Connected majority domatic number of a graph

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Abstract

This paper introduces the concept connected majority domatic number of a graph G , denoted as $d_{CM}(G)$. Also, exact values for some particular classes of graphs and bounds of $d_{CM}(G)$ are established.

Keywords

Connected Majority Dominating Set, Connected Majority Domination Number, Connected Majority Domatic Number.

AMS Subject Classification

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1. Introduction

In this research article all graphs are simple, finite and undirected. The concept of domination arises naturally from location problems in operations research. The domination problem in a graph is to find a minimum sized vertex set S such that every vertex not in S is adjacent to at least one vertex in S . Dominants and its variants have been extensively studied during the past decades. This paper considers a newly defined parameter namely connected majority domatic number of graph G . It is to find a maximum number of sets in the partition of $V(G)$ into connected majority dominating sets of a graph G . For terminology and notation not specifically defined here, one can refer the reader [2].

A subset S of $V(G)$ is said to be a **dominating set** [1],[2] of G if every vertex in $(V-S)$ is adjacent to at least one vertex in S . A dominating set is called **minimal dominating set** if no proper subset of S is a dominating set. The minimum cardinality of the minimal dominating set of G is called the **domination**

number of G , denoted by $\gamma(G)$. A domatic partition of a graph $G(V, E)$ is a partition of $V(G)$ all of whose classes are dominating sets of G . The maximum number of classes of a domatic partition of G is called the **domatic number** of G , denoted by $d(G)$.

A dominating set S is said to be a **connected dominating set** [2],[8] if the subgraph $\langle S \rangle$ induced by S is connected in G . The minimum cardinality of the minimal connected dominating set of G is called the **connected domination number**, denoted by $\gamma_c(G)$. A connected domatic partition of a graph $G(V, E)$ is a partition of $V(G)$ all of whose classes are connected dominating sets of G . The maximum number of classes of connected domatic partition of G is called the **connected domatic number** [7] of G , denoted by $d_c(G)$.

A subset S of $V(G)$ is said to be a **Majority Dominating set** [4] if at least half of the vertices of $V(G)$ are either in S or adjacent to elements of S i.e. $|N[S]| \geq \lceil \frac{V(G)}{2} \rceil$. A majority dominating set S is minimal if no proper subset of S is a majority dominating set. The minimum cardinality of a minimal majority dominating set is called **Majority Domination number**, denoted by $\gamma_M(G)$. A majority domatic partition of a graph $G(V, E)$ is a partition of $V(G)$ all of whose classes are majority dominating sets of G . The maximum number of classes of majority domatic partition of G is called the **Majority Domatic number** [4] of G , denoted by $d_M(G)$.

2. Preliminaries

Definition 2.1. [3] A set $S \subseteq V(G)$ is a **Connected Majority Dominating (CMD) set** if

majority domatic partition for G . This implies that $d_{CM}(G) \geq 2$. By proposition 3.4, $d_{CM}(G) \leq \lfloor \frac{p}{\gamma_{CM}} \rfloor = \lfloor \frac{p}{\lceil \frac{p}{2} \rceil - 2} \rfloor$, where $\gamma_{CM}(G) = \lceil \frac{p}{2} \rceil - 2$. This implies that $d_{CM}(G) \leq 2$. Hence $d_{CM}(G) = 2$, if $p \geq 9$. $\square$

Proposition 3.6. For Cycle $C_p, p \geq 3$

$$d_{CM}(C_p) = \begin{cases} p, & \text{if } p = 4, 5, 6 \\ 4, & \text{if } p = 8 \\ 3, & \text{if } p = 3, 7, 9, 10 \\ 2, & \text{if } p \geq 11 \end{cases}$$

Proposition 3.7. For Wheel

$$W_p = C_{p-1} \cup K_1, p \geq 5$$

$$d_{CM}(W_p) = \begin{cases} p, & \text{if } p = 5, 6, 7, 8 \\ 5, & \text{if } p = 9, 10 \\ 4, & \text{if } p = 11, 12, 13, 14, 16 \\ 3, & \text{if } p = 15 \text{ and } p \geq 17 \end{cases}$$

Proof. Let $\{v_1, v_2, \dots, v_p\}$ be the set of vertices of $G = W_p, p \geq 5$. The vertex $\{v_p\}$ be the centre of G and the remaining vertices $\{v_1, v_2, \dots, v_{p-1}\}$ form a outer cycle of a wheel graph $G = W_p$. Therefore $d(v_p) = p - 1$ and $d(v_i) = 3$, for $i = 1, 2, \dots, p - 1$.

