S –vertex colouring on categorical product of S –valued graphs

P. Victor^{1*} and M. Chandramouleeswaran²

Abstract

In this paper, we reformulate the definition of S –vertex colouring of S –valued graphs, and study the S –vertex colouring in categorical product of S –valued graphs and prove some simple results.

Keywords

S –vertex colouring, S –valued Graphs, categorical Product of S –valued graphs.

AMS Subject Classification

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¹Department of Mathematics, Mohamed Sathak Engineering College, Kilakarai-623806, Tamil Nadu, India.

²Department of Mathematics, Sri Ramanas College of Arts and Science for Women, Aruppukottai-626134, Tamil Nadu, India.

*Corresponding author: ¹ antonyraj96@gmail.com; ² moulee59@gmail.com

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1. Introduction

Recently in 2015 [4], the authors Chandramouleeswaran et.al introduced the notion of S –valued graphs. In this paper, we reformulate the definition for vertex colouring in [5] and the definition of edge colouring in [6] and discuss the colouring on categorical product of two S –valued graphs. Also we prove some fundamental results on S –vertex colouring on S –valued graphs.

2. Preliminaries

In this section, we recall the basic definitions that are needed for our sequel

Definition 2.1. [3] A semiring $(S, +, \cdot)$ is an algebraic system with a non-empty set S together with two binary operations $+$ and $\cdot$ such that

1. $(S, +, \cdot)$ is a monoid.
2. $(S, \cdot)$ is a semigroup.

3. For all $a, b, c \in S$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$

4. $0 \cdot x = x \cdot 0 = 0 \quad \forall x \in S$.

The element $0 \in S$ is the additive identity as well as the zero of the semiring S .

Definition 2.2. [3] Let $(S, +, \cdot)$ be a semiring. $\preceq$ is said to be a canonical pre-order if for $a, b \in S$, $a \preceq b$ if and only if there exists $c \in S$ such that $a + c = b$.

Definition 2.3. [4] Let $G = (V, E \subset V \times V)$ be a given graph with $V, E \neq \emptyset$. For any semiring $(S, +, \cdot)$, a semiring valued graph (or a S -valued graph) G^S is defined to be the graph $G^S = (V, E, \sigma, \psi)$ where $\sigma : V \rightarrow S$ and $\psi : E \rightarrow S$ is defined by $\psi(x, y) = \begin{cases} \min\{\sigma(x), \sigma(y)\} & \text{if } \sigma(x) \preceq \sigma(y) \text{ or } \sigma(y) \preceq \sigma(x) \\ 0 & \text{otherwise} \end{cases}$ for every unordered pair (x, y) of $E \subset V \times V$. We call σ , a S –vertex set and ψ a S –edge set of the S –valued graph G^S .

Notation: Any S –valued graph $G^S = (V, E, \sigma, \psi)$ can be also denoted by $G^S = (V_S, E_S)$, where $V_S = \{(v, \sigma(v)) \mid v \in V\}$ and $E_S = \{(uv, \psi(uv)) \mid uv \in E\}$.

Definition 2.4. [1] The open neighbourhood of v_i in G^S is defined as $N_S(v_i) = \{(v_j, \sigma(v_j)) \mid (v_i, v_j) \in E, \psi(v_i, v_j) \in S\}$. The closed neighbourhood of v_i in G^S is defined as $N_S[v_i] = N_S(v_i) \cup \{(v_i, \sigma(v_i))\}$.

Definition 2.5. [2] Let $G^S = (V, E, \sigma, \psi)$ be a given S –valued graph. Let $D \subseteq V$ then the scalar cardinality of D , denoted by $|D|_S$, is defined by $|D|_S = \sum_{v \in D} \sigma(v)$.

Definition 2.6. [5] Consider the S -valued graph $G^S = (V, E, \sigma, \psi)$. Let $\mathcal{C} = \{c_1, c_2, \dots\}$ be a set of colours. A colouring of G^S is given by a function $f: V \times V \rightarrow S \times \mathcal{C}$ such that for all $v \in V$, $f(u, v) = (\sigma(v), C(v))$, $C: V \rightarrow \mathcal{C}$ such that $C(v) \in \mathcal{C}$.

Definition 2.7. [5] Let $G^S = (V, E, \sigma, \psi)$ be a S -valued graph. The vertex chromatic number of G^S denoted by $\chi_S(G^S)$, is defined to be $\chi_S(G^S) = \left(\min_{v \in V} \sigma(v), \min |\mathcal{C}| \right)$.

