2-Weight domination on S -valued graphs

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Abstract

In this paper, we introduce the notion of distance 2-weight dominating vertex sets in a S -valued graphs and study some of its properties. .

Keywords

Distance 2-dominating sets, S -valued graphs, Domination in S -valued graphs.

AMS Subject Classification

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1. Introduction

In crisp graph theory, one of the main area of research is the domination in graphs. The concept of distance k -dominating sets $k \geq 1$ was introduced by Meir and Moon under the name k -covering. Distance 2-dominating sets are identified in School bus routing, radio stations, communication networks and mobile adhoc networks. In 2015, Chandramouleeswaran and others studied the notion of semiring-valued graphs. In our paper, we have introduced the notion of vertex domination on S -valued graphs. In this paper, we study distance 2-domination on S -valued graphs.

2. Preliminaries

In this section, we recall some basic definitions that are needed for our work.

Definition 2.1. [6] A semiring $(S, +, \cdot)$ is an algebraic system with a non-empty set S together with two binary operations $+$ and $\cdot$ such that

1. $(S, +, 0)$ is a monoid.
2. $(S, \cdot)$ is a semigroup.

3. For all $a, b, c \in S$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.

4. $0 \cdot x = x \cdot 0 = 0 \forall x \in S$.

Definition 2.2. Let $(S, +, \cdot)$ be a semiring. A Canonical Pre-order $\preceq$ in S defined as follows : for $a, b \in S$, $a \preceq b$ if and only if, there exists $c \in S$ such that $a + c = b$.

Definition 2.3. [7] Let $G = (V, E \subset V \times V)$ be the underlying graph with both $V, E \neq \emptyset$. For any semiring $(S, +, \cdot)$, a Semiring-valued graph (or an S -valued graph) G^S is defined to be the graph $G^S = (V, E, \sigma, \psi)$ where $\sigma : V \rightarrow S$ and $\psi : E \rightarrow S$ is defined to be

$$\psi(x, y) = \begin{cases} \min \{ \sigma(x), \sigma(y) \} & \text{if } \sigma(x) \preceq \sigma(y) \text{ or } \sigma(y) \preceq \sigma(x) \\ 0 & \text{otherwise} \end{cases}$$

for every unordered pair (x, y) of $E \subset V \times V$. We call σ , a S -vertex set and ψ , a S -edge set of S -valued graph G^S .

Definition 2.4. [7] Consider the S -valued graph $G^S = (V, E, \sigma, \psi)$. For $v_i \in V$,

- The open neighbourhood of v_i in G^S is defined as a subset of $V \times S$ such that,

$$N_S(v_i) = \{ (v_j, \sigma(v_j)) / (v_i, v_j) \in E, \psi(v_i, v_j) \in S \}.$$

- the closed neighbourhood of v_i in G^S is defined to be the subset of $V \times S$ such that,

$$N_S[v_i] = N_S(v_i) \cup \{ (v_i, \sigma(v_i)) \}$$

- Let $D \subseteq V$. The open neighbourhood of D denoted by $N_S(D) = \bigcup_{v \in D} N_S(v)$

- The closed neighbourhood of D defined as $N_S[D] = \bigcup_{v \in D} N_S[v]$.

Definition 2.5. [7] The degree of the vertex v_i of the S -valued graph G^S is defined as,

$$\deg_S(v_i) = \left(\sum_{(v_i, v_j) \in E} \psi(v_i, v_j), l \right)$$

where l is the number of edges incident with v_i .

That is, $\deg_S(v_i) = (|N_S(v_i)|_S, |N_S(v_i)|)$

Definition 2.6. In the S -valued graph $G^S = (V, E, \sigma, \psi)$, to compare the degree of two vertices $v, w \in G^S$, we define the $\preceq$ as follows:

$$* (\sigma(v), \deg(v)) \preceq (\sigma(w), \deg(w)) \iff \sigma(v) \preceq \sigma(w)$$

and $\deg(v) \leq \deg(w)$.

$$* \text{ if } \sigma(v) \preceq \sigma(w) \text{ and } \deg(v) \geq \deg(w),$$

the comparison is with respect to the S -values.