Case (i): Let $p = 5, 6, 7, 8$. Each vertex $\{v_i\}$ is a CMD set of G .

Case (ii): Let $p = 9, 10$.

When $p = 9$. Let $S_1 = \{v_9\}, S_2 = \{v_1, v_2\}, S_3 = \{v_3, v_4\}, S_4 = \{v_5, v_6\}, S_5 = \{v_7, v_8\}$

When $p = 10$. Let $S_1 = \{v_{10}\}, S_2 = \{v_1, v_2\}, S_3 = \{v_3, v_4\}, S_4 = \{v_5, v_6\}, S_5 = \{v_7, v_8\} \cup \{v_9\}$.

Therefore, S_1, S_2, S_3, S_4 and S_5 are the connected majority domatic partitions of $V(G)$. Hence, $d_{CM}(G) = 5$ if $p=9, 10$.

Case (iii): Let $p = 11, 12, 13, 14, 16$.

For $p = 11$. Let $S_1 = \{v_{11}\}, S_2 = \{v_1, v_2, v_3\}, S_3 = \{v_4, v_5, v_6\}, S_4 = \{v_7, v_8, v_9\} \cup \{v_{10}\}$

For $p = 12$. Let $S_1 = \{v_{12}\}, S_2 = \{v_1, v_2, v_3\}, S_3 = \{v_4, v_5, v_6\}, S_4 = \{v_7, v_8, v_9\} \cup \{v_{10}, v_{11}\}$

For $p = 13$. Let $S_1 = \{v_{13}\}, S_2 = \{v_1, v_2, v_3, v_4\}, S_3 = \{v_5, v_6, v_7, v_8\}, S_4 = \{v_9, v_{10}, v_{11}, v_{12}\}$

For $p = 14$. Let $S_1 = \{v_{14}\}, S_2 = \{v_1, v_2, v_3, v_4\}, S_3 = \{v_5, v_6, v_7, v_8\}, S_4 = \{v_9, v_{10}, v_{11}, v_{12}\} \cup \{v_{13}\}$

For $p = 16$. Let $S_1 = \{v_{16}\}, S_2 = \{v_1, v_2, v_3, v_4, v_5\}, S_3 = \{v_6, v_7, v_8, v_9, v_{10}\}, S_4 = \{v_{11}, v_{12}, v_{13}, v_{14}, v_{15}\}$.

For the above numbers, $|N[S_k]| \geq \lceil \frac{p}{2} \rceil$ and $d(v_i, v_j) = 1 \forall v_i, v_j \in S_k$ for $k = 1, 2, 3, 4$. Therefore $V(G)$ is partitioned into four connected majority dominating sets namely, S_1, S_2, S_3 and S_4 . Hence $d_{CM}(G) = 4$.

Case (iv): Let $p = 15$ and $p \geq 17$.

Let $p = 15$.

Let $S_1 = \{v_{15}\}, S_2 = \{v_1, v_2, v_3, v_4, v_5\}, S_3 = \{v_6, v_7, v_8, v_9, v_{10}\}, R = \{v_{11}, v_{12}, v_{13}, v_{14}\}$ Such that $|N[S_i]| \geq \lceil \frac{p}{2} \rceil$ for $i=1, 2, 3$. But $|N[R]| < \lceil \frac{p}{2} \rceil$. This implies that $V(W_p)$ can be partitioned into three connected majority dominating sets $\{S_1, S_2, S_3 \cup R\}$. Hence, $d_{CM}(W_p) = 3$.

Let $p \geq 17$.

Let $S_1 = \{v_p\}, S_2 = \{v_1, v_2, \dots, v_{\lceil \frac{p}{2} \rceil - 3}\}$ and $S_3 = \{v_{\lceil \frac{p}{2} \rceil - 2},$

