Edge bi-magic total labeling on signed necklace

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Abstract

Gracefulness and Skolem gracefulness on signed graphs are discussed in [6] and [7]. No other types of labeling are studied on signed graphs. Here we have introduced a new type of labeling namely, edge bi-magic total labeling on signed graphs and investigates such labeling on some classes of signed graphs.

An injective labeling $f : V \cup E^+ \cup E^- \rightarrow \{0, 1, 2, \dots, p+q\}$ on a signed graph $S(V, E, \sigma)$ is said to be edge bi-magic total if $f(u) + f(v) + \sigma(uv) \cdot f(uv) = k_1$ for all $uv \in E^+$ and $f(u) + f(v) + \sigma(uv) \cdot f(uv) = k_2$ for all $uv \in E^-$, where, $k_1 > 0$ and $k_2 > 0$. The positive magic weight of f is $w^+ = k_1$ and $w^- = k_2$ is the negative magic weight of f on the signed graph S .

The positive upper strength (S^+) of a signed graph is equal to $\max\{w_1^+, w_2^+, w_3^+, \dots, w_m^+\}$, where $w_1^+, w_2^+, w_3^+, \dots, w_m^+$ are positive magic weights of $f_1, f_2, f_3, \dots, f_m$ on S respectively. The positive lower strength (s^+) of a signed graph is equal to $\min\{w_1^+, w_2^+, w_3^+, \dots, w_m^+\}$.

Similarly the negative upper strength (S^-) of a signed graph is equal to $\max\{w_1^-, w_2^-, w_3^-, \dots, w_n^-\}$, where $w_1^-, w_2^-, w_3^-, \dots, w_n^-$ are negative magic weights of $f_1, f_2, f_3, \dots, f_n$ on S respectively. The negative lower strength (s^-) of a signed graph is equal to $\min\{w_1^-, w_2^-, w_3^-, \dots, w_n^-\}$. This is total in nature because all the vertices and edges are labeled. All positive and negative magic weights are always greater than or equal to zero, no negative weights are taken in to account.

In this paper, we introduce and investigate the edge bi-magic total labeling on signed necklace.

Keywords

Edge bi-magic total labeling, Positive magic strength, Negative magic strength, Signed Necklace.

AMS Subject Classification

05C78.

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Article History: Received 21 December 2018; Accepted 11 February 2019

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1. Introduction

Signed graphs form an interesting area in graph theory. Labeling of signed graphs is a new area. Edge bi-magic total labeling of signed graphs is attempted in this paper.

A labeling of a graph G is a mapping that carries a set of graph elements, usually the vertices and edges into a set of numbers, usually integers. Many kinds of labeling have been studied and an excellent survey of graph labeling can be found in [3].

Sedlacek [11] introduced the concept of magic labeling. Suppose that G is a graph with q edges. We shall say that G is magic if the edges of G can be labeled by the numbers $1, 2, 3, \dots, q$ so that the sum of labels of all the edges incident with any vertex is the same. For concepts of magic type labeling, refer to [2, 12].

For a graph G with vertex set $V(G)$ and edge set $E(G)$, an edge-magic total labeling is a bijection $\lambda : V(G) \cup E(G)$ to the set of integers $1, 2, \dots, |V(G) \cup E(G)|$ with the property that, for each edge xy , $\lambda(x) + \lambda(xy) + \lambda(y) = k$, for a fixed integer k . Such a labeling is called super if the smallest possible integers are assigned on the vertices. A graph is called edge-magic total if it admits an edge-magic total labeling [2] and super edge-magic total if it admits a super edge-magic total labeling.

The notion of edge-magic total graphs was introduced and studied by Kotzig and Rosa [5] under the name of graphs

with magic valuation.

Heider's (1946) work on Psychology led Harary (1953) to the notion of signed graphs (or sigraphs in short) and later on Cartwright and Harary (1956) developed the notion of signed digraphs. B. D. Acharya and J. R. Wrens applied the same sigraph theory to balanced Psychology [1] and [13]. Through 4-sigraphs [9] E. Sampathkumar generalized the sigraph and obtained the generalized graph structure [10].

