k -Isolate domination number of simple graphs

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Abstract

In this paper we introduce k -isolate dominating set and minimal k -isolate dominating set and it is defined as follows: A dominating set S of a graph G is said to be a k -isolate dominating set if $\langle S \rangle$ has at least k -isolated vertices. The k -isolate dominating set S is said to be a minimal k -isolate dominating set if any proper subset of S is not an isolate dominating set. The minimum and the maximum cardinality of a minimal k -isolate dominating set of G are called k -isolate domination number denoted by $\gamma_{ki}(G)$ and the upper k -isolate domination number by $\Gamma_{ki}(G)$ respectively. Also k -isolate domination number for some simple graphs such as path, cycle and wheel graph have been found.

Keywords

Dominating set, Isolate dominating set, k -isolate dominating set.

AMS Subject Classification

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Contents

1	Introduction	113
2	Main Results	113
3	Conclusion	115
	References	115

1. Introduction

In a graph $G = (V, E)$, the degree of a vertex v in $V(G)$ is the number of edges incident with v and is denoted by $\deg(v)$. A dominating set for a graph G is a subset S of $V(G)$ such that every vertex in $V(G) - S$ is adjacent to at least one vertex in S . A dominating set S is called minimal dominating set if no proper subset of S is a dominating set. The domination number $\gamma(G)$ is the number of vertices in a smallest dominating set for G . The maximum cardinality of a minimal dominating set of a graph is called upper domination number. It is denoted by $\Gamma(G)$.

A detailed study of domination can be seen in [1–3]. In [5, 6], Sahul Hamid et.al introduced isolate dominating set and it is defined as dominating set S is such that the subgraph $\langle S \rangle$ induced by S has at least one isolated vertex is called an isolate dominating set. An isolate dominating set S is said to be a minimal isolate dominating set if no proper subset of S is an isolate dominating set. The minimum and the maximum cardinality of a minimal isolate dominating set are

called the isolate domination number $\gamma_0(G)$ and upper isolate domination number $\Gamma_0(G)$ respectively.

In this paper we introduced a new variant based on isolate domination number which is called k -isolate domination defined as follows:

A dominating set S of a graph G is said to be a k -isolate dominating set if $\langle S \rangle$ has at least k -isolated vertices. The k -isolate dominating set S is said to be a minimal k -isolate dominating set if proper subset of S is not an isolate dominating set. The minimum and the maximum cardinality of a minimal k -isolate dominating set of G are called k -isolate domination number denoted by $\gamma_{ki}(G)$ and the upper k -isolate domination number by $\Gamma_{ki}(G)$ respectively.

Theorem 1.1. [6] For the path P_n and the cycle C_n , we have $\gamma_0(P_n) = \gamma_0(C_n) = \lceil \frac{n}{3} \rceil$

2. Main Results

In this section, we determine the k -isolate domination number for some standard graphs such as paths, cycles and wheels.

Theorem 2.1. For the path P_n , we have

$$\gamma_{ki}(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil, & k \leq \lceil \frac{n}{3} \rceil \\ k, & \lceil \frac{n}{3} \rceil < k \leq \lceil \frac{n}{2} \rceil \\ \text{does not exist,} & k > \lceil \frac{n}{2} \rceil \end{cases}$$

Figure 1. P_{13}

Proof.

Let P_n be the path on n vertices namely $\{v_1, v_2, \dots, v_n\}$. In path, the degree of the first and the last vertex is one and the remaining vertices are of degree two.

Case (i): $k \leq \lceil \frac{n}{3} \rceil$

Obviously $\gamma_{ki}(P_2) = 1$, we have $S' = \{v_2, v_5, v_8, \dots, v_{n-1}\}$, the minimal isolate dominating set when n is a multiple of 3 and $S'' = \{v_2, v_5, v_8, \dots, v_{n-1}\}$, the minimal isolate dominating set, if n is not a multiple of 3. Hence by Theorem 1.1, For the path P_n , $\gamma_0(P_n) = \lceil \frac{n}{3} \rceil$. Also every minimal isolate dominating set is the minimal k -isolate dominating set when $k \leq \lceil \frac{n}{3} \rceil$. Hence $\gamma_{ki}(P_n) = \lceil \frac{n}{3} \rceil$

Case (ii): $\lceil \frac{n}{3} \rceil < k \leq \lceil \frac{n+1}{2} \rceil$

In particular from Case (i), when $k = \lceil \frac{n}{3} \rceil$, we have $\gamma_{ki}(P_n) = \lceil \frac{n}{3} \rceil$ with k vertices. From the minimal k -isolate dominating set, we construct the new set S_1 by replacing the vertex v_2 as v_1 and v_3 , and we get $S_1 = \{v_1, v_3, v_5, v_8, \dots, v_{n-1}\}$ with $k+1$ vertices if n is a multiple of 3, or $S_1 = \{v_1, v_3, v_5, v_8, \dots, v_n\}$ with $k+1$ vertices if n is not a multiple of 3. Hence $\gamma_{k+1}(P_n) = k+1$ where $\lceil \frac{n}{3} \rceil < k+1 \leq \lceil \frac{n+1}{2} \rceil$. Continuing in this process by replacing the vertex which is labeled with even number say v_{2i} by v_{2i-1} and v_{2i+1} , we have $\gamma_{ki}(P_n) = k$.