Definition 2.7. A S -Spider is a S -valued graph $G_S = (V, E, \sigma, \psi)$ with a vertex $(v_0, \sigma(v_0))$ such that $\deg_S(v_0) \succeq (|N_S(v_0)|_S, 3)$ and other vertices $(v_i, \sigma(v_i))$ such that $\deg_S(v_i) \succeq (|N_S(v_i)|_S, 2)$ where $i = 1, 2, \dots, n$. Here $(v_0, \sigma(v_0))$ is called S -Pole of the S -Spider.

Definition 2.8. [5] Consider the S -valued graph G^S . For $i = 1, 2, \dots, n$ let $e_i^{i+1} = ((u_i, \sigma(u_i)), (u_{i+1}, \sigma(u_{i+1}))) \in E_S$. A S -path $P^S(uv)$ between $u, v \in V$ in G^S , is defined to be the sequence of vertices and edges in G^S , given by $\{(u_i, \sigma(u_i))(e_i^{i+1}, \psi(e_i^{i+1}))(u_{i+1}, \sigma(u_{i+1}))\}_{i=1}^n$ where $u = (u_1, \sigma(u_1))$ and $v = (u_{n+1}, \sigma(u_{n+1}))$. Then

- The length of the path $P^S(uv)$, denoted by $l(P^S(uv)) = n$, is defined to be the number of edges of the path $P^S(uv)$.
- The weight of the S -path $P^S(uv)$ is defined as

$$wt(P^S(uv)) = \sum_{e \in P^S(uv)} \psi(e).$$

Definition 2.9. [5] Consider the S -valued graph G^S . Let $u, v \in V$. Let $P_i^S(uv)$ be the i^{th} path that connects u and v , $i = 1, 2, \dots$. Let the set of all paths $P_i^S(uv)$ that connects u and v along with their weights and lengths be denoted by P_{uv}^S . That is,

$$P_{uv}^S = \{ (wt(P_i^S), l(P_i^S)) / P_i \text{ is any path connecting } u \text{ and } v, \}$$

for all $i = 1, 2, \dots, k$ where $l(P_i)$ denote the length of the path P_i^S .

Definition 2.10. [5] Consider the S -valued graph $G^S = (V, E, \sigma, \psi)$. The distance between the vertices u and v of a S -valued graph G^S is defined as $\text{dist}_S(u, v) =$

$$\min(P_{uv}^S) = \min \{ (wt(P_i^S), l(P_i^S)) / P_i^S \in P_{uv}^S, i = 1, 2, \dots, k \}$$

where the minimum runs over all S -paths between u and v .

Definition 2.11. [3] A vertex v in G^S is said to be a weight dominating vertex if

$$\sigma(u) \preceq \sigma(v), \forall u \in N_S[v].$$

Definition 2.12. [3] A subset $D \subseteq V$ is said to be a weight dominating vertex set if,

$$\text{for each } v \in D, \sigma(u) \preceq \sigma(v), \forall u \in N_S[v].$$

Definition 2.13. [3] The cardinality of the minimal weight dominating vertex set $D \subseteq V$ is called the weight domination vertex number of G^S . It is denoted by $\gamma_V^S(G^S)$. That is,

$$\gamma_V^S(G^S) = \min \{ |D|_S, |D| / D \text{ is a weight dominating vertex set of } G^S \}.$$

Definition 2.14. A set of vertices D of a graph G is a distance 2-dominating set of G if the distance between each vertex $u \in (V(G) - D)$ and D is at most two. Let $\gamma_2(G)$ denote the size of a smallest distance 2-dominating set of G .

3. 2-Weight domination on S -valued graphs

In this section, we discuss the notion of 2-Weight dominating vertex on S -valued graph G^S .

