On the hyperbola $X^2 + 4XY + Y^2 - 2X + 2Y - 8 = 0$

S. Vidhyalakshmi,^{1*} M.A. Gopalan² and S. Aarthi Thangam³

Abstract

In this work, we search for the lattice points of the hyperbola $X^2 + 4XY + Y^2 - 2X + 2Y - 8 = 0$. Various connections among the solutions are given. Given a solution, solutions for other forms of hyperbolas and parabolas are determined.

Keywords

Binary quadratic, Hyperbola, Parabola, Pell equation, Integral solutions.

AMS Subject Classification

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^{1,2,3}Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620002, Tamil Nadu, India.

*Corresponding author: ¹ vidhyasigc@gmail.com; ² mayilgopalan@gmail.com; ³ aarthithangam@gmail.com

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1. Introduction

Every student of mathematics is familiar with the subject of analytical geometry which is a study of geometry using a co-ordinate system. Linear equations involving x and y specify lines while quadratic equations specify conic sections. The hyperbola, a special conic, represented by the pell equation $y^2 = Dx^2 + N$ ($D > 0$ and square free) for various values of D and N are studied in [6, 8, 9]. The hyperbola represented by an equation of the form $x^2 + Axy + y^2 + Bx = 0$ is analyzed for various values of A and B in [2-5]. In [1,7], the hyperbola represented by an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is considered for particular values A, B, C, D, E and F . It seems that much work has not been done in this choice. It is therefore towards this end, the hyperbola represented by $X^2 + 4XY + Y^2 - 2X + 2Y - 8 = 0$ is considered in this paper for determining its non-zero distinct integer solutions. Employing the solutions of the given equation, integer solutions to special hyperbolas and parabolas are obtained.

2. Method of Analysis

The diophantine equation under consideration is

$$X^2 + 4XY + Y^2 - 2X + 2Y - 8 = 0 \quad (2.1)$$

It is to be noted that (1) represents a hyperbola. Substituting

$$X = x - 1, Y = y + 1 \quad (2.2)$$

in (1), we get

$$x^2 + y^2 + 4xy - 6 = 0 \quad (2.3)$$

Again setting

$$x = M + N, y = M - N \quad (2.4)$$

in (3), it simplifies to the equation

$$N^2 = 3M^2 - 3 \quad (2.5)$$

whose initial solution is $M_0 = 2, N_0 = 3$ Now consider the fundamental positive pell equation

$$N^2 = 3M^2 + 1 \quad (2.6)$$

whose general solution $(\tilde{M}_s, \tilde{N}_s)$ is given by

$$\tilde{N}_s = \frac{1}{2}f_s, \tilde{M}_s = \frac{1}{2\sqrt{3}}g_s$$

where

$$\begin{aligned} f_s &= (2 + \sqrt{3})^{s+1} + (2 - \sqrt{3})^{s+1}, \\ g_s &= (2 + \sqrt{3})^{s+1} - (2 - \sqrt{3})^{s+1}, \\ s &= -1, 0, 1, 2, \dots \end{aligned}$$

To obtain the solutions of (5), we employ the lemma known as Brahmagupta lemma stated as follows:

If (x_0, y_0) and (x_1, y_1) represent the solutions of the pell equations $y^2 = Dx^2 + k_1$ and $y^2 = Dx^2 + k_2$ ($D > 0$ and square free) respectively, the $(x_0y_1 + y_0x_1, y_0y_1 + Dx_0x_1)$ represents the solution of the pell equation $y^2 = Dx^2 + k_1k_2$.

Applying the above Brahmagupta lemma between (M_0, N_0) and $(\tilde{M}_s, \tilde{N}_s)$, the other integer solutions of (5) are given by

$$M_{s+1} = f_s + \frac{\sqrt{3}}{2} g_s$$

$$N_{s+1} = \frac{3}{2} f_s + \sqrt{3} g_s, s = -1, 0, 1, 2, \dots$$

Taking the advantage of (2) and (4), the sequence of integral solutions of (1) can be written as

$$\begin{cases} X_{s+1} = \frac{5}{2} f_s + \frac{3\sqrt{3}}{2} g_s - 1 \\ Y_{s+1} = -\frac{1}{2} f_s - \frac{\sqrt{3}}{2} g_s + 1, s = 0, 1, 2, \dots \end{cases} \quad (2.7)$$

Thus (7) represent non-zero distinct integer solutions to (1). The recurrence relations satisfied by the values of X_{s+1} and Y_{s+1} are given by

$$X_{s+3} - 4X_{s+2} + X_{s+1} = 2$$

$$Y_{s+3} - 4Y_{s+2} + Y_{s+1} = -2$$

A few numerical examples are given in the Table: 1 below:

Table 1: Numerical Examples

s	X_{s+1}	Y_{s+1}
0	18	-4
1	70	-18
2	264	-70
3	988	-264
4	3690	-988
5	13774	-3690

A few interesting relations among the solutions are given below:

- $X_{s+2} - 4X_{s+1} - Y_{s+1} = 2$
- $X_{s+3} - 15X_{s+1} - 4Y_{s+1} = 10$
- $Y_{s+3} + 4X_{s+1} + Y_{s+1} = -2$
- $X_{s+3} + 4Y_{s+3} + X_{s+1} = 2$
- $Y_{s+2} + X_{s+1} = 0$
- If (X_0, Y_0) is any solution of (1), then $X_1 = Y_0 - 2, Y_0 = X_0 + 2$ also satisfy (1)
- Each of the following expressions represents a perfect square

$$\bigcirc X_{2s+2} + 3Y_{2s+2}$$

$$\bigcirc 3X_{2s+3} - 11X_{2s+2} - 6$$

$$\bigcirc \frac{1}{4}(3X_{2s+4} - 41X_{2s+2} - 30)$$

- Each of the following expressions represents a Nasty number

$$\bigcirc 6X_{2s+2} + 18Y_{2s+2}$$

$$\bigcirc 18X_{2s+3} - 66X_{2s+2} - 36$$

$$\bigcirc \frac{3}{2}(3X_{2s+4} - 41X_{2s+2} - 30)$$

- Each of the following expressions represents a Cubical integer

$$\bigcirc X_{3s+3} + 3Y_{3s+3} + 3X_{s+1} + 9Y_{s+1} - 8$$

$$\bigcirc 3X_{3s+4} - 11X_{3s+3} + 9X_{s+2} - 33X_{s+1} - 32$$

$$\bigcirc \frac{1}{4}(3X_{3s+5} - 41X_{3s+3} + 9X_{s+3} - 123X_{s+1} - 152)$$

- Each of the following expressions represents a Bi-quadratic integer

$$\bigcirc X_{4s+4} + 3Y_{4s+4} + 4X_{2s+2} + 12Y_{2s+2} - 4$$

$$\bigcirc 3X_{4s+5} - 11X_{4s+4} + 12X_{2s+3} - 44X_{2s+2} - 34$$

$$\bigcirc \frac{1}{4}(3X_{4s+6} - 41X_{4s+4} + 12X_{2s+4} - 164X_{2s+2} - 166)$$

- Each of the following expressions represents a quintic integer

$$\bigcirc X_{5s+5} + 3Y_{5s+5} - 2 + 30P_{f_s-1}^3, f_s = X_{s+1} + 3Y_{s+1} - 2, \\ P_n^3 - \text{Tetrahedral number of rank } n$$

$$\bigcirc 3X_{5s+6} - 11X_{5s+5} - 8 + 30P_{f_s-1}^3, \\ f_s = 3X_{s+2} - 11X_{s+1} - 8$$

$$\bigcirc \frac{1}{4}(3X_{5s+7} - 41Y_{5s+5} - 38 + 30P_{f_s-1}^3), \\ f_s = \frac{1}{4}(3X_{s+3} - 41X_{s+1} - 38)$$

- Employing linear combinations among the solutions of (2.1), one may generate integer solutions for other choices of hyperbola which are presented in Table: 2 below:

Table 1. Hyperbolas

S.No	Hyperbolas	(P,Q)
1	$3P^2 - Q^2 = 12$	$(X_{s+1} + 3Y_{s+1} - 2, 4 - X_{s+1} - 5Y_{s+1})$
2	$3P^2 - Q^2 = 12$	$(3X_{s+2} - 11X_{s+1} - 8, 19X_{s+1} - 5X_{s+2} + 14)$
3	$3P^2 - Q^2 = 192$	$(3X_{s+3} - 41X_{s+1} - 38, 71X_{s+1} - 5X_{s+3} + 66)$

Table 2. Parabolas

S.No	Parabolas	(P,Q)
1	$3P - Q^2 = 12$	$(X_{2s+2} + 3Y_{2s+2}, 4 - X_{s+1} - 5Y_{s+1})$
2	$3P - Q^2 = 12$	$(3X_{2s+3} - 11X_{2s+2} - 6, 19X_{s+1} - 5X_{s+2} + 14)$
3	$12P - Q^2 = 192$	$(3X_{2s+4} - 41X_{2s+2} - 30, 71X_{s+1} - 5X_{s+3} + 66)$

- Employing linear combinations among the solutions of (2.1), one may generate integer solutions for other choices of parabola which are presented in Table: 3 below:

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