Definition 2.8. [7] Let $G_1^S = (V_1, E_1, \sigma_1, \psi_1)$ where $V_1 = \{v_i \mid i = 1, \dots, p_1\}$, $E_1 \subseteq V_1 \times V_1$ and $G_2^S = (V_2, E_2, \sigma_2, \psi_2)$ where $V_2 = \{u_j \mid j = 1, \dots, p_2\}$, $E_2 \subseteq V_2 \times V_2$ be two given S -valued graphs.

Let $V_1 \times V_2 = \{w_{ij} = (v_i, u_j) \mid i = 1, \dots, p_1; j = 1, \dots, p_2\}$; and $E = E_1 \times E_2 \subseteq V_1 \times V_2$.

The Categorical product of two S -valued graphs G_1^S and G_2^S is defined by $G_{\times}^S = G_1^S \times G_2^S = (V, E, \sigma, \psi)$

where $V = V_1 \times V_2 = \{w_{ij} = (v_i, u_j) \mid v_i \in V_1 \text{ and } u_j \in V_2\}$. The two vertices $w_{ij} = (v_i, u_j), w_{kl} = (v_k, u_l)$ are adjacent if $v_i v_k \in E_1$ and $u_j u_l \in E_2$. Then,

$E = \{e_{ij}^{kl} = (w_{ij}, w_{kl}) \mid e_i^k = v_i v_k \in E_1 \text{ and } e_j^l = u_j u_l \in E_2\}$

The S -valued function $\sigma: V \rightarrow S$ is defined by

$$\sigma(w_{ij} = (v_i, u_j)) = \min \{\sigma_1(v_i), \sigma_2(u_j)\}$$

and the S -valued function $\psi: E \rightarrow S$ is defined by

$$\psi(e_{ij}^{kl} = w_{ij} w_{kl}) = \min \{\psi_1(e_i^k), \psi_2(e_j^l)\}$$

3. S -vertex colouring on S -valued graphs

In this section, we redefine the definition of S -vertex colouring in S -valued graphs and extend this concept into categorical product of two S -valued graphs.

Definition 3.1. Let $G^S = (V_S, E_S)$ be a given S -valued graph. Let $\mathcal{C} = \{c_1, c_2, \dots\}$ be a given collection of colours. A S -vertex colouring of G^S is defined as a mapping $f_S: V_S \rightarrow S \times \mathcal{C}$ given by $f(v, \sigma(v)) = (\sigma(v), C(v))$, where $C: V \rightarrow \mathcal{C}$.

There are two cases of S -vertex colouring of S -valued graphs.

Definition 3.2. A S -vertex colouring f_S is said to be an equi-weight proper vertex colouring if $\sigma(u) = \sigma(v)$ and $C(u) \neq C(v)$ for all $uv \in E(G^S)$.

Example 3.3. Consider the semiring $S = (\{0, a, b, c, d, e\}, +, \cdot)$ with the binary operations $' + '$ and $' \cdot '$ defined by the following

Cayley tables.

+	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	a	b	c	d	e
b	b	b	b	d	d	e
c	c	c	d	c	d	e
d	d	d	d	d	d	e
e	e	e	e	e	e	e

$\cdot$	0	a	b	c	d	e
0	0	0	0	0	0	0
a	0	0	0	a	a	b
b	0	a	b	a	b	b
c	0	0	0	c	c	e
d	0	a	b	c	d	e
e	0	c	e	c	e	e

In S , we define a canonical pre-order by $0 \preceq 0, a, b, c, d, e$; $a \preceq a, b, c, d, e$; $b \preceq b, d, e$; $c \preceq c, d, e$; $d \preceq d, e$; $e \preceq e$. Consider the following S -vertex colouring on G^S .

Here, the S -valued graph G^S is vertex regular and it is coloured with distinct colouring for adjacent vertices. Thus, the S -vertex colouring on G^S is the equi weight proper vertex colouring of G^S .

Definition 3.4. A S -vertex colouring f_S is said to be a total weight proper vertex colouring if $\sigma(u) \neq \sigma(v)$ and $C(u) \neq C(v)$ for all $uv \in E(G^S)$.

Example 3.5. Consider the semiring $S = (\{0, a, b, c, d, e\}, +, \cdot)$ as in example 3.3.

Consider the following S -valued graph G^S and its S -vertex colouring.

Here, the first figure shows that G^S is a S -irregular and second figure is the S -vertex proper colouring of G^S . Hence, the S -vertex colouring of G^S is the total weight proper vertex colouring.

Definition 3.6. Consider the S -valued graph $G^S = (V_S, E_S)$. Then the S -vertex chromatic number of G^S is defined to be $\chi_S(G^S) = \left(\min_{v_i \in V} |N_S[v_i]|_S, \chi(G) \right)$ where, $\chi(G)$ is a vertex chromatic number of a crisp graph G .