Definition 3.1. Consider a S -valued graph $G^S = (V, E, \sigma, \psi)$. For $v \in V$, the 2-neighbourhood of G^S is defined as

$$N_2^S[v] = \{u \in V / \text{dist}_S(u, v) \preceq (\sigma(v), 2)\}.$$

Definition 3.2. Consider a S -valued graph $G^S = (V, E, \sigma, \psi)$. A subset $D \subseteq V$ is said to be a distance 2-weight dominating vertex set in G^S , if for any $v \in D$ there exist a $u \in V - D$ such that $\sigma(u) \preceq \sigma(v), \forall u \in N_2^S[v]$.

Definition 3.3. The minimum cardinality of a distance 2-weight dominating vertex set in G^S is called a distance 2-weight dominating vertex number and is denoted by $\gamma_{\preceq 2V}^S(G^S)$. That is,

$$\gamma_{\preceq 2V}^S(G^S) = \min \{ (|D|_S, |D|) / D \text{ is a 2-weight dominating vertex set.} \}$$

Example 3.4. Let $(S = \{0, a, b, c\}, +, \cdot)$ be a semiring with the following Cayley tables:

+	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	b
c	c	c	b	b

·	0	a	b	c
0	0	0	0	0
a	0	0	a	0
b	0	a	b	c
c	0	0	c	c

Let $\preceq$ be a canonical pre-order in S , given by

$$0 \preceq 0, 0 \preceq a, 0 \preceq b, 0 \preceq c, a \preceq a, a \preceq b, a \preceq c, b \preceq b, c \preceq c, c \preceq b.$$

Consider the S -valued graph $G^S = (V, E, \sigma, \psi)$.

Consider the subset $D = \{v_2(b)\}$, then

$\text{dist}_S(v_1, v_2) = (a, 1) \preceq (b, 1)$ and
 $\text{dist}_S(v_3, v_2) = (c, 1) \preceq (b, 1)$; $\text{dist}_S(v_4, v_2) = (b, 1) \preceq (b, 1)$
 $\text{dist}_S(v_5, v_2) = (a, 2) \preceq (b, 1)$; $\text{dist}_S(v_6, v_2) = (a, 1) \preceq (b, 1)$
 $\text{dist}_S(v_7, v_2) = (a, 2) \preceq (b, 1)$; $\text{dist}_S(v_8, v_2) = (c, 1) \preceq (b, 1)$
Hence $\text{dist}_S(u, v) \preceq (\sigma(v), 2)$.

Also for any $u \in V - D$, $\sigma(u) \preceq \sigma(v)$, $\forall u \in N_2^S[v]$.

Hence $D = \{v_2(b)\}$ is a distance 2-weight dominating vertex set of G^S and also D is a minimal distance 2-weight dominating vertex set of G^S .

Thus $\gamma_{\leq 2V}^S(G^S) = (b, 1)$.

Remark 3.5. For any S -valued graph $G^S = (V, E, \sigma, \psi)$, we have $\gamma_{\leq 2V}^S(G^S) \preceq \gamma_V^S(G^S)$. But the converse of the above result need not be true.

From the above example the subset $D = \{v_2(b)\}$ is a minimal 2-weight dominating vertex set of G^S .

However D is not a weight dominating vertex set of G^S .

For, minimal weight dominating vertex set will be $D' = \{v_2(b), v_4(b), v_7(b)\}$.

Hence $\gamma_{\leq 2V}^S(G^S) = (b, 1)$ and $\gamma_V^S(G^S) = (b, 3)$.

Theorem 3.6. For the S -valued graph K_p^S , $\gamma_{\leq 2V}^S(K_p^S) = (\sigma(v), 1)$.

Proof. Consider the S -valued complete graph $K_p^S = (V, E, \sigma, \psi)$.

Let $D \subseteq V$ be a distance 2-weight dominating vertex set.

Since $G^S = K_p^S$, the complete graph on p vertices with $p \geq 2$ and the degree of v are all equal for all $v \in V$.

Let $v \in D$ be such that $\sigma(v) \succeq \sigma(u)$ for all $u \in V - \{v\}$.