$v_{\lceil \frac{p}{2} \rceil - 1}, \dots, v_{2(\lceil \frac{p}{2} \rceil - 3)}\}$ $S_3 = \{v_{\lceil \frac{p}{2} \rceil - 2}, v_{\lceil \frac{p}{2} \rceil - 1}, \dots, v_{2(\lceil \frac{p}{2} \rceil - 3)}\}$ be the three sets with $d(v_i, v_j) = 1$, for all $v_i, v_j \in S_k$ for $k=2, 3$ and $i \neq j$ and the remaining vertices $R = \{v_{2(\lceil \frac{p}{2} \rceil - 5)}, v_{2(\lceil \frac{p}{2} \rceil - 4)}, \dots, v_{p-1}\}$. Since $\{v_p\}$ is a full degree vertex, $|N[S_1]| > \lceil \frac{p}{2} \rceil$. Since $S_2, S_3 \subseteq V(C_{p-1})$ and $|N[S_2]| = \lceil \frac{p}{2} \rceil - 3 + 3 = \lceil \frac{p}{2} \rceil$ and $|N[S_3]| = 2(\lceil \frac{p}{2} \rceil - 3) - (\lceil \frac{p}{2} \rceil - 3) + 3 = \lceil \frac{p}{2} \rceil$. This implies that $S_k, k = 1, 2, 3$ are all CMD sets of G . But we obtain the vertices $R = \{v_{2(\lceil \frac{p}{2} \rceil - 5)}, v_{2(\lceil \frac{p}{2} \rceil - 4)}, \dots, v_{p-1}\}$, each vertex with $d(v_i, v_j) = 1$, for all $v_i, v_j \in R$ $|N[R]| = (p - 1) - (2(\lceil \frac{p}{2} \rceil - 4) + 3) < \lceil \frac{p}{2} \rceil$. Therefore R is not a CMD set of G always. Hence, the vertex set of G can be partitioned into 3 CMD sets $\{S_1, S_2, S_3 \cup R\}$. Hence, $d_{CM}(W_p) = 3$. $\square$

Proposition 3.8. Let $K_{m,n}, m \leq n$ be a complete bipartite graph with $m + n = p$. Then

$$d_{CM}(G) = \begin{cases} p, & \text{if } n = m, m + 1, m + 2 \\ m, & \text{if } m < n \end{cases}$$

Proof. Let $V_1(G) = \{u_1, u_2, \dots, u_m\}$ and $V_2(G) = \{v_1, v_2, \dots, v_n\}$ be bipartitions of $V(G)$.

Case (i): Let $n = m, m + 1, m + 2$. Then $|N[u_i]| \geq \lceil \frac{p}{2} \rceil$ and $|N[v_i]| = \lceil \frac{p}{2} \rceil$. Therefore each $\{u_i\}, i = 1, 2, \dots, m$ and $\{v_i\}, i = 1, 2, \dots, n$ are CMD sets of G . Hence

$$d_{CM}(G) = m + n = p \text{ if } n = m, m + 1, m + 2.$$

Case (ii): Let $m < n$. Then $n \geq m + 3$.

In $V_1(G)$, each vertex u_i of degree $d(u_i) = n$ and in $V_2(G)$, each vertex v_i of degree $d(v_i) = m$.

If $p = |V_1| + |V_2| = n + m$, then $\lceil \frac{p}{2} \rceil = \lceil \frac{m+3+m}{2} \rceil = \lceil \frac{2m+3}{2} \rceil = m + 2$, when $n=m+3$. Therefore, each u_i covers at least $|N[u_i]| = n + 1 = m + 4$, vertices, when $n=m+3$.

Since every vertex $\{u_i\}$ is connected and $|N[u_i]| > \lceil \frac{p}{2} \rceil$, each $\{u_i\}, i = 1, 2, \dots, m$ in $V_1(G)$ is a CMD set for G . The vertices in $V_2(G)$ form a majority dominating set for G but the induced graph could not be connected. So, the vertices of $V_2(G)$ could not form a connected majority dominating set for G . Hence, $d_{CM}(G) = m$ if $n \geq m + 3$ and $m < n$. $\square$

4. Bounds For $d_{CM}(G)$

Theorem 4.1. For any graph G with p vertices, $d_{CM}(G) = p$ if and only if $\delta(G) \geq \lceil \frac{p}{2} \rceil - 1$.

Proof. Let, $d_{CM}(G) = p$. Then every vertex $u \in V(G)$ forms a connected majority dominating set for G . Thus $d(u) \geq \lceil \frac{p}{2} \rceil - 1$, for every $u \in V(G)$. Hence $\delta(G) \geq \lceil \frac{p}{2} \rceil - 1$. Conversely, suppose $\delta(G) \geq \lceil \frac{p}{2} \rceil - 1$. Then $d(u) = \delta(G)$, for any $u \in V(G)$ and $d(v) \geq \delta(G) \geq \lceil \frac{p}{2} \rceil - 1$, $d(u) = \delta(G)$, for any $u \in V(G)$ and $d(v) \geq \delta(G) \geq \lceil \frac{p}{2} \rceil - 1$, for every $v \in V(G)$. It implies that each vertex v of G is a CMD set of G . Hence $d_{CM}(G) = p$. $\square$

Result 4.2. Given any positive integer k , there exists a graph G which is not complete for which $d_{CM}(G) = k$.

