Digital Hilbert transformer designing approach using discrete Sine transform method

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Abstract
Designing approach for non-integer order digital Hilbert transformer using discrete sine transform (DST) Methods are presented in this paper. First, review about the Hilbert transformer definition for non-integer order. Then, given discrete-time signal is used to reconstruct a continuous-time signal with the help of the different definitions of DST methods. Next, the filters coefficients and transfer function for the proposed design method are obtained using index mapping technique. Finally, computational problem are discussed for showing the effectiveness of DST methods.

Keywords
Non-integer order digital Hilbert transformer, Lanczos window, discrete sine transform methods.

1. Introduction
During several decades, Hilbert transformer are evolved after including non-integer order for enhancing the phase response. One of the best applications of Hilbert transformer is used for the generation of single side band and in the VLSI also by using some dedicated architectures, where fractional order is used for reconstructing or demodulation the signal [1]. Another application of fractional Hilbert transformer (FHT) in the field of digital image processing where it works as to determine position, edges of images etc. All this types of applications can be controlled with the fractional order derivative [2][3]. There are several techniques for the designing of fractional order hilbert transformer [1]-[5]. The work has been done already on the designing technique of variable FHT by the adjustable phase response [6].

Next the theory of DST is discussed for highly correlated signal i.e. devolved by the Karhunen-Loeve transform [7]. DST is used successfully in the application of speech processing [8], signal interpolation [9], image coding and encryption [10]. The organized way of paper as following: In section 2, the definition of Hilbert transformer is discussed . In section 3, Different transfer function for DST interpolation methods are discussed using Lanczos window technique. In section 4, computational problem and result analysis is discussed. At last section, conclusion is discussed.

2. Definition and properties of Hilbert transformer
The definition and properties of the digital Hilbert transformer will be discussed. So Hilbert transformer is defined in frequency domain as [11]

\[
G_{\beta}(\omega) = \begin{cases} 
  e^{-j\pi/2} & 0 < \omega < \infty \\
  e^{j\pi/2} & -\infty < \omega < 0
\end{cases} = \begin{cases} 
  -j & 0 < \omega < \infty \\
  j & -\infty < \omega < 0
\end{cases}
\]
Where order is represented by a symbol $\beta$. Various Hilbert transformer properties are given below as:

1. When the frequency response of Hilbert transformer i.e. $G_{\beta}(\omega) = 1$ with order $\beta = 0$. It means there is no changes in the output signal after passing input signal through non-integer Hilbert transformer.

2. When the phase of Hilbert transformer i.e. $\Psi = \frac{\pi}{2}$ with order $\beta = 1$, it means the output signal get distorted after passing through conventional non-integer Hilbert transformer.

3. The additive property for Hilbert transformer is given as $G_{\beta_1+\beta_2}(\omega) = G_{\beta_1}(\omega)G_{\beta_2}(\omega)$.

4. Periodic property for frequency $F_{\beta}(\omega)$ is shown as 8 the $G_{\beta+8}(\omega) = G_{\beta}(\omega)$ with period 8.

5. When a signal is applied on a system i.e. non-integer Hilbert transformer then its output signal comes out as

$$FHT [\cos(\omega,t + \phi)] = \cos(\omega,t + \phi - \Psi)$$

$$FHT [\sin(\omega,t + \phi)] = \sin(\omega,t + \phi - \Psi)$$

### 3. Main Results

#### 3.1 Design method using DST-I

DST interpolation approach is defined in various ways as [10]-[11]. The definition of DST interpolation methods will be discussed in this paper. The input signal $g(t)$ i.e. continuous in nature are scattered into a small data points (sampled) as $g(0), g(1), \ldots, g(R-1)$ i.e. discrete-time sequences. Different DST approach is used to reshape the original signal $y(t)$ i.e., continuous signal from a given discrete signal. So, DST-I definition can be express as

$$G(k) = \sum_{m=0}^{R-1} g(m) \sin \left( \frac{(m+1)(k+1)\pi}{R+1} \right)$$

(3.1)

$$g(m) = \sum_{k=0}^{R-1} G(k) \sin \left( \frac{(m+1)(k+1)\pi}{R+1} \right)$$

(3.2)

