$(\in, \in \vee q)$ -Fuzzy ideals in ternary semirings

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Abstract

In this paper we introduce the notion of $(\in, \in \vee q)$ -fuzzy ternary subsemirings and $(\in, \in \vee q)$ -fuzzy ideals in ternary semirings which can be regarded as the generalization of fuzzy ternary subsemirings and fuzzy ideals in ternary semirings.

Keywords

$(\in, \in \vee q)$ -fuzzy ternary subsemirings, $(\in, \in \vee q)$ -fuzzy ideals, $(\in, \in \vee q)$ -level set.

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1. Introduction

The notion of ternary algebraic system was introduced by Lehmer [14] in 1932. He investigated certain ternary algebraic systems called triplexes. In 1971, Lister [15] characterized additive semigroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring. Dutta and Kar [3] introduced a notion of ternary semirings which is a generalization of ternary rings and semirings, and they studied some properties of ternary semirings [3–10].

The concept of fuzzy subsets was introduced by Zadeh [20] in 1965. Since then many researchers have explored generalizations of fuzzy subsets. Many papers on fuzzy subsets have appeared showing the importance of the concept and its application to logic, set theory, group theory, ring theory, real analysis, topology, measure theory, etc. It was first applied to the theory of groups by A. Rosenfeld [18]. In [19], Yuan et al. introduced the definition of fuzzy subgroup with thresholds, which is a generalization of Rosenfeld's fuzzy subgroup and Bhakat and Das's fuzzy subgroup. Murali [16] proposed a definition of fuzzy point belonging to a fuzzy subset under a natural equivalence on a fuzzy subset. The idea of quasi

coincidence of a fuzzy point with a fuzzy subset, which is mentioned in [17], played a vital role in generating some different types of fuzzy subgroup. Bhakat and Das [1, 2] gave the concept of (α, β) -fuzzy subgroups by using the "belongs to" relation $\in$ and "quasi-coincidence with" relation q between a fuzzy point and a fuzzy subgroup and introduced the concept of an $(\in, \in \vee q)$ -fuzzy subgroup. In particular, the $(\in, \in \vee q)$ -fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. We see the fuzzification of different concepts of ternary semirings in [11–13]. In this paper we introduce the notion of $(\in, \in \vee q)$ -fuzzy ternary subsemirings and $(\in, \in \vee q)$ -fuzzy ideals in ternary semirings which can be regarded as the generalization of fuzzy ternary subsemirings and fuzzy ideals in ternary semirings.

2. Preliminaries

In this section, we refer to some elementary aspects of the theory of ternary semirings and fuzzy algebraic systems that are necessary for this paper.

Definition 2.1. A nonempty set S together with a binary operation called, addition $+$ and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if $(S, +)$ is a commutative semigroup satisfying the following conditions:

(i) $(abc)de = a(bcd)e = ab(cde)$,

(ii) $(a+b)cd = acd + bcd$,

(iii) $a(b+c)d = abd + acd$

and (iv) $ab(c+d) = abc + abd$ for all $a, b, c, d, e \in S$.

We can see that any semiring can be reduced to a ternary

semiring. However, a ternary semiring does not necessarily reduce to a semiring by this example. We consider Z_0^- , the set of all non-positive integers under usual addition and multiplication, we see that Z_0^- is a commutative semigroup which is closed under the triple multiplication but is not closed under the binary multiplication. Moreover, Z_0^- is a ternary semiring but is not a semiring under usual addition and multiplication.

Definition 2.2. Let S be a ternary semiring. If there exists an element $0 \in S$ such that $0 + x = x = x + 0$ and $0xy = x0y = xy0 = 0$ for all $x, y \in S$, then 0 is called the zero element or simply the zero of the ternary semiring S . In this case we say that S is a ternary semiring with zero.

Throughout this paper S denotes a ternary semiring with zero.

Definition 2.3. An