In a sigraph $S = (V, E, \sigma)$, an edge with a positive (negative) sign is called positive edge (negative edge) and $E^+(S)$ ($E^-(S)$) is the set of all positive (negative) edges in the sigraph S .

2. Edge bi-magic total labeling on signed graphs

Let $S(V, E, \sigma)$ be a signed graph where V is the set of vertices. $E = E^+ \cup E^-$ is the disjoint union of the set of positive edges E^+ and the set of negative edges E^- and $\sigma : E \rightarrow \{+, -\}$ is a map. Let $SG^{(l,m)}$ denote the signed graph with consecutive l -positive edges and consecutive m -negative edges, where $l + m = q$, the size of the graph. Some concepts of signed graphs can be found in [4, 8, 13].

3. Edge bi-magic total labeling on signed necklace

Definition 3.1. An n -point contact of K_2 is NK_n (i. e) $NK_n = K_2 + nK_1$ along with an edge $K_2 = uv$ is called a **necklace** with rope uv and is denoted by NK_n with rope uv .

Definition 3.2. If the edges of a necklace are signed with either $+$ or $-$ then it is called a **signed necklace**.

Note 3.3. In a signed necklace if $\sigma(uv) = -1$ then it is called a necklace with negative rope and is denoted by NK_n^- .

Note 3.4. In a signed necklace if $\sigma(uv) = 1$ then it is called a necklace with positive rope and is denoted by NK_n^+ .

Definition 3.5. $SNK_{(n-)}^-$ is signed necklace with a negative rope uv along with $E^+ = \{uw_1, uw_2, uw_3, \dots, uw_n\}$ $E^- = \{w_1v, w_2v, w_3v, \dots, w_nv\} \cup \{uv\}$

Definition 3.6. $SNK_{(-n+)}^+$ is signed necklace with a positive rope uv along with $E^+ = \{w_1v, w_2v, w_3v, \dots, w_nv\} \cup \{uv\}$ $E^- = \{uw_1, uw_2, uw_3, \dots, uw_n\}$

Definition 3.7. $SNK_{(-n+)}^-$ signed necklace with a negative rope uv along with $E^+ = \{w_1v, w_2v, w_3v, \dots, w_nv\}$ $E^- = \{uw_1, uw_2, uw_3, \dots, uw_n\} \cup \{uv\}$

Note 3.8. The labeling order in $SNK_{(+n-)}^-$ can be adopted for $SNK_{(-n+)}^-$ and therefore $SNK_{(-n+)}^-$ is called the dual of $SNK_{(+n-)}^-$ and vice-versa.

Example:

Definition 3.9. SNK_n^+ is symmetric if $SNK_{(+n+)}^+$, $SNK_{(-n-)}^+$, $SNK_{(+n-)}^+$, $SNK_{(-n+)}^+$, $SNK_{(-n+)}^-$, and $SNK_{(+n-)}^-$ are the only signed necklaces which are symmetric.

The **converse** of a sigraph $S = (V, E, \sigma)$ is the sigraph $S^* = (V, E, \sigma^*)$, having the same vertex set V and edge set E and $\sigma^* : E \rightarrow \{+, -\}$ is defined by $\sigma^*(e) = -\sigma(e)$, for all $e \in E = E^+ \cup E^-$.

Note 3.10. $(SNK_{(-n+)}^+)^* = (SNK_{(+n-)}^-)$
 $(SNK_{(+n-)}^+)^* = (SNK_{(-n+)}^-)$
 Therefore $((SNK_{(-n+)}^+)^*)^* = SNK_{(-n+)}^+$.

Note 3.11. $(SNK_{(+n+)}^+)^* = (SNK_{(-n-)}^-)$
 $(SNK_{(-n-)}^-)^* = (SNK_{(+n+)}^+)$.

Note 3.12. Dual of SNK_n with positive or negative rope is not its converse.

Note 3.13. Upon the weights, $S(SNK_{(+n-)}^+) \neq S(DSNK_{(+n-)}^+)$
 $s(SNK_{(+n-)}^+) \neq s(DSNK_{(+n-)}^+)$.
 In general the positive (negative) upper (lower) strength of a

signed graph need not be equal to the corresponding positive (negative) upper (lower) strength of its dual.