Putting the value of eq. (3.1) into the eq. (3.2), we get

$$g(m) = \sum_{n=0}^{R-1} g(n) \left\{ \frac{2}{R+1} \sum_{k=0}^{R-1} \sin \left( \frac{(n+1)(k+1)\pi}{R+1} \right) \sin \left( \frac{(m+1)(k+1)\pi}{R+1} \right) \right\}$$

Substitute $t$ in place of $m$, then interpolated signal $y(t)$ can be define as

$$g(t) = \sum_{n=0}^{R-1} g(n)f_1(n,t)$$

(3.3)

Interpolated basis function is expressed by $f_1(n,t)$

$$g_1(n,t) = \sum_{k=0}^{R-1} \frac{2}{R+1} \left\{ \sin \left( \frac{(n+1)(k+1)\pi}{R+1} \right) \sin \left( \frac{(t+1)(k+1)\pi}{R+1} \right) \right\}$$

(3.4)

when the definition of non-integer Hilbert transformer is applied on a given input signal $g(t)$, then the expression for the system output is written as

$$FHTg(t) = \sum_{n=0}^{R-1} g(n)FHTf_1(n,t)$$

(3.5)

Due to linear property of FHT eq. 3.5

$$FHTf_1(n,t) = \sum_{k=0}^{R-1} \sin \left( \frac{(n+1)(k+1)\pi}{R+1} \right) \sin \left( \frac{(t+1)(k+1)\pi}{R+1} - \frac{\pi\beta}{2} \right)$$

(3.6)

Substitute the equation(6) into the equation(5), then

$$FHTg(t) = \sum_{n=0}^{R-1} g(n)b_n(t)$$

(3.7)

where

$$b_n(t) = \sum_{k=0}^{R-1} \sin \left( \frac{(n+1)(k+1)\pi}{R+1} \right) \sin \left( \frac{(t+1)(k+1)\pi}{R+1} - \frac{\pi\beta}{2} \right)$$

(3.8)

The transfer function of DST-I method approximates to the ideal frequency response of Hilbert transformer as:

$$H_d(\omega) = B_{\beta}(\omega)e^{-j\omega l}$$

(3.9)

The parameter delay value is expressed by a symbol $l$. The transfer function for a finite impulse response (FIR) filter is written as:

$$H(z) = \sum_{v=0}^{R-1} h(v)z^{-v}$$

(3.10)

when a discrete time input signal $x(m)$ is passed through a filter (FIR) then the filter will generates the output signal with some sample delay values as $x(m), x(m-1), x(m-2), \ldots, x(m-R+1)$, then

$$x(m) = \sum_{v=0}^{R-1} h(v)x(m-v)$$

(3.11)

Now our task is to calculate the coefficients values $h(v)$ from eq. (3.10). So $g(m)$ approximate to the FHT[x(m)] with delay values as

$$g(m) \approx FHT[x(m-l)]$$

(3.12)
The value of the filter coefficients is obtained after putting the value of eq. (3.8) into eq. (3.14),

\[ s(n) = x(m - (R - 1) + n) \quad 0 \leq n \leq R - 1 \tag{3.14} \]

Equate \( s(t) = x(m - (R - 1) + t) \) into eq. (3.7)

\[ FHT[x(m - (R - 1) + t)] = \sum_{n=0}^{R-1} x(m - (R - 1) + n) b_n(t) \tag{3.15} \]

\[ h(v) = b_{R-1-v}(R - 1 - 1) \tag{3.16} \]

The simplified expression of above matrix is given below as:

\[
\begin{bmatrix}
  x(m) = s(R - 1) \\
  x(m - 1) = s(R - 2) \\
  \vdots \\
  x(m - R + 1) = s(0)
\end{bmatrix}
\]

\[ (3.13) \]

The value of the filter coefficients is obtained after putting the value of eq. (3.8) into eq. (3.14),

\[ h(v) = \frac{2}{R + 1} \sum_{k=0}^{R-1} \sin \left( \frac{(R - r - 1)(k + 1)\pi}{R + 1} \right) \sin \left( \frac{(R - l - 1)(k + 1)\pi}{R + 1} - \frac{\pi R}{2} \right) \tag{3.17} \]