Case (iii): $k > \lceil \frac{n+1}{2} \rceil$

Clearly for the path P_n , we have only $\frac{n}{2}$ vertices for the dominating set. Hence when k exceeds more than $\frac{n}{2}$ vertices, $\gamma_{ki}(P_n)$ does not exist. $\square$

Example 2.2. In P_{13} , as shown in Figure 1, when $k = 1, 2, 3, 4, 5$, k -isolate dominating set is $\{v_2, v_5, v_8, v_{11}, v_{13}\}$
 $\gamma_{1i} = \gamma_{2i} = \gamma_{3i} = \gamma_{4i} = \gamma_{5i} = 5$

6-isolate dominating set is $\{v_1, v_3, v_5, v_8, v_{11}, v_{13}\}$ $\gamma_{6i} = 6$

7-isolate dominating set is $\{v_1, v_3, v_5, v_7, v_9, v_{11}, v_{13}\}$
 $\gamma_{7i} = 7$

For $k > 7$, k -isolate dominating set doesn't exist.

Theorem 2.3. If C_n is a cycle with n vertices ($n \geq 3$), then

$$\gamma_{ki}(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil, & k \leq \lceil \frac{n}{3} \rceil \\ k, & \lceil \frac{n}{3} \rceil < k < \lceil \frac{n+1}{2} \rceil \\ \text{does not exist}, & k \geq \lceil \frac{n+1}{2} \rceil \end{cases}$$

Proof. Let C_n be the cycle of n vertices with n edges whose degree of each vertex be two. Let the vertex of C_n be $\{v_1, v_2, \dots, v_n\}$. And v_i is adjacent to v_{i-1} and v_{i+1} , also v_n is adjacent to v_1 where $i = 2, 3, 4, \dots, n-1$

Case (i): $k \leq \lceil \frac{n}{3} \rceil$

Let $S = \{v_2, v_5, v_8, \dots, v_{n-4}, v_{n-1}\}$ be the minimal isolate dominating set. By Theorem (1.1), For the cycle C_n with n vertices, the minimum cardinality of a minimal isolate dominating set of C_n , $\gamma_0(C_n)$ is $\lceil \frac{n}{3} \rceil$. Hence there must be

minimum $\lceil \frac{n}{3} \rceil$ isolated vertices. Therefore $\gamma_{ki}(C_n)$ is $\lceil \frac{n}{3} \rceil$ when $k \leq \lceil \frac{n}{3} \rceil$

Case (ii): $\lceil \frac{n}{3} \rceil < k < \lceil \frac{n+1}{2} \rceil$

Subcase (i): $n \equiv 0 \pmod{3}$

By Case (i) we have the set $S = \{v_2, v_5, v_8, \dots, v_{n-4}, v_{n-1}\}$, the minimal k -isolate dominating set when $k \leq \lceil \frac{n}{3} \rceil$. Hence $\gamma_{ki}(C_n) = \lceil \frac{n}{3} \rceil$ which is same as $\lceil \frac{n}{3} \rceil$ since n is a multiple of 3. Construct the new set S_1 as $S_1 = \{v_2, v_4, v_6, v_8, v_{11}, \dots, v_{n-4}, v_{n-1}\}$ by replacing the vertex v_5 as v_4 and v_6 which is a minimal $(k+1)^{th}$ isolate dominating set containing $(k+1)$ vertices.

Hence $\gamma_{(k+1)i}(C_n) = k+1$, where $\lceil \frac{n}{3} \rceil < k+1 < \lceil \frac{n+1}{2} \rceil$. Again construct the new set S_2 from S_1 as $S_2 = \{v_2, v_4, v_6, v_8, v_{10}, v_{12}, v_{14}, \dots, v_{n-4}, v_{n-1}\}$ by replacing the vertex v_{11} by v_{10} and v_{12} which contains $(k+2)$ vertices where $\lceil \frac{n}{3} \rceil < k+2 < \lceil \frac{n+1}{2} \rceil$. Continuing the above process by replacing the vertex which is labeled with odd number say v_{2i+1} by v_{2i} and v_{2i+2} , we construct the minimal k -isolate dominating set. Continue this process upto $k < \lceil \frac{n+1}{2} \rceil$. Hence we get $\gamma_{ki}(C_n) = k$.

