Distance measure and its application

Thangaraj Beaula^{1*} and Selvakumari²

Abstract

In this paper new distance measure between triangular fuzzy numbers is established, using this distance measure, critical path is computed for fuzzy project network, further illustrated by a numerical example.

Keywords

Distance Measure, triangular fuzzy number, Critical path.

AMS Subject Classification

90B10, 90B15, 90B50, 90C06.

^{1,2}P.G and Research Department of Mathematics, TBML College, Porayar-609307, India.

*Corresponding author: ¹ edwinbeaula@yahoo.co.in

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1. Introduction

The theory of fuzzy sets was first introduced by Zadeh in 1965 which opened new gates to the field of applied mathematics. The concept of fuzzy numbers was also introduced by Zadeh which is widely spread in the area of research in fuzzy mathematics. The distance between the fuzzy numbers plays an important role in fields like remote sensing, data mining, pattern recognition. Usually the fuzzy distance is computed in terms of crisp distance. Voxman[6] was the first to coin fuzzy distance between fuzzy numbers.

In this paper a new distance measure between triangular fuzzy numbers is established and using the new distance measure critical path is computed for fuzzy project network. A new algorithm is developed to obtain the critical path, illustrated by a numerical example.

2. Preliminaries

Definition 2.1. A fuzzy set on a set X is a function $\mu : X \rightarrow [0, 1]$. The support of μ is the closure of the set $\{x \in X / \mu(x) > 0\}$ usually denoted by $\text{supp}\mu$.

Definition 2.2. A fuzzy number is a fuzzy set $\mu : \mathbb{R} \rightarrow [0, 1]$ on $\mathbb{R}$ satisfying

1. μ is upper semi-continuous.
2. $\text{supp}\mu$ is a closed and bounded interval.
3. If $\text{supp}\mu = [a, b]$ then there exist $c, d, a \leq c \leq d \leq b$ such that μ is increasing on the interval $[a, c]$, equal to 1, on the interval $[c, d]$ and decreasing on the interval $[d, b]$.

Definition 2.3. Let F denote the family of all fuzzy numbers. If $\mu \in F$ then for each $\alpha, 0 < \alpha \leq 1$, the α -cut of μ is defined by $\mu_\alpha = \{x \in X / \mu(x) \geq \alpha\}$

The α -cut representation of μ is the pair of functions $(L(\alpha), R(\alpha))$ defined by

$$L(\alpha) = \begin{cases} \inf\{x / x \in \mu_\alpha\}, & \text{if } \alpha > 0 \\ \inf\{x / x \in \text{supp}\mu\}, & \text{if } \alpha = 0. \end{cases}$$

and

$$R(\alpha) = \begin{cases} \sup\{x / x \in \mu_\alpha\}, & \text{if } \alpha > 0 \\ \sup\{x / x \in \text{supp}\mu\}, & \text{if } \alpha = 0. \end{cases}$$

Definition 2.4. A fuzzy number $\tilde{A} = (a, b, c)$ is said to be a triangular fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x=b \\ \frac{c-x}{c-b}, & b \leq x \leq c \\ 0, & \text{otherwise.} \end{cases}$$

The α -cut representation of a triangular fuzzy number $\tilde{A}$ is $A_\alpha = [A^L(\alpha), A^R(\alpha)]$

$$A_\alpha = [a + \alpha(b-a), c - \alpha(c-b)]$$

2.5 Fuzzy Distance (Voxman)

Describe the fuzzy distance measure by Voxman[6] as a function $\Delta: F \times F \rightarrow F$

$$\Delta(\mu, \nu)(z) = \text{Supmin}\{\mu(x), \nu(y) \mid |x - y| = z\}$$

For each pair of fuzzy numbers μ and ν , let $\Delta_{\mu\nu}$ denote the fuzzy number $\Delta(\mu, \nu)$. If the α -cut representations of μ and ν are $(A_1^L(\alpha), A_1^R(\alpha))$ and $(A_2^L(\alpha), A_2^R(\alpha))$ respectively then the α -cut representation of $\Delta_{\mu\nu}$, $(L(\alpha), R(\alpha))$ is given by