Proof. Let $G = K_{\lfloor \frac{k}{2} \rfloor, \lceil \frac{k}{2} \rceil}$. Since each vertex of G is a connected majority dominating set, $d_{CM}(G) = k$. $\square$

Proposition 4.3. Let T be a tree with $p \geq 3$ and $e > 2$, pendants. If $diam(T) = 2$ or 3 , then $d_{CM}(T) = 1$ or 2 .

Proof. Let $diam(T) = 2$. Then the tree $T = K_{1,p-1}$ and $\gamma_{CM}(T) = 1$. Therefore let $S = \{u\}$, where u is the central vertex of T . Since all other vertices are pendants, $V - \{u\}$ could not form a CMD set for G . Hence, $d_{CM}(G) = 1$. Let $diam(T) = 3$. Then T is a double star. Let x and y be the centers of T and $\{u_1, u_2, \dots, u_m\}$ be the set of pendants adjacent to x and $\{v_1, v_2, \dots, v_n\}$ be the set of pendants adjacent to y . Let $|V(T)| = m + n + 2$.

Case (i): Let $m < n$ and $n \geq m + 3$ is even.

Let $S_1 = \{y\}$ and $S_2 = \{x\}$. Then $|N[S_1]| \geq \lceil \frac{p}{2} \rceil$ and $|N[S_2]| < \lceil \frac{p}{2} \rceil$. Here S_1 is the only CMD set of G and S_2 could not be a connected majority dominating set of G . Also, all pendant sets $\{u_i\}$ and $\{v_i\}$ could not form a connected majority dominating set of G . It implies that there is only one connected majority domatic set in $V(G)$. Hence $d_{CM}(T) = 1$.

Case (ii): Let $n = m, m + 1, m + 2$.

The only connected majority domatic partitions of $V(T)$ are the centres of T i.e., $S_1 = \{x\}$ and $S_2 = \{y\}$ such that $|N[S_1]| \geq \lceil \frac{p}{2} \rceil$ and $|N[S_2]| \geq \lceil \frac{p}{2} \rceil$. This implies that $d_{CM}(T) = 2$. $\square$

Proposition 4.4. If G is a complete graph then $d_{CM}(G) = d_M(G) = d_C(G)$.

Proof. Since every vertex of a complete graph G is a full degree vertex, each vertex is a connected dominating set or majority dominating set or connected majority dominating set of G . Hence $d_{CM}(G) = d_M(G) = d_C(G) = p$ if $G = K_p$. $\square$

Proposition 4.5. If a graph G has exactly one full degree and others are pendant vertices then $d_{CM}(G) = d_C(G)$.

Corollary 4.6. For a subdivision graph $S(K_{1,p-1})$ with each edge is divided by any number of times, then $d_{CM}(G) = d_C(G) = 1$.

Proof. Since the full degree vertex v of G is a connected dominating set and also connected majority dominating set of G , the remaining all vertices could not form a connected dominating set or connected majority dominating set for G . Therefore, the vertex set $V(G)$ cannot be partitioned into either connected dominating set or connected majority dominating set of G . Hence, $d_{CM}(G) = d_C(G) = 1$. $\square$

Proposition 4.7. For any graph G ,

(i). $d_{CM}(G) \leq d_M(G)$ and

(ii). $d_{CM}(G) \geq d_C(G)$ (i) $d_{CM}(G) \geq d_M(G)$ and (ii) $d_{CM}(G) \geq d_C(G)$.

Proof. (i). Since every connected majority dominating set of G is a Majority Dominating set of G , $\gamma_{CM}(G) \geq \gamma_M(G)$. Hence, $d_{CM}(G) \leq d_M(G)$.

(ii) Since every Connected Dominating set of G is a Connected Majority Dominating set of G , $\gamma_C(G) \geq \gamma_{CM}(G)$. Hence, $d_C(G) \leq d_{CM}(G)$. $\square$

Proposition 4.8. For any graph G , $d_C(G) \leq d_{CM}(G) \leq d_M(G)$.