Example 3.7. For the graph G^S given in the example 3.3, the S -vertex chromatic number $\chi_S(G^S) = \left(\min_{v_i \in V} |N_S[v_i]|_S, \chi(G) \right) = \left(\min_{v_i \in V} \{ |N_S[v_i]|_S, \chi(G) \}, i = 1, 2, \dots, 6. \right)$

$$\text{Now, } |N_S[v_1]|_S = \left(\sigma(v_1) + \sigma(v_2) + \sigma(v_5) + \sigma(v_6) + \sigma(v_4) \right) = (b + b + b + b + b) = b$$

Similarly, we can find, $|N_S[v_2]|_S = |N_S[v_3]|_S = b$.

Also, $|N_S[v_4]|_S = |N_S[v_5]|_S = |N_S[v_6]|_S = b$.

Thus, $\chi_S(G^S) = (\min\{b, b, b, b, b, b\}, 3) = (b, 3)$

For the graph G^S given in the example, 3.5,

$$\chi_S(G^S) = \left(\min_{v_i \in V} |N_S[v_i]|_S, \chi(G) \right) = \left(\min_{v_i \in V} \{ |N_S[v_1]|_S, |N_S[v_2]|_S, |N_S[v_3]|_S, \chi(G) \}, \chi(G) \right)$$

$$\text{Now } |N_S[v_1]|_S = \left(\sigma(v_1) + \sigma(v_2) + \sigma(v_3) \right) = (a + b + d) = d$$

Similarly, $|N_S[v_2]|_S = |N_S[v_3]|_S = d$.

Thus, $\chi_S(G^S) = (\min\{d, d, d\}, 3) = (d, 3)$.

Definition 3.8. A S -valued graph G^S is said to be (a, k) -colourable for some $a \in S$, if for all $v_i \in V$, $\sigma(v_i) = a$ and $\chi(G) \leq k$.

Remark 3.9. Every equi-weight proper vertex colouring S -valued graph is a (a, k) -colourable graph for some $a \in S$.

In the theory of crisp graphs, if a graph G is k -colourable if and only if $\chi(G) \leq k$. But in the theory of semiring valued graphs that condition does not hold. This is illustrated by the following example.

Example 3.10. Consider the semiring $S = (\{0, a, b, c\}, +, \cdot)$ with the binary operations '+' and '·' defined by the following Cayley tables.

+	0	a	b	c
0	0	a	b	c
a	a	b	c	c
b	b	c	c	c
c	c	c	c	c

·	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	c	c
c	0	c	c	c

The canonical pre-order $\preceq$ in S is given by

$$0 \preceq 0, a, b, c; a \preceq a, b, c; b \preceq b, c; c \preceq c$$

Consider the following (a, k) -colourable S -valued graph G^S :

Here, all the vertices have the same S -value a and also it is 3-colourable. Thus, G^S is a $(a, 3)$ -colourable graph.

$$\text{But } \chi_S(G^S) = \left(\min_{v_i \in V} |N_S[v_i]|_S, \chi(G) \right) = \left(\min\{c, c, c\}, 3 \right) = (c, 3) \not\preceq (a, 3).$$

Lemma 3.11. For any S -valued graph G^S , if $\chi_S(G^S) \preceq (a, k)$, where $a \neq 0 \in S$ is the minimum among the S -values. Then G^S is a (a, k) -colourable graph with $\chi(G) \leq k$.

Proof. From the definition of canonical preorder in the semiring S , we can easily prove that,

$\chi_S(G^S) \preceq (a, k)$ implies G^S is a (a, k) -colourable. $\square$

Definition 3.12. Let $G_\times^S = (V, E, \sigma, \psi) = (V_S, E_S)$ be the categorical product of two S -valued graphs $G_1^S = (V_1, E_1, \sigma_1, \psi_1) = (V_{1S}, E_{1S})$ and $G_2^S = (V_2, E_2, \sigma_2, \psi_2) = (V_{2S}, E_{2S})$. Consider $f_1 : V_{1S} \rightarrow S \times \mathcal{C}$ by $f_1(v_i, \sigma_1(v_i)) = (\sigma_1(v_i), C(v_i))$ and $f_2 : V_{2S} \rightarrow S \times \mathcal{C}$ given by $f_2(u_j, \sigma_2(u_j)) = (\sigma_2(u_j), C(u_j))$ are S -vertex colourings of G_1^S and G_2^S respectively. Then, the S -vertex colouring on $G_\times^S$ is a function $f : V_S \rightarrow S \times \mathcal{C}$ defined by $f(w_{ij}, \sigma(w_{ij})) = (\sigma(w_{ij}), C(w_{ij}))$ where, $C(w_{ij}) = C(v_i)$ is a vertex colouring of a graph G_1^S .