$$L(\alpha) = \max\{A_2^L(\alpha) - A_1^R(\alpha), 0\} \text{ if } 1/2(A_1^L(1) + A_1^R(1)) \leq 1/2(A_2^L(1) + A_2^R(1))$$

$$= \max\{A_1^L(\alpha) - A_2^R(\alpha), 0\} \text{ if } 1/2(A_2^L(1) + A_2^R(1)) \leq 1/2(A_1^L(1) + A_1^R(1))$$

and

$$R(\alpha) = \max\{A_1^R(\alpha) - A_2^L(\alpha), A_2^R(\alpha) - A_1^L(\alpha)\}$$

3. New distance between two fuzzy numbers

Definition 3.1. Let A_1 and A_2 be two arbitrary fuzzy numbers with α -cut representation $\alpha_{A_1} = [A_1^L(\alpha) - A_1^R(\alpha)]$ and $\alpha_{A_2} = [A_2^L(\alpha) - A_2^R(\alpha)]$ respectively. The new distance between A_1 and A_2 is defined as $d(\alpha_{A_1}, \alpha_{A_2}) = [(a.c - A_1^L(\alpha))^2 + (A_1^R - a.c)^2 - [(a.c - A_2^L(\alpha))^2 + (A_2^R - a.c)^2]]^{1/2}$ and

$$D(A_1, A_2) = \int_0^{1/2} (1 - \alpha)d(\alpha_{A_1}, \alpha_{A_2})d\alpha + \int_{1/2}^1 \alpha d(\alpha_{A_1}, \alpha_{A_2})d\alpha \quad (3.1)$$

$$\text{where } a.c = \frac{L(\alpha) + R(\alpha)}{2}$$

Property 3.2. $d^2(\alpha_{(A_1+A_2)}, \tilde{0}) < d^2(\alpha_{A_1}, \tilde{0}) + d^2(\alpha_{A_2}, \tilde{0})$ for every $\alpha_{A_1}, \alpha_{A_2} \in F$

Proof.

$$\begin{aligned} d^2(\alpha_{(A_1+A_2)}, \tilde{0}) &= (a.c - (A_1^L + A_2^L))^2 + ((A_2^R + A_2^R) - a.c)^2 \\ &= (a.c)^2 + (A_1^L + A_2^L)^2 - 2a.c(A_1^L + A_2^L) \\ &\quad + (A_1^R + A_2^R)^2 + (a.c)^2 - 2a.c(A_1^R + A_2^R) \\ &= (a.c)^2 + (A_1^L)^2 + (A_2^L)^2 + 2A_1^L A_2^L \\ &\quad - 2a.cA_1^L - 2a.cA_2^L \\ &\quad + (A_1^R)^2 + (A_2^R)^2 + 2A_1^R A_2^R + (a.c)^2 \\ &\quad - 2a.cA_1^R - 2a.cA_2^R \\ &= (a.c)^2 + (A_1^L)^2 - 2a.cA_1^L + (A_1^R)^2 + (a.c)^2 \\ &\quad - 2a.cA_1^R - (a.c)^2 + (a.c)^2 + (A_2^L)^2 \\ &\quad - 2a.cA_2^L - (a.c)^2 + (a.c)^2 + (A_2^R)^2 - 2a.cA_2^R \\ &\quad + 2(A_1^L A_2^L + A_1^R A_2^R) \\ &= (a.c - A_1^L)^2 + (A_1^R - a.c)^2 + (a.c - A_2^L)^2 \\ &\quad + (A_2^R - a.c)^2 - 2[(a.c)^2 + A_1^L A_2^L + A_1^R A_2^R] \end{aligned}$$

$$d^2(\alpha_{(A_1+A_2)}, \tilde{0}) < d^2(\alpha_{A_1}, \tilde{0}) + d^2(\alpha_{A_2}, \tilde{0}) \quad \square$$

