Topological *BCH* – groups

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Abstract

In our paper [7], we introduced topological structures on BCH-algebras. In our paper [9], we discussed the topologies on set of Left maps on BCH- algebras. In this paper we introduce the notion of BCH- groups and discuss some basic topological concepts on it

Keywords

BCH-groups, *BCH* – subgroups, Topological *BCH* – groups.

AMS Subject Classification

06F35, 54A10, 54A05.

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Article History: Received 21 December 2018; Accepted 11 February 2019

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1. Introduction

Y. Imai and K. Iseki introduced two classes of abstract algebras: BCK -algebras and BCI- algebras ([4], [5]). In ([2], [3]) Q.P.Hu and X.Li introduced a wider class of abstract algebras : BCH-algebras . K. H. Dar and M.Akram [9] discussed some algebraic concepts on BCH- algebras . Groups recur throughout mathematics and the methods of group theory have influenced many parts of algebra. Almost all structures of abstract algebra are special case of groups. Motivated by this, we discuss the topological concepts in BCH-groups.

In this paper, we introduce the notion of topological BCH-groups and discuss some basic interesting results.

2. Preliminaries

In this section, we recall some definitions and basic results that are required for our work.

Definition 2.1. [3] A *BCH*-algebra $(X, *, 0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following conditions:

1. $x * x = 0$
2. $(x * y) * z = (x * z) * y$
3. $x * y = 0$ and $y * x = 0 \Rightarrow x = y$ for all $x, y, z \in X$.

Definition 2.2. [8] Let $(X, *, 0)$ be a *BCH*-algebra and A be any subset of X is said to be a *BCH*-subalgebra, if for every $x, y \in X$ $x * y \in A$.

Definition 2.3. [1] Let X be a *BCH*-algebra and U, V be any non empty subsets of X . We define a subset $U * V$ of X as follows $U * V = \{x * y / x \in U, y \in V\}$.

Definition 2.4. [8] Let $(X, *)$ be a *BCH*-algebra, and $a \in X$. Define a left map $L_a : X \rightarrow X$ by, $L_a(x) = a * x$, for all $x \in X$.

Remark 2.5. [6] The members of τ are called open sets in X . The complement of $A \in \tau$, that is $X \setminus A$ is called a closed set in X .

The open set of an element $x \in X$ is a member of τ containing x .

Definition 2.6. [7] Let $(X, *)$ be a *BCH*-algebra and τ a topology on X . Then $X = (X, *, \tau)$ is called a topological *BCH* – algebra, if the operation $'*'$ is continuous, or equivalently, for any $x, y \in X$ and any open set W of $x * y$ then there exist two open sets U and V of x and y respectively such that $U * V \subseteq W$.

Definition 2.7. [6] A topological group G is a group that is also a topological space, such that the map of $G \times G$ into G sending $x \times y$ into $x * y$, and the map G into G sending x into x^{-1} are continuous.

3. Topolpgical BCH – groups

In this section, we introduce the concept of BCH-groups and discuss some topological properties on it.

Theorem 3.1. *In a BCH-algebra $(X, *, 0)$ if $0 * x = x$, for all $x \in X$ then $(X, *, 0)$ forms an abelian group and every element has involution.*

Proof. 1.commutative: $x * y = 0 * (x * y)$
 $= (x * x) * (x * y)$ since $x * x = 0$
 $= (x * (x * y)) * x$ since $(x * y) * z = (x * z) * y$
 $= y * x$ since $(x * (x * y)) = y$
 $\therefore x * y = y * x$ (1)
 2. Closure: Since $*$ is a binary operation, $x * y \in X$
 3. Associative : $(x * y) * z = (0 * (x * y)) * z$ since $0 * x = 0$
 $= ((x * x) * (x * y)) * z$ since $x * x = 0$
 $= ((x * (x * y)) * x) * z$ since $(x * y) * z = (x * z) * y$
 $= (y * x) * z$ since $x * (x * y) = y$
 $= (y * z) * x$ since $(x * y) * z = (x * z) * y$
 $= x * (y * z)$ by(1)
 $\therefore (x * y) * z = x * (y * z)$
 4.Identity: Clearly, $0 * x = x * 0 = x$, for all $x \in X$.
 Here, 0 is an identity element of X.
 5.Inverse: Now, $x * x = 0$, for all $x \in X$
 If there exist another element $y \in X$, such that
 $x * y = y * x = 0$, then $x = y$
 That is Every element in X has involution. $\square$

Notation :

The abelian group $(X, *, 0)$ is called a BCH-group.

Example 3.2. Let $A = \{5, 7\}$ and $X = \{\phi, \{5\}, \{7\}, A\}$, the powerset of A. The set (X, Δ, ϕ) forms a BCH-group with the binary operation $B \Delta C = (B \setminus C) \cup (C \setminus B)$, where $B, C \subseteq A$.

Remark 3.3. In general, every BCH-group is a group. But a group is not a BCH-group.

Example 3.4. Let $\mathbb{R}$, the set of all real numbers with the binary operation $+$ $\mathbb{R}$ is a group but not a BCH-group under $+$.

Theorem 3.5. Let X be a BCH-group and A, any subset of X. Then A is closed under the operation $*$ if and only if A is a BCH-subgroup.

Proof. Let X be a BCH-group and A a subset of X be closed under the operation $*$.

Since for every $x \in A$, $x * x = 0$, $0 \in A$.

Clearly, $0 * x = x * 0 = x$.

$\Rightarrow$ A has an identity element.

In X, each element has involution $\Rightarrow$ each element in A also has an involution.

Therefore there exist a unique inverse for all $a \in A$.

