Multiplicative functional on elementary fuzzy matrices

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Abstract

In this paper some new elementary fuzzy matrices are defined, some of its properties using multiplicative functional are studied.

Keywords

Fuzzy matrices, elementary fuzzy matrices, multiplicative functional, fuzzy addition and multiplication.

AMS Subject Classification

15B15, 15B10, 03E73, 04A72.

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1. Introduction

Fuzzy sets and fuzzy matrices play an important role in scientific development. Fuzzy sets were presented by L.A. Zadeh[12] in 1965, and fuzzy matrices were introduced by Thomason[10] in 1977. Various properties relative to fuzzy set and fuzzy matrix were subsequently studied by many authors, for example, [1]-[3] presented a number of results on the convergence of the power sequence of fuzzy matrices respectively. In 1980, K.H. Kim and F.W. Roush [5] studied the canonical form of an idempotent matrix.

In 1995, Ragab et al.[9] presented some properties of the min-max composition discussed triangular fuzzy matrices. With regard to determinant theory of fuzzy matrix J.B. Kim[5], [6] discussed some properties of determinant theory for fuzzy and Boolean matrices in 1988. M.Z. Ragab and E.G. Emam[8] presented the theory of the determinant and adjoint of a square fuzzy matrix in 1994. R. Hemasinha, N.R. Pal and J.C. Bezdek[2] studied the determinant of a fuzzy matrix with respect to t-norm and co-norms in 1997.

2. Preliminaries

Definition 2.1. Let $F[0,1]$ denote the fuzzy algebra over the interval $[0,1]$ with fuzzy addition $+$ and fuzzy multiplication operation $\times$ defined as

$$a+b=\max\{a,b\} \text{ and } a\times b=\min\{a,b\}$$

for all $a, b \in [0,1]$ and the standard order $\geq$.

Definition 2.2. If $A = [a_{ij}]$ is a $m \times n$ matrix and every $a_{ij} \in F[0,1]$, the matrix A is said to be a fuzzy matrix.

Definition 2.3. The fuzzy addition and fuzzy multiplication of $m \times n$ fuzzy matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are respectively defined as

$$A+B=[a_{ij}+b_{ij}]=[\max\{a_{ij},b_{ij}\}]$$

and

$$A \times B = [\max\{\min\{a_{ik},b_{kj}\}, 1 \leq k \leq m\}]$$

Definition 2.4. If $f : F_n, n \rightarrow F[0,1]$,

$$f(A \times B)=f(A) f(B)$$

for any $A, B \in F_{n,n}$ then f is said to be multiplicative functional, if a functional

$f : F[0,1] \rightarrow F[0,1]$ satisfies

$$f(ab)=f(a)f(b) \text{ for any } a, b \in F$$

Definition 2.5. Let $A = (a_{ij})$ be a fuzzy matrix of order n then the matrix (A^k) is obtained from the matrix A by adding λ times i^{th} row(column) to the k^{th} row (column) of A .

Definition 2.6. Let $A = (a_{ij})$ be a fuzzy matrix of order 'n' then the matrix $({}_KA)$ is defined as,
 $({}_KA) = {}_K a_{ij}$, where ${}_K a_{ij} = \lambda_{ij}^a$, when $K = i$
 $= 0$, otherwise

Definition 2.7. Let $A = (a_{ij})$ be a fuzzy matrix of order n then the relation between (A^K) and $({}_KA)$ is obtained by multiplying all the elements of i^{th} row by a non-zero number λ .
 $A^K = A + {}_KA$

3. Some results of multiplicative functional on elementary fuzzy matrices

Theorem 3.1. Let $f : F_{n,n} \rightarrow F[0, 1]$ be multiplicative linear functional then,

- (i) $f(A^K) = f(A) + f({}_KA)$
- (ii) $f(A^K + B^K) = f(A^K) + f(B^K)$
- (iii) $f(A^K B^K) = f((AB^K) + \lambda(\lambda + 1)f(AB))$