Then this $v \in D$ will dominate every other vertices in K_p^S and every vertex in $V - \{v\}$ is within distance 2 of $v \in D$.

Hence $D = \{v\} \Rightarrow |D|_S = \sigma(v)$

Thus $\gamma_{\leq 2V}^S(K_p^S) = (|D|_S, |D|) = (\sigma(v), 1)$. $\square$

Analogously, we can prove the following theorems.

1. For the S -valued graph W_n^S , for $n \geq 4$ $\gamma_{\leq 2V}^S(W_n^S) = (\sigma(v), 1)$, if the pole has the maximum S -value.
2. For the S -star $K_{1,n}^S$, for $n \geq 2$ $\gamma_{\leq 2V}^S(K_{1,n}^S) = (\sigma(v), 1)$, if the pole has the maximum S -value.

Theorem 3.7. A distance 2-weight dominating vertex set D of a S -valued graph G^S is a minimal distance 2-weight dominating vertex set of G^S iff every vertex $v \in D$ such that there exist a vertex $u \in V - D$ such that $N_{\leq 2V}^S(u) \cap (D \times S) = \{(v, \sigma(v))\}$.

Proof. Let $v \in D$. Assume that v is distance 2 to no vertex of D , then $D - \{v\}$ cannot be a distance 2-weight dominating vertex set of G^S .

This implies that D is a minimal distance 2-weight dominating vertex set of G^S .

On the other hand, if for any $v \in D$ there exist a $u \in V - D$ such that $N_{\leq 2V}^S(u) \cap (D \times S) = \{(v, \sigma(v))\}$

Then u is distance 2- to $v \in D$ and no other vertex of D .

In this case also, $D - \{v\}$ cannot be a distance 2-weight dominating vertex set of G^S .

Hence D is a minimal distance 2-weight dominating vertex set of G^S .

Conversely, assume that D is a minimal distance 2-weight dominating vertex set of G^S .

Then for each $v \in D$, $D - \{v\}$ is not a distance 2-weight dominating vertex set of G^S .

Therefore there exist a $u \in V - (D - \{v\})$ that is adjacent to no vertex of $(D - \{v\})$.

If $u = v$, then v is adjacent to no vertex of D .

If $u \neq v$, then D is a distance 2-dominating set and $u \notin D$. This implies u is adjacent to atleast one vertex of D . However u is not adjacent to any vertex of $D - \{v\}$.

This implies that $N_{\leq 2V}^S(u) \cap D \times S = \{(v, \sigma(v))\}$. $\square$

Theorem 3.8. The distance 2-weight dominating vertex number of the S -valued graph $K_{m,n}^S$, is given by $\gamma_{\leq 2V}^S(K_{m,n}^S) = (\sum_{v \in D} \sigma(v), m + n - 2)$.

Proof. Let $G^S = K_{m,n}^S$ be a S -valued complete bipartite graph.

If $m = 1$ or $n = 1$, G^S will be a S -spider.

Then any vertex which is the root of the S -spider will be the distance 2-weight dominating vertex set with no edges.

$\therefore \gamma_{\leq 2V}^S(K_{m,n}^S) = (|D|_S, |D|) = (\sigma(v), 1)$.

Thus the theorem is true for the case $m = 1$ or $n = 1$.

Assume now that both $m, n \geq 2$.

Without loss of generality, assume that $m \geq n$. Then $\Delta(G) = n$.

Let $D \subseteq V$ be a distance 2-weight dominating vertex set in G^S .

Since G^S is a S -valued complete bipartite graph, the end vertices of any edge in G^S will dominate whole graph G^S , provided the S -values of the end vertices will have equal and maximal weight.

$\therefore |D|_S = \sum_{v \in D} \sigma(v)$ and $|D| = m + n - \Delta(G)$.

The above discussion holds good even if we assume that $n \geq m$.

$\therefore \gamma_{\leq 2V}^S(K_{m,n}^S) = (|D|_S, |D|) = (\sum_{v \in D} \sigma(v), m + n - \Delta(G))$ for any m, n . $\square$

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