Using the window techniques for the modification of filter coefficients. So in this paper Lanczos window technique will be used and it’s transfer function is defined as

\[ w(v) = \sin \left( \frac{2v}{R - 1} - 1 \right) \tag{3.18} \]

Modified coefficients of the FIR filter using window techniques is

\[ h_w(v) = h(v)w(v) \tag{3.19} \]

The system performance for the Hilbert transformer with fractional order can be evaluated by integral square design error formula i.e. given in the frequency domain as

\[ E = \sqrt{\int_{0.1\pi}^{2\pi} |H(e^{j\omega}) - H_d(\omega)|^2 d\omega} \tag{3.20} \]

Performance can be checked for the proposed method with the ideal response of Hilbert transformer using error formula \( E \).

### 3.2 Design Method of Hilbert transformer using DST-II

DST-II function is defined as

\[ G(k) = \sqrt{\sum_{m=0}^{R-1} c_m g(m) \sin \left( \frac{(2m + 1)(k + 1)\pi}{2R} \right) \tag{3.21} } \]

The inverse function of \( G(k) \) can be written as

\[ g(m) = \sqrt{\sum_{m=0}^{R-1} c_m G(k) \sin \left( \frac{(2m + 1)(k + 1)\pi}{2R} \right) \tag{3.22} } \]

\[ c_k = c_m = \begin{cases} \frac{1}{\sqrt{2}} & 0 \leq k \leq R - 1 \\ 0 & \text{otherwise} \end{cases} \]

By using the definition of fractional order Hilbert transformer, filter coefficients comes

\[ h(v) = \frac{2}{R} \sum_{k=0}^{R-1} \sin \left( \frac{(2R - 2r - 1)(k + 1)\pi}{2R} \right) \sin \left( \frac{(2R - 2l - 1)(k + 1)\pi}{2R} - \frac{\pi R}{2} \right) \tag{3.23} \]

Coefficients of the FIR filter is modified by the Lanczos window technique. So Lanczos window can defined as

\[ w(v) = \sin \left( \frac{2v}{R - 1} - 1 \right) \tag{3.24} \]

Optimization results of filter is obtained by

\[ h_w(v) = h(v)w(v) \tag{3.25} \]

### 3.3 Design Method of Hilbert transformer using DST-III

DST-III function is defined as

\[ G(k) = \sqrt{\sum_{m=0}^{R-1} c_m g(m) \sin \left( \frac{(m + 1)(2k + 1)\pi}{2R} \right) \tag{3.26} } \]

\[ g(m) = \sqrt{\sum_{k=0}^{R-1} c_k G(k) \sin \left( \frac{(m + 1)(2k + 1)\pi}{2R} \right) \tag{3.27} } \]

\[ c_k = c_m = \begin{cases} \frac{1}{\sqrt{2}} & 0 \leq k \leq R - 1 \\ 0 & \text{otherwise} \end{cases} \]

By using the definition of fractional order Hilbert transformer, filter coefficients comes

\[ h(v) = \frac{2}{R} \sum_{k=0}^{R-1} \sin \left( \frac{(R - r)(2k + 1)\pi}{2R} \right) \sin \left( \frac{(R - l)(2k + 1)\pi}{2R} - \frac{\pi R}{2} \right) \tag{3.28} \]

Coefficients of the filter FIR is modified by using the Lanczos window technique. So Lanczos window can defined as

\[ w(v) = \sin \left( \frac{2v}{R - 1} - 1 \right) \tag{3.29} \]

Optimization results of filter is obtained by

\[ h_w(v) = h(v)w(v) \tag{3.30} \]
3.4 Design Method of Hilbert transformer using DST-IV

DST-IV function is defined as

\[ G(k) = \sqrt{\frac{2}{R}} \sum_{m=0}^{R-1} g(m) \sin \left( \frac{(2m+1)(2k+1)\pi}{4R} \right) \]  (3.31)

\[ g(m) = \sqrt{\frac{2}{R}} \sum_{k=0}^{R-1} G(k) \sin \left( \frac{(2m+1)(2k+1)\pi}{4R} \right) \]  (3.32)