Proof. It follows from Proposition 4.7. $\square$

Proposition 4.9. Let G be any 3-regular graph with ‘ p ’ vertices. Then

$$d_{CM}(G) = \begin{cases} p, & \text{if } p = 4, 6, 8 \\ 5, & \text{if } p = 12 \\ 4, & \text{if } p = 10 \text{ and } p \geq 14. \end{cases}$$

Proof. Let $V(G) = \{v_1, \dots, v_p\}$ be the set of ‘ p ’ vertices in G .

Case (i) : Let $p = 4, 6, 8$. Since each vertex $\{v_i\}$ is a CMD set of G , $d_{CM}(G) = p$.

Case (ii): Let $p = 12$. Since $\gamma_{CM}(G) = \lfloor \frac{p-1}{4} \rfloor$, consider $S_1 = \{v_1, v_6\}$, $S_2 = \{v_2, v_7\}$, $S_3 = \{v_3, v_8\}$, $S_4 = \{v_4, v_9\}$, $S_5 = \{v_5, v_{10}\}$ with $d(v_i, v_j) = 1$, for every $v_i, v_j \in S_k$, $k = 1, \dots, 5$ and $i \neq j$. Then $|N[S_k]| \geq \lceil \frac{p}{2} \rceil$, for every $k = 1, \dots, 5$. Therefore the sets S_k are the only CMD sets of G . In this structure, the remaining vertices $R = \{v_{11}, v_{12}\}$ with distance $d(v_{11}, v_{12}) = 2$, which is not connected. Therefore, there are $\{S_1, S_2, S_3, S_4\}$,

$S_5 \cup R\}$ are the five connected majority domatic partitions of G . Hence, $d_{CM}(G) = 5$.

Case (iii): (a) Let $p = 10$. Since $\gamma_{CM}(G) = \lfloor \frac{p-1}{4} \rfloor$, consider $S_1 = \{v_1, v_5\}$, $S_2 = \{v_2, v_6\}$, $S_3 = \{v_3, v_7\}$, $S_4 = \{v_4, v_8\}$ with $d(v_i, v_j) = 1$, for every $v_i, v_j \in S_k$, $k = 1, \dots, 4$ and $i \neq j$. Then $|N[S_k]| \geq \lceil \frac{p}{2} \rceil$, for every $k = 1, \dots, 4$. Then the sets S_k are the only CMD sets of G . In this structure, the remaining vertices $R = \{v_9, v_{10}\}$ with distance $d(v_9, v_{10}) \geq 2$, which is not connected. Therefore, there are $\{S_1, S_2, S_3, S_4 \cup R\}$ are the four connected majority domatic partitions of G . Hence, $d_{CM}(G) = 4$.

Case (iii): (b) Let $P \geq 14$. In the structure, v_1 is adjacent to three vertices namely, $v_{\frac{p}{2}}, v_{\frac{p}{2}+1}, v_{\frac{p}{2}+2}$. Similarly, v_2 is adjacent to three vertices $v_{\frac{p}{2}+1}, v_{\frac{p}{2}+2}, v_{\frac{p}{2}+3}$. Similarly, v_3 is adjacent to three vertices $v_{\frac{p}{2}+2}, v_{\frac{p}{2}+3}, v_{\frac{p}{2}+4}$ and so on. $S_1 = \{v_1, v_{\frac{p}{2}}, v_{p-1}, v_{\frac{p}{2}-2}, \dots\}$, $S_2 = \{v_2, v_{\frac{p}{2}+1}, v_p, v_{\frac{p}{2}-1}, \dots\}$, $S_3 = \{v_3, v_{\frac{p}{2}+4}, v_5, v_{\frac{p}{2}+6}, \dots\}$, $S_4 = \{v_4, v_{\frac{p}{2}+5}, v_6, v_{\frac{p}{2}+7}, \dots\}$ such that $d(v_i, v_j) = 1$, for every $v_i, v_j \in S_k$, $k = 1, \dots, 4$ and $i \neq j$ and $|N[S_k]| \geq \lceil \frac{p}{2} \rceil$. Therefore S_k are the only CMD sets of G and the remaining vertices form the set $R = V(G) - \bigcup_{i=1}^4 S_i$ at a distance 2 which is not connected. Therefore, there are $\{S_1, S_2, S_3, S_4 \cup R\}$ are the four connected majority domatic partitions of G . Hence, $d_{CM}(G) = 4$. $\square$

5. Conclusion

In this article, researcher introduced and discussed new type of parameter of a graph G . Connected majority domatic number of a graph is defined and it is determined for some

classes of graphs. This concept is useful in communication network and in many real-life situations. Also, this concept can be extended to product graphs.

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