Proof. (i) By definition,

$A^K = (a_{ij}^k)$ where $(a_{ij}^k) = a_{kj} + a_{ij}$ $k = 1, 2, 3, \dots, n$

$$\begin{aligned} (A^k) &= f(a_{ij}^k) \\ &= f(a_{kj} + \lambda_{ij}) \\ &= f(a_{kj} + f(\lambda a_{ij})) \\ &= f(A) + f({}_KA) \end{aligned}$$

where ${}_K a_{ij} = \lambda_{ij}^a$, when $K = i$
 $= 0$, otherwise

Therefore, $f(A^K) = f(A) + f({}_KA)$.

(ii)

$$\begin{aligned} f(A^k + B^k) &= f(a_{kj} + \lambda a_{ij} + b_{kj} + \lambda b_{ij}) \\ &= f(a_{kj} + \lambda a_{ij}) + f(b_{kj} + \lambda b_{ij}) \\ &= f(A^k) + f(B^k) \end{aligned}$$

(iii)

$$\begin{aligned} f(A^k B^k) &= f(a_{ij}^k b_{ij}^k) \\ &= f\left(\sum_k a_{ik} + \lambda a_{ik} + b_{kj} + \lambda b_{kj}\right) \\ &= f\left(\sum_k a_{ik} b_{kj} + \lambda \sum_k a_{ik} b_{kj} + \lambda \sum_k a_{ik} b_{kj} + \lambda^2 \sum_k a_{ik} b_{kj}\right) \\ &= f\left\{\sum_k (a_{ik} b_{kj} + \lambda a_{ik} b_{kj}) + [\lambda \sum_k a_{ik} b_{kj} (1 + \lambda)]\right\} \\ &= f\left(\sum_k (a_{ik} b_{kj} + \lambda a_{ik} b_{kj}) + \lambda(\lambda + 1)f\left(\sum_k a_{ik} b_{kj}\right)\right) \\ f(A^k B^k) &= f(AB^K) + \lambda(\lambda + 1)f(AB) \end{aligned}$$

□

Theorem 3.2. Let $A = (a_{ij})$ be a fuzzy matrix of order 'n' let $f : F_{n,n} \rightarrow F[0, 1]$ be the multiplicative functional and $F_0(K) = f(A^K)$ then

- (i) $F_0(1) = f({}_1A) + f({}_KA)$
- (ii) $F_0(k_1 k_2) = f(A^{k_1})f(A^{k_2}) + f({}_KA)$

Proof. (i) Since,

$$\begin{aligned} F_0(K) &= f(A^K) \\ &= f(a_{ij}^k) \end{aligned}$$

$$F_0(K) = f(a_{kj} + \lambda a_{ij})$$

Take $k=1$

$$\begin{aligned} F_0(1) &= f(a_{1j} + \lambda a_{ij}) \\ &= f(a_{1j}) + f(\lambda a_{ij}) \\ &= f(1^a) + f({}_KA) \text{ where } {}_KA = \lambda a_{ij} \end{aligned}$$

(ii)

$$\begin{aligned} F_0(K_1 K_2) &= f(A^{K_1 K_2}) \\ &= f(a_{ij}^{K_1 K_2}) \\ &= f(a_{k_1 k_2 j} + \lambda a_{ij}) \\ &= f(a_{k_1 j})f(a_{k_2 j} + f(\lambda a_{ij})) \\ F_0(K_1 K_2) &= f(A^{k_1})f(A^{k_2}) + f({}_KA). \end{aligned}$$

□

Theorem 3.3. Let $f : F_{n,n} \rightarrow F[0, 1]$ be multiplicative functional and $H_0(\lambda) = f(A^K)$ then,