Associative and commutative properties holds in X. This implies that the associative condition also holds in A. Therefore A is a BCH – group under the same operation in X. Conversely, Let A be a BCH-subgroup of X. Trivially, A is closed under binary operation $*$. $\square$

Definition 3.6. A topological BCH-group G is a BCH-group that is also a topological space, such that the map of $G \times G$ into G sending $x \times y$ into $x * y$, and the map G into G sending x into x^{-1} are continuous.

Example 3.7. Consider a BCH-group $(X = \{0, a, b, c\}, *, 0)$ with the following Cayley table,

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

And $\tau = \{X, \phi, \{0, 1\}, \{2, 3\}\}$.

Clearly, the maps $x \times y$ into $x * y$ and x into x^{-1} are continuous on X. Therefore, $(X, *, \tau)$ is a topological BCH-group.

Theorem 3.8. Let $(X, *, \tau)$ be a topological BCH-algebra with $0 * x = x$, for all $x \in X$. Then $(X, *, \tau)$ is a topological BCH-group.

Proof. Let X be a topological BCH-algebra with $0 * x = x$, for all $x \in X$.

From theorem 3.1, X is a BCH-group.

Now define a map $f : X \times X \rightarrow X$ by,

$$(x, y) \rightarrow x * y.$$

Since X is a topological BCH-algebra, f is continuous.

Again define the map $g : X \rightarrow X$ by,

$$x \rightarrow x^{-1}.$$

Since $x^{-1} = x$, for all x implies g is a constant function.

$\Rightarrow$ g is continuous. $\square$

Remark 3.9. In general, every element in a BCH-group is involutive and so the map x into x^{-1} in the definition 3.6 is always continuous in any BCH-group. Therefore it is sufficient to show that the map $x \times y$ into $x * y$ is continuous, which provides a topological BCH – group.

Theorem 3.10. Every subalgebra in a topological BCH-group, under subspace topology, forms a topological BCH – group.

Proof. Let X be a topological BCH-group, and A be any subalgebra of X.

Since Every BCH-subalgebra of X is a BCH-group, we want to prove in the map f from $A \times A$ into A given by $f(x, y) = x * y$ is continuous.

For any $U \in \tau$ and $a, b \in A$ and $U \cap A$ is an arbitrary open set of $a * b$ in A, then there exist an open set V and W of a and b in X such that $V * W \subseteq U$.

Thus $V \cap A$ and $W \cap A$ are the open sets of a and b in A such that
 $(V \cap A) * (W \cap A) \subseteq (U \cap A)$. Hence f is continuous. $\square$

Definition 3.11. Let X be a BCH-group and A be any subset of X . The left coset of A is defined as,

$$a * A = \{a * h \mid h \in A\}, \quad a \in X.$$

Theorem 3.12. Let X be a topological BCH-group, and A be a BCH-subgroup of X . If any left coset of A is open, then A is open.

Proof. Let X be a topological BCH-group, and A be a any BCH-subgroup of X .

Given $x * A$ is open for any $x \in X$.

We want to show that, A is open.

Let $a \in A$,

$$\Rightarrow x * a \in x * A$$

Since X is a topological BCH-group, for any open set W of $x * a$, there exist a two open sets U and V of x and a respectively such that $U * V \subseteq W$.

Since $x * A$ is open, $W \subseteq x * A$.

This implies that, $U * V \subseteq x * A$.

Claim: $V \subseteq A$

Suppose $V \not\subseteq A$.

Then there exist an atleast one element $b \in V$ but $b \notin A$.

$$\Rightarrow x * b \in U * V \subseteq x * A = \{x * a \mid a \in A\}$$

$$\Rightarrow b \in A.$$

which is a contradiction to our assumption.

$$\Rightarrow V \subseteq A. \text{ Hence, } A \text{ is open.} \quad \square$$

Theorem 3.13. Let X be a topological BCH-group and A, B be any two BCH-subgroup of X . If $A * B$ is open then both A and B are open.

Proof. Let X be a topological BCH-group and A, B be BCH-subgroups of X .

Let $a \in A$ and $b \in B$, so that $a * b \in A * B$.

Consider an open set W of $a * b$. Since $A * B$ is open, we can prove that there exist open sets U and V of a and b respectively such that $U * V \subseteq W$.

Since $A * B$ is open, $W \subseteq A * B$.

$$\Rightarrow U * V \subseteq A * B$$

Claim: $U \subseteq A$

Suppose not. That is assume that $U \not\subseteq A$.

Then there exist an element $c \in U$ and $c \notin A$

$$\Rightarrow c * V \subseteq U * V \subseteq A * B.$$

$$\Rightarrow c \in A$$

which contradicts the fact that $c \notin A$. Therefore, $U \subseteq A$.

Similarly, we can prove that $V \subseteq B$.

Hence, both A and B are open. $\square$

Theorem 3.14. In a topological BCH-group, every open BCH-subgroup is closed.

Proof. Let X be a topological BCH-group and A be an open BCH-subgroup.

Let $a \in X \setminus A$,

Since $x * x = 0$ and A is open, there exist an open set U of X such that $U * U \subseteq A$

Claim : $U \subseteq A$.

Suppose $U \not\subseteq A$. Then $U \subseteq X \setminus A$, there exist an element $y \in U \cap A$.

$$\Rightarrow z * y \in U * U \subseteq A, \text{ for all } z \in U.$$

Since A is a BCH-subgroup, $z = y * (y * z) \in A$.

$$\Rightarrow U \subseteq A, \text{ contradicting } U \not\subseteq A.$$

Hence, A is closed. $\square$

4. Conclusion

In this paper we discuss only preliminary concepts on topological BCH-groups. If we extend this concept we can get interesting results on it.

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ISSN(P):2319 – 3786

Malaya Journal of Matematik

ISSN(O):2321 – 5666