|}, \frac{\|x-Tx\|\|y-Ty\|}{1+\|Tx-Ty\|}\right\}$$

(ii) If $x\in X$ with $\alpha(x)\geq 1$ then $\alpha(Tx)\geq 1$

(iii) There exists $x_0\in X$ such that $\alpha(x_0)\geq 1$

(iv) T is continuous Then T has a fixed point.

Proof. It follows from Theorem 3.3 by taking the function $\beta: X\times X\to[0,+\infty)$ to be α. □

Corollary 3.7. *Let $(X,\|\cdot\|)$ be a Banach space, $T: X\to X$ be a mapping and $\alpha: X\times X\to[0,+\infty)$ be a function. Suppose that below conditions are followed:*

(i) There exists $\zeta\in Z$ such that, if $x,y\in X$ with $\alpha(x)\beta(y)\geq 1$ then $\zeta(\|Tx-Ty\|,L(x,y))\geq 0$, where

$$L(x,y) = \max\left\{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \frac{\|x-Tx\|\|y-Ty\|}{1+\|x-y\|}, \frac{\|x-Tx\|\|y-Ty\|}{1+\|Tx-Ty\|}\right\}$$

(ii) If $x \in X$ with $\alpha(x) \geq 1$ then $\alpha(Tx) \geq 1$

(iii) There exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$

(iv) If $\{x_n\}$ is a sequence in X converges to $x \in X$ with $\alpha(x_n) \geq 1$ for all $n \in N$, then $\alpha(x) \geq 1$ for all $n \in N$. Then T has a fixed point.

Proof. It follows from Theorem 3.4 by taking the function $\beta : X \times X \to [0,+\infty)$ to be α. □

Corollary 3.8. *Let $(X, \|\cdot\|)$ be a Banach space, $T : X \to X$ be a mapping and $\alpha, \beta : X \times X \to [0,+\infty)$ be two functions. Assume the following conditions hold:*

(i) T is a (α,β) - cyclic.

(ii) There exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$.

(iii) There exists $k \in [0,1)$ such that, if $x, y \in X$ with $\alpha(x)\beta(y) \geq 1$,

$$\|Tx - Ty\| \leq k \max\left\{ \|x-y\|, \|x-Tx\|, \|y-Ty\|, \frac{\|x-Tx\|\|y-Ty\|}{1+\|x-y\|}, \frac{\|x-Tx\|\|y-Ty\|}{1+\|Tx-Ty\|} \right\}$$

(iv) T is continuous. Then T has a fixed point $x_0 \in X$.

Proof. Suppose there exists $k \in [0,1)$ such that condition (ii) holds. Define the simulation function $\zeta : [0,+\infty) \times [0,+\infty) \to R$ by $\zeta(t,s) = ks - t$. Note thatif $x, y \in X$ with $\alpha(x)\beta(y) \geq 1$, then

$$\zeta\left(\|Tx-Ty\|, \max\left\{ \|x-y\|, \|x-Tx\|, \|y-Ty\|, \frac{\|x-Tx\|\|y-Ty\|}{1+\|x-y\|}, \frac{\|x-Tx\|\|y-Ty\|}{1+\|Tx-Ty\|} \right\} \right) \geq 0.$$

The last inequality together with conditions (i) ensure that T is (α,β,Z) -contraction. Then T satisfies all conditions of theorem 3.3 and hence T has a fixed point. The continuity of T in corollary 3.8 can be replaced by a new suitable condition. □

Corollary 3.9. *Let $(X, \|\cdot\|)$ be a Banach space, $T : X \to X$ be a mapping and $\alpha, \beta : X \times X \to [0,+\infty)$ be two functions. Consider the below conditions is followed:*

(i) T is a $(\alpha,\beta)-$ cyclic.

(ii) There exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$.

(iii) There exists $k \in [0,1)$ such that, if $x, y \in X$ with $\alpha(x)\beta(y) \geq 1$, then

$$\|Tx - Ty\| \leq k \max\left\{ \|x-y\|, \|x-Tx\|, \|y-Ty\|, \frac{\|x-Tx\|\|y-Ty\|}{1+\|x-y\|}, \frac{\|x-Tx\|\|y-Ty\|}{1+\|Tx-Ty\|} \right\}$$

(iv) If $\{x_n\}$ is a sequence in X converges to $x \in X$ with $\alpha(x_n) \geq 1$ (or $\beta(x_n) \geq 1$) for all $n \in N$ then $\beta(x) \geq 1$ (or $\alpha(x) \geq 1$) for all $n \in N$ Then T has a fixed point $x_0 \in X$.

Proof. Follows from theorem 3.4 by following the same technique of the proof of corollary 3.8. □

Corollary 3.10. *Let $(X, \|\cdot\|)$ be a Banach space, $T : X \to X$ be a mapping and $\alpha, \beta : X \times X \to [0,+\infty)$ be two functions. Assume the following conditions hold:*

(i) T is a $(\alpha,\beta)-$ cyclic.

(ii) There exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$.

(iii) There exists a lower semi-continuous function $\varphi : R^+ \to R^+$ with $\varphi(x) > 0$ for all $t > 0$ and $\varphi(0) = 0$ such that, if $x, y \in X$ with

(iv) T is continuous. Then T has a fixed point $x_0 \in X$.

Proof. It follows from theorem 3.3 by defining $\zeta : [0,+\infty) \times [0,+\infty) \to R$ via $\zeta(t,s) = s - \varphi(s) - t$ and follow the same technique as in corollary 3.8. □

Corollary 3.11. *Let $(X, \|.\|)$ be a Banach space, $T : X \to X$ be a mapping and $\alpha, \beta : X \times X \to [0,+\infty)$ be two functions. Consider the below conditions are followed:*

(i) T is a (α,β) - cyclic.

(ii) There exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$.

(iii) There exists a lower semi-continuous function $\varphi : R^+ \to R^+$ with $\varphi(x) > 0$ for all $t > 0$ and $\varphi(0) = 0$ such that, if $x, y \in X$ with $\alpha(x)\beta(y) \geq 1$, then

$$\|Tx - Ty\| \leq \max\left\{ \|x-y\|, \|x-Tx\|, \|y-Ty\|, \frac{\|x-Tx\|\|y-Ty\|}{1+\|x-y\|}, \frac{\|x-Tx\|\|y-Ty\|}{1+\|Tx-Ty\|} \right\} - \varphi\left(\max\left\{ \|x-y\|, \|x-Tx\|, \|y-Ty\|, \frac{\|x-Tx\|\|y-Ty\|}{1+\|x-y\|}, \frac{\|x-Tx\|\|y-Ty\|}{1+\|Tx-Ty\|} \right\} \right)$$

(iv) If $\{x_n\}$ is a sequence in X converges to $x \in X$ with $\alpha(x_n) \geq 1$ (or $\beta(x_n) \geq 1$) for all $n \in N$ then $\beta(x) \geq 1$ (or $\alpha(x) \geq 1$) for all $n \in N$. Then T has a fixed point $x_0 \in X$.

Proof. It follows from theorem 3.4 by defining $\zeta : [0,+\infty) \times [0,+\infty) \to R$ via $\zeta(t,s) = s - \varphi(s) - t$ and follow the same technique as in corollary 3.8 . □

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