- (i). $H_0(0) = f(A)$ and
- (ii). $H_0(K_1 + K_2) = H_0(K_1) + H_0(K_2)$

Proof. (i) Since $H_0(0) = f(A^K)$
 $= f(a_{kj} + \lambda a_{ij})$

put $\lambda = 0$

$$H_0(0) = f(a_{kj} + 0.a_{ij})$$

$$= f(a_{kj}) \text{ where } k = 1, 2, 3, \dots, n \text{ and } j = 1, 2, 3, \dots, n$$

$$H_0(0) = f(A)$$

(ii)

$$\begin{aligned} H_0(K_1 + K_2) &= f(A^{k_1 + k_2}) \\ &= f(a_{(k_1 + k_2)j} + \lambda(a_{k_1 j} + a_{k_2 j})) + \lambda a_{k_2 j} \\ &= f\left[(a_{k_1 j} + \lambda a_{k_1 j}) + (a_{k_2 j} + \lambda a_{k_2 j})\right] \\ &= f(a_{k_1 j} + \lambda a_{k_1 j}) + f(a_{k_2 j} + \lambda a_{k_2 j}) \\ &= f(A^{k_1}) + f(A^{k_2}) \\ &= H_0(K_1) + H_0(K_2) \end{aligned}$$

□

Theorem 3.4. Let $A = (a_{ij})$ be a fuzzy matrix of order n then $A^{(k_1 + k_2)}(\lambda_1 \lambda_2) = A^{k_1 \lambda_1} + A^{k_2 \lambda_2} - \lambda_1(1 - \lambda_2)A - \lambda_2(1 - \lambda_1)A$.

Proof.

$$\begin{aligned} A^{(k_1 + k_2)}(\lambda_1 \lambda_2) &= a_{(k_1 + k_2)j} + \lambda_1 \lambda_2 a_{(k_1 + k_2)j} \\ &= a_{k_1 j} + a_{k_2 j} + \lambda_1 \lambda_2 (a_{k_1 j} + a_{k_2 j}) \\ &= a_{k_1 j} + a_{k_2 j} + \lambda_1 a_{k_1 j} + \lambda_2 a_{k_2 j} - \lambda_1 a_{k_1 j} \\ &\quad - \lambda_2 a_{k_2 j} + \lambda_1 \lambda_2 (a_{k_1 j} + a_{k_2 j}) \\ &= (a_{k_1 j} + \lambda_1 a_{k_1 j}) + (a_{k_2 j} + \lambda_2 a_{k_2 j}) \end{aligned}$$

$$\begin{aligned}
 & -\lambda_1 a_{k_1 j}(1 - \lambda_2) - \lambda_2(1 - \lambda_1)a_{k_2 j} \\
 & = A^{k_1 \lambda_1} + A^{k_2 \lambda_2} - \lambda_1(1 - \lambda_2)A - \lambda_2(1 - \lambda_1)A. \quad \square
 \end{aligned}$$

Theorem 3.5. Let $A = (a_{ij})$ be a fuzzy matrix of order 2. Let $f : F_{2 \times 2} \rightarrow F[0, 1]$ be the multiplicative functional such that the matrix A can be partitioned into two fuzzy matrices of order 2 then

$$f(A^K) = f(A^{(1)}(i, j)) + f(A^{(2)}(i, j)) + f({}_K A)$$

Proof. Let $A = (a_{ij})$ be a fuzzy matrix of order 2. The matrix A can be partitioned into two fuzzy matrices $A^{(1)}(i, j)$ and $A^{(2)}(i, j)$ where $A^{(1)}(i, j)$ be the fuzzy matrix whose elements are in the first column only and $A^{(2)}(i, j)$ be the fuzzy matrix having elements in the second column only.

$$\text{Let, } \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix} + \begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \end{pmatrix}$$

$$A = A^{(1)}(i, j) + A^{(2)}(i, j)$$

$$\text{But } A^K = A + ({}_K A)$$

$$\text{Therefore } f(A^K) = f(A + ({}_K A))$$

$$= f(A^{(1)}(i, j)) + f(A^{(2)}(i, j)) + f({}_K A)$$

$$f(A^K) = f(A^{(1)}(i, j)) + f(A^{(2)}(i, j)) + f({}_K A) \quad \square$$

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