Property 3.3. $d^2(kA_1, kA_2) = |k|^2 d^2(A_1, A_2)$ for every $A_1, A_2 \in F$ and $k \in R$

$$\begin{aligned} \text{Proof. } d^2(kA_1, kA_2) &= (ka.c - kA_1^L)^2 + (kA_1^R - ka.c)^2 - [(ka.c - kA_2^L)^2 + (kA_2^R - ka.c)^2] \\ &= (ka.c)^2 + (kA_1^L)^2 - 2k^2 a.c A_1^L + (kA_1^R)^2 + (ka.c)^2 - 2k^2 a.c A_1^R \\ &\quad - (ka.c)^2 - (kA_2^L)^2 + 2k^2 a.c A_2^L - (kA_2^R)^2 - (ka.c)^2 + 2k^2 a.c A_2^R \\ &= k^2 [(a.c)^2 + A_1^L^2 - 2a.c A_1^L + A_1^R^2 + (a.c)^2 - 2a.c A_1^R - ((a.c)^2 + A_2^L^2 \\ &\quad - 2a.c A_2^L + A_2^R^2 + (a.c)^2 - 2a.c A_2^R)] \\ &= k^2 [(a.c - A_1^L)^2 + (A_1^R - a.c)^2 - (a.c - A_2^L)^2 - (A_2^R - a.c)^2] \\ &= |k|^2 d^2(A_1, A_2) \end{aligned}$$

Here $\tilde{L}_1 > \tilde{L}_2$ if and only if $A(\tilde{L}_{max} > \tilde{L}_1) > A(\tilde{L}_{max} > \tilde{L}_2)$

Since $b_i < x < c_i$,

$$y_d = \frac{(c_i - x)}{(c_i - b_i)} \Rightarrow (c_i - b_i)y_d = c_i - x \Rightarrow x = c_i - (c_i - b_i)y_d \quad (3.2)$$

Since $a < x < b$,

$$y_d = \frac{(x - a)}{(b - a)} \Rightarrow (b - a)y_d = x - a \Rightarrow x = a + (b - a)y_d \quad (3.3)$$

Equating (3.2) & (3.3)

$$c_i - (c_i - b_i)y_d = a + (b - a)y_d$$

$$y_d = \frac{(c_i - a)}{(b - a) + (c_i - b_i)} \quad \square$$

4. Application in critical path method

Notation 4.1.

N = The set of all nodes in a project network.

A_{ij} = The Activity between nodes i and j .

$F\tilde{E}T_{ij}$ = The fuzzy activity time of A_{ij} .

$F\tilde{E}S_j$ = The earliest starting fuzzy time of node j .

$F\tilde{L}F_i$ = The latest finishing fuzzy time of node i .

$F\tilde{T}S_{ij}$ = The total slack fuzzy time of A_{ij} .

Definition 4.2. (Forward Pass Calculation)

The earliest starting fuzzy time in the project network is calculated as follows:

Set the initial node to zero by starting (i.e) $F\tilde{E}S_1 = (0, 0, 0)$ and

$$F\tilde{E}S_j = \max_i \{F\tilde{E}S_i(+) F\tilde{E}T_{ij}\}, j \neq i, j \in N$$

where i = number of preceding nodes.

Ranking value is utilized to identify the maximum value.

Earliest finishing fuzzy time = Earliest starting fuzzy time (+) fuzzy activity time.

Definition 4.3. (Backward Pass Calculation)

To calculate the latest finishing time in the project network.

Set $F\tilde{L}F_n = F\tilde{E}S_n$ and $F\tilde{L}F_i = \min_j \{F\tilde{L}F_j(-) F\tilde{E}T_{ij}\}, i \neq n, i \in N$ and

where j = number of succeeding nodes.