Subcase (ii): $n \equiv 1 \pmod{3}$

As in Case (i), for $n \equiv 1 \pmod{3}$, we have the set $S_3 = \{v_2, v_5, v_8, \dots, v_{n-5}, v_{n-2}, v_n\}$, the minimal k -isolate dominating set when $k = \frac{n}{3}$ containing $(k+1)$ vertices. Since the set S_3 have $(k+1)$ vertices, we choose the same set for the minimal $(k+1)$ -isolate dominating set say S_4 . Hence $\gamma_{(k+1)i}(C_n) = k+1$.

Then construct the set $S_5 = \{v_2, v_4, v_6, v_8, \dots, v_{n-5}, v_{n-2}, v_n\}$ by replacing the vertex v_5 as v_4 and v_6 containing $(k+2)$ vertices. Hence $\gamma_{(k+2)i}(C_n) = k+2$. Continuing in this manner by fixing the last two vertices namely v_{n-2} and v_n permanently and by replacing the vertex which is labeled with odd number say v_{2i+1} by v_{2i} and v_{2i+2} , we construct the minimal k -isolate dominating set. Continuing the above process upto $k < \lceil \frac{n+1}{2} \rceil$, we get $\gamma_{ki}(C_n) = k$.

Subcase (iii): $n \equiv 2 \pmod{3}$

The proof is similar as in Subcase (i). Hence from the above three sub cases we have proved that $\gamma_{ki}(C_n) = k$.

Case (iii): $k \geq \lceil \frac{n+1}{2} \rceil$

In this case we have only $(k-1)$ isolate vertices in C_n . Hence $\gamma_{ki}(C_n)$ does not exist in this case. $\square$

Example 2.4. For cycle C_{12} , $\gamma_{1i} = \gamma_{2i} = \gamma_{3i} = \gamma_{4i} = 4$

$\gamma_{5i} = 5$

$\gamma_{6i} = 6$

When $k \geq 7$, k -isolate dominating set does not exist

Theorem 2.5. If W_n is the wheel on n vertices then

$$\gamma_{ki}(W_n) = \begin{cases} 1, & k = 1 \\ \lceil \frac{n-1}{3} \rceil, & 1 < k \leq \lceil \frac{n-1}{3} \rceil \\ k, & \lceil \frac{n-1}{3} \rceil < k < \lceil \frac{n}{2} \rceil \\ \text{does not exist}, & k \geq \lceil \frac{n}{2} \rceil \end{cases}$$

Proof. Let W_n be the wheel graph with at least 4 vertices. Let the vertices of W_n be $\{u, v_1, v_2, v_3, \dots, v_{n-1}\}$.

Figure 2. C_{12}

Figure 3. W_{15}

Case(i): $k = 1$

The set $S = \{u\}$ is a minimal 1-isolate dominating set since u is adjacent to all the other vertices. Hence $\gamma_i(W_n) = 1$

Case (ii): $1 < k \leq \lfloor \frac{n-1}{3} \rfloor$

Now we have the dominating set of vertices without the center vertex which is connected to all the other $(n-1)$ vertices in the W_n graph. If the center vertex is removed we have the cycle C_{n-1} with $(n-1)$ vertices. By Theorem (2.2), for the cycle C_{n-1} , $\gamma_{ki}(C_{n-1}) = \lceil \frac{n-1}{3} \rceil$. That is $\gamma_{ki}(W_n) = \lceil \frac{n-1}{3} \rceil$

Case (iii): $\lceil \frac{n-1}{3} \rceil < k < \lceil \frac{n}{2} \rceil$

As in Case (ii) we have the dominating set of vertices without the center vertex with $(n-1)$ vertices say $\{v_1, v_2, v_3, \dots, v_{n-1}\}$ in W_n which is a cycle C_{n-1} . By Theorem (2.2) for the cycle C_{n-1} , $\gamma_{ki}(C_{n-1}) = k$. Hence $\gamma_{ki}(W_n) = k$

Case (iv): $k \geq \lceil \frac{n}{2} \rceil$.

In this case we have only $(k-1)$ isolated vertices in W_n . Hence $\gamma_{ki}(W_n)$ does not exist in this case. $\square$

Example 2.6. In W_{15} , 1-isolate dominating set is $\{u\}$ so, $\gamma_i(W_{15}) = 1$

When $k = 2, 3, 4$, k -isolate dominating set of W_{15} is $\{v_2, v_5, v_8, v_{11}, v_{14}\}$. therefore $\gamma_2(W_{15}) = \gamma_3(W_{15}) = \gamma_4(W_{15}) = 5$

5-isolate dominating set of W_{15} is $\{v_2, v_5, v_8, v_{11}, v_{14}\}$, thus $\gamma_5(W_{15}) = 5$.

6,7-isolate dominating set of W_{15} are $\{v_2, v_4, v_6, v_8, v_{11}, v_{14}\}$ and $\{v_2, v_4, v_6, v_8, v_{10}, v_{12}, v_{14}\}$ respectively. Hence $\gamma_6(W_{15}) = 6$ and $\gamma_7(W_{15}) = 7$.

3. Conclusion

In this paper we introduced the concept of k -isolate dominating set and we have found k -isolate domination number for some simple graphs.

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