Definition 3.14. A signed graph which admits an edge bimagic total labeling is called edge bi-magic.

Theorem 3.15. Let $G = SNK_n$ be symmetric. Then G is edge bi-magic.

Proof. Consider SNK_n with rope uv . $V(G) = \{u, v, w_1, w_2, \dots, w_n\}$. $E(G) = \{uv, \{uw_1, uw_2, \dots, uw_n, w_1v, w_2v, \dots, w_nv\}\}$. Given that SNK_n is symmetric.

Case(i):

Let $SNK_{(+n+)}^+$ be a signed graph with positive rope uv .

Here $V(SNK_{(+n+)}^+) = \{u, v, w_1, w_2, \dots, w_n\}$

$E(SNK_{(+n+)}^+) = \{uw_1, uw_2, \dots, uw_n, uv, w_1v, w_2v, \dots, w_nv\}$

Define $f_1 : V \cup E^+ \cup E^- \rightarrow \{0, 1, 2, \dots, p+q\}$ as follows:

$$f_1(u) = 3n+3; f_1(v) = 2n+2.$$

$$f_1(w_i) = 3n+3-i, i = 1, 2, \dots, n; f_1(uv) = n.$$

$$f_1(uw_i) = i-1, i = 1, 2, \dots, n.$$

$$f_1(w_iv) = n+i, i = 1, 2, \dots, n.$$

Here f_1 is a trivial edge bi-magic total labeling where the positive magic weight of f_1 is $w_1^+ = 6n+5$ and the negative magic weight does not exist.

$$\text{Hence } S^+(G) = 6n+5 = s^+(G).$$

Note that the label 0 is assigned to an edge and the label $2n+1$ is omitted and therefore there is no possibility for another nontrivial edge bi-magic total labeling.

Case(ii):

The bi-magic constants of $SNK_{(+n+)}^-$ is $S^+ = s^+ = 6n+5$ and $S^- = s^- = 4n+5$.

For:

Assign the same label as in the above case(i) as f_2 . Therefore we get $w_2^+(SNK_{(+n+)}^-) = 6n+5$, $w_2^-(SNK_{(+n+)}^-) = 4n+5$. Note that the label 0 assigned to an edge and the label $2n+1$ is omitted and therefore there is no possibility for another non-trivial edge bi-magic total labeling. Hence $S^+ = s^+ = 6n+5$ and $S^- = s^- = 4n+5$.

Case(iii):

The bi-magic constants of $SNK_{(-n-)}^+$ are $S^+ = s^+ = 6n+5$ and $S^- = s^- = 4n+5$. For:

Define $f_3 : V \cup E^+ \cup E^- \rightarrow \{0, 1, 2, \dots, p+q\}$ as follows:

$$f_3(u) = 3n+3.$$

$$f_3(v) = 2n+2.$$

$$f_3(w_i) = 3n+3-i, i = 1, 2, \dots, n.$$

$$f_3(uv) = n.$$

$$f_3(uw_i) = 2n+1-i, i = 1, 2, \dots, n.$$

$$f_3(w_iv) = n-i, i = 1, 2, \dots, n.$$

Here f_3 is a trivial edge bi-magic total labeling where the positive magic weight of f_3 is $w_3^+ = 6n+5$ and the negative magic weight of f_3 is $w_3^- = 4n+5$.

Case(iv):

The bi-magic constants of $SNK_{(-n-)}^-$ are $S^+ = s^+ = 4n+5$.

For:

Assign the same label as in the above case(iii) as f_4 . Therefore we get $w_4^-(SNK_{(-n-)}^-) = 4n+5$. w_4^+ does not exist.

Note that the label 0 assigned to an edge and the label $2n+1$ is omitted and therefore there is no possibility for another non-trivial edge bi-magic total labeling. Hence $S^- = s^- = 4n+5$.