4.4 New algorithm to find critical path

1. **Step1:** Construct an acyclic network $G(V,E)$, where V is the set of vertices and E is the set of edges. Let the starting node always be $(0,0,0)$ denoted by a triangular fuzzy number.
2. **Step2:** Calculate the ES_i for each node $i = 1, 2, \dots, n$
3. **Step3:** Calculate the α -cut for each ES_i of each node.
4. **Step4:** Calculate the distance between the two nodes using Definition(3.1) which yields the duration for each activity which is a crisp number.
5. **Step5:** The fuzzy network converted to a crisp network
6. **Step6:** Compute ES_i , LF_j and total float.
7. **Step7:** The activity with zero total float is the critical activity.
8. **Step8:** The path constituting critical activities form the critical path.

4.5 Numerical Example

Consider the example

The corresponding fuzzy duration times are calculated using distance Definition(3.1) as follows

$$\begin{aligned}
 D^2(A_1, A_2) &= \int_0^{1/2} (1-\alpha)[(a.c - A_1^L(\alpha))^2 \\
 &\quad + (A_1^R(\alpha) - a.c)^2 - [(a.c - A_2^L(\alpha))^2 + (A_2^R(\alpha) - a.c)^2]] d\alpha \\
 &\quad + \int_{1/2}^1 \alpha[(a.c - A_1^L(\alpha))^2 + (A_1^R(\alpha) - a.c)^2 \\
 &\quad - [(a.c - A_2^L(\alpha))^2 + (A_2^R(\alpha) - a.c)^2]] d\alpha \\
 &= \int_0^{1/2} (1-\alpha) \left[-\left(\frac{7-\alpha}{2} - a\alpha - 2\right)^2 - \left(5 - 2\alpha - \frac{7}{2} + \frac{\alpha}{2}\right)^2 \right] d\alpha \\
 &\quad + \int_{1/2}^1 \alpha \left[-\left(\frac{7-\alpha}{2} - a\alpha - 2\right)^2 - \left(5 - 2\alpha - \frac{7}{2} + \frac{\alpha}{2}\right)^2 \right] d\alpha \\
 D_2(A_1, A_2) &= -1.1719 \\
 D_2(A_2, A_3) &= \int_0^{1/2} (1-\alpha) \left[\left(\frac{-\alpha+7}{2} - (\alpha+2)\right)^2 + \left(5 - 2\alpha - \left(\frac{-\alpha+7}{2}\right)\right)^2 \right. \\
 &\quad \left. - \left(\frac{3\alpha-39}{2} - (7\alpha+4)\right)^2 - \left(25 - 4\alpha - \left(\frac{3\alpha+39}{2}\right)\right)^2 \right] d\alpha \\
 &\quad + \int_{1/2}^1 \alpha \left[\left(\frac{-\alpha+7}{2} - (\alpha+2)\right)^2 + \left(5 - 2\alpha - \left(\frac{-\alpha+7}{2}\right)\right)^2 \right. \\
 &\quad \left. - \left(\frac{3\alpha+39}{2} - (7\alpha+14)\right)^2 - \left(25 - 4\alpha - \left(\frac{3\alpha+39}{2}\right)\right)^2 \right] d\alpha \\
 &= -14.583
 \end{aligned}$$

Similarly,

$$D_2(A_2, A_4) = -2.0837$$

$$D_2(A_3, A_4) = 12.5$$

$$D_2(A_1, A_4) = 3.255$$

The Critical path is calculated as follows

Activity	Normal time	Earliest		Latest		TF	FF
		Start	Finish	Start	Finish		
1-2	1.17	0	1.17	0.01	1.18	0.02	0
1-3	3.26	0	3.26	0	3.26	0	0
2-4	14.58	1.17	15.75	1.18	15.76	0.01	0
2-3	2.08	1.17	3.25	1.18	3.26	0.01	0
3-4	12.5	3.26	15.76	3.26	15.76	0	0

The critical path is 1-3-4.

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