Remark 3.13. Since $C(w_{ij}) = C(v_i) \neq C(v_k)$ for all edges $e_{ij}^k \in E_1(G_1^S)$, $C(w_{ij}) = C(v_i) \neq C(v_k) = C(w_{kl})$ for all edges $e_{ij}^{kl} \in E(G_\times^S)$. Thus, $C(w_{ij}) \neq C(w_{kl})$ for all adjacent vertices $w_{ij}, w_{kl} \in E(G_\times^S)$.

Example 3.14. Consider the semiring $S = (\{0, a, b, c, d\})$ as in example 3.3.

Consider the following S -vertex colouring graphs G_1^S and G_2^S .

Then the S -vertex colouring of $G_\times^S$ is :

Theorem 3.15. *The categorical product of two equi-weight proper vertex colouring S -valued graphs is a equi-weight proper vertex colouring S -valued graph.*

Proof. Let $f_1 : V_{1S} \rightarrow S \times \mathcal{C}$ be a equi weight proper vertex colouring of $G_1^S = (V_{1S}, E_{1S})$ such that $f_1(v_i, \sigma_1(v_i)) = (\sigma_1(v_i), C(v_i)) = (a, C(v_i))$ for some $a \in S$. Similarly, let $f_2 : V_{2S} \rightarrow S \times \mathcal{C}$ be a equi weight proper vertex colouring of G_2^S such that $f_2(u_j, \sigma_2(u_j)) = (\sigma_2(u_j), C(u_j)) = (b, C(u_j))$ for some $b \in S$.

Then, from the definition of categorical product

$$G_\times^S = (V_S, E_S) = G_1^S \times G_2^S,$$

$$\sigma(w_{ij}) = \min\{\sigma_1(v_i), \sigma_2(u_j)\} = \min\{a, b\}$$

Thus $\sigma(w_{ij}) = a$ or $\sigma(w_{ij}) = b$ for all $w_{ij} \in V(G_\times^S) \dots (1)$.

Also, from the definition of S -vertex colouring of $G_\times^S$
 $C(w_{ij}) = C(v_i) \neq C(v_k) = C(w_{kl})$ for all adjacent vertices $w_{ij}, w_{kl} \in E(G_\times^S) \dots (2)$.

Thus from (1) and (2), for all vertices $w_{ij} \in V(G_\times^S)$,

$f : V_S \rightarrow S \times \mathcal{C}$ by

$$f(w_{ij}, \sigma(w_{ij})) = (\sigma(w_{ij}), C(w_{ij})) = (a, C(v_i)) \text{ or } (b, C(v_i))$$

This implies that $f : V_S \rightarrow S \times \mathcal{C}$ is a equi weight proper vertex colouring of $G_\times^S$. $\square$

In the theory of semiring valued graphs, the categorical product of two S -irregular graphs is S -irregular. By this, one can analogously prove the following.

Theorem 3.16. *The categorical product of two total weight proper S -vertex colouring S -valued graphs is a total weight proper S -vertex colouring S -valued graph.*

Conjecture 3.17. *For any S -valued graphs G_1^S and G_2^S*

$$\chi_S(G_1^S \times G_2^S) \preceq \min\{\chi_S(G_1^S), \chi_S(G_2^S)\}.$$

References

- [1] Jeyalakshmi. S. and Chandramouleeswaran. M, Connected S -Valued graphs, *Mathematical Sciences International Research Journal*, 4(2)(2015), 323–325.
- [2] Jeyalakshmi. S. and Chandramouleeswaran. M, Vertex Domination on S -Valued graphs, *IOSR Journal of Mathematics.*, 12(2016), 08–12.
- [3] Jonathan S Golan *Semirings and Their Applications*, Kluwer Academic Publishers, London, 2003.
- [4] Rajkumar. M, Jeyalakshmi. S. and Chandramouleeswaran. M, Semiring Valued Graphs, *International Journal of Math.Sci. and Engg. Appls.*, 9(3)(2015), 141–152.
- [5] Shriprakash.T.V.G and Chandramouleeswaran.M., Colouring of S -valued graphs, *IJPAM*, 112(5)(2017), 123–129.
- [6] Shriprakash.T.V.G and Chandramouleeswaran.M., Edge Coloring of S -valued graphs, *Proceedings of ICADM'18, M.K.University*, (2018), 434–447.
- [7] Victor. P and Chandramouleeswaran. M, Categorical Product of two S -valued graphs, *International J. Math.Sci. Engg. Appls.*, 13(III)(2017), 113–122.

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