Case(v):

The bi-magic constants of $SNK_{(+n-)}^+$ are

$$S^+ = s^+ = 6n+5 \text{ and } S^- = s^- = 3n+4.$$

For:

Define $f_5 : V \cup E^+ \cup E^- \rightarrow \{0, 1, 2, \dots, p+q\}$ as follows:

$$f_5(u) = 3n+3.$$

$$f_5(v) = 2n+2.$$

$$f_5(w_i) = 3n+3-i, i = 1, 2, \dots, n.$$

$$f_5(uv) = n.$$

$$f_5(uw_i) = i-1, i = 1, 2, \dots, n.$$

$$f_5(w_iv) = 2n+1-i, i = 1, 2, \dots, n.$$

Here f_5 is a trivial edge bi-magic total labeling where the positive magic weight of f_5 is $w_5^+ = 6n+5$ and the negative magic weight of f_5 is $w_5^- = 3n+4$.

Case(vi):

The bi-magic constants of $SNK_{(+n-)}^-$ are $S^+ = s^+ = 7n+6$ and $S^- = s^- = 4n+5$.

For:

Define $f_6 : V \cup E^+ \cup E^- \rightarrow \{0, 1, 2, \dots, p+q\}$ as follows:

$$f_6(u) = 3n+3.$$

$$f_6(v) = 2n+2.$$

$$f_6(w_i) = 3n+3-i, i = 1, 2, \dots, n.$$

$$f_6(uw_i) = n+i, i = 1, 2, \dots, n.$$

$$f_6(w_iv) = n-i, i = 1, 2, \dots, n.$$

$$f_6(uv) = n.$$

Here f_6 is a trivial edge bi-magic total labeling where the positive magic weight of f_6 is $w_6^+ = 7n+6$ and the negative magic weight of f_6 is $w_6^- = 4n+5$.

Case(vii):

The bi-magic constants of $SNK_{(-n+)}^+$ are $S^+ = s^+ = 6n+5$ and $S^- = s^- = 5n+6$.

For:

Define $f_7 : V \cup E^+ \cup E^- \rightarrow \{0, 1, 2, \dots, p+q\}$ as follows:

$$f_7(u) = 3n+3.$$

$$f_7(v) = 2n+2.$$

$$f_7(w_i) = 3n+3-i, i = 1, 2, \dots, n.$$

$$f_7(uv) = n.$$

$$f_7(uw_i) = n-i, i = 1, 2, \dots, n.$$

$$f_7(w_iv) = n+i, i = 1, 2, \dots, n.$$

Here f_7 is a trivial edge bi-magic total labeling where the positive magic weight of f_7 is $w_7^+ = 6n+5$ and the negative magic weight of f_7 is $w_7^- = 5n+6$.

Case(viii):

The bi-magic constants of $SNK_{(-n+)}^-$ are $S^+ = s^+ = 5n+4$ and $S^- = s^- = 4n+5$.

For:

Define $f_8 : V \cup E^+ \cup E^- \rightarrow \{0, 1, 2, \dots, p+q\}$ as follows:

$$f_8(u) = 3n+3.$$

$$f_8(v) = 2n + 2.$$

$$f_8(w_i) = 3n + 3 - i, \quad i = 1, 2, \dots, n.$$

$$f_8(uv) = n.$$

$$f_8(uw_i) = 2n + 1 - i, \quad i = 1, 2, \dots, n.$$

$$f_8(w_iv) = i - 1, \quad i = 1, 2, \dots, n.$$

Here f_8 is a trivial edge bi-magic total labeling where the positive magic weight of f_8 is $w_8^+ = 5n + 4$ and the negative magic weight of f_8 is $w_8^- = 4n + 5$. Hence SNK_n is edge bi-magic. $\square$

Examples:

Here $n = 7$

$$S^+ = s^+ = 47$$

Here $n = 7$

$$S^+ = s^+ = 47; \quad S^- = s^- = 33$$

Here $n = 8$

$$S^+ = s^+ = 53; \quad S^- = s^- = 37$$

Here $n = 8$

$$S^- = s^- = 37$$

Here $n = 9$

$$S^+ = s^+ = 59; \quad S^- = s^- = 31 \quad S^+ = s^+ = 69; \quad S^- = s^- = 41$$

Here $n = 9$

Here $n = 10$

$$S^+ = s^+ = 65; \quad S^- = s^- = 56$$

Here $n = 10$

$$S^+ = s^+ = 54; \quad S^- = s^- = 45$$

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ISSN(P):2319 – 3786

Malaya Journal of Matematik

ISSN(O):2321 – 5666