By using the definition of fractional order Hilbert transformer, filter coefficients comes

\[ h(v) = \frac{2}{R} \sum_{k=0}^{R-1} \sin \left( \frac{(2R-2r-1)(2k+1)\pi}{4R} \right) \]
\[ \sin \left( \frac{(2R-2I-1)(2k+1)\pi}{4R} - \frac{\pi \beta}{2} \right) \]  (3.33)

Coefficients of the filter FIR is modified by using the Lanczos window technique. So Lanczos window can defined as

\[ w(v) = \text{sinc} \left( \frac{2v}{R-1} - 1 \right) \]  (3.34)

Optimization results of filter is obtained by

\[ h_w(v) = h(v)w(v) \]  (3.35)

4. Design Examples

**Example:** The frequency response are shown in this example for the digital Hilbert transformer with non-integer order for various definitions of DST methods. The Hilbert transformer performance can be calculated by design error formula of eq.(20) for the different approach of DST [12][13]. The size of error \( E \) defines the effectiveness of fractional order Hilbert transformer for the different DST methods. The optimum design values are chosen as \( R = 60, I = 30, \beta = 0.3, 0.5, 0.7 \) for the different DST methods.

<table>
<thead>
<tr>
<th>Orders</th>
<th>( \beta = 0.3 )</th>
<th>( \beta = 0.5 )</th>
<th>( \beta = 0.7 )</th>
<th>( \beta = 0.9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_{\text{DST-I}} )</td>
<td>0.1594</td>
<td>0.2587</td>
<td>0.3597</td>
<td>0.4556</td>
</tr>
<tr>
<td>( E_{\text{DST-II}} )</td>
<td>0.1595</td>
<td>0.2588</td>
<td>0.3599</td>
<td>0.4537</td>
</tr>
<tr>
<td>( E_{\text{DST-III}} )</td>
<td>0.1534</td>
<td>0.2550</td>
<td>0.3599</td>
<td>0.4577</td>
</tr>
<tr>
<td>( E_{\text{DST-IV}} )</td>
<td>0.1331</td>
<td>0.2548</td>
<td>0.3596</td>
<td>0.4572</td>
</tr>
</tbody>
</table>

Error values \( E \) are shown in the above table for the different orders with different DST interpolation methods. In the DST-IV case, the orders from \( \beta = 0.3 \) to \( \beta = 0.7 \) the size of error \( E \) is smaller than the other DST methods.

Fig. 1, 2, 3, 4, 5, 6, 7, 8 shows the response of the different DST methods for orders \( \beta = 0.3, 0.5, 0.7, 0.9 \). The ideal magnitude response is defined as \(-2 * \left[ \text{angle}(H(e^{jw})) + wI \right] / \pi \) with \( \beta \) phase response.

But order \( \beta = 0.9 \) the error size \( E \) of DST-II is smaller than the other DST methods. So with the help of error table we can evaluate which DST method will well suit for the designing of fractional order Hilbert transformer.

**Fig.1:** Magnitude response graph of order \( \beta = 0.3 \) for different DST methods.

**Fig.2:** Phase response graph of order \( \beta = 0.3 \) for different DST methods.
Fig. 3: Magnitude response graph of order $\beta = 0.5$ for different DST methods.

Fig. 4: Phase response graph of order $\beta = 0.5$ for different DST methods.

Fig. 5: Magnitude response graph of order $\beta = 0.7$ for different DST methods.

Fig. 6: Phase response graph of order $\beta = 0.7$ for different DST methods.

Fig. 7: Magnitude response graph of order $\beta = 0.9$ for different DST methods.

Fig. 8: Phase response graph of order $\beta = 0.9$ for different DST methods.
5. Conclusion

In this paper comparison of DST methods of different orders for non-integer Hilbert transformer are presented. Computational problem is shown for the different orders for checking the effectiveness of DST methods by using integral square error formula and values of error for different orders have been shown in the form of table. In this paper, DST-IV method has shown smaller size of error values for orders $\beta = 0.3$ to $\beta = 0.7$ as compare to other methods. So, DST-IV is well suited technique method for the designing of fractional Hilbert transformer. This work can also be extended for 2-D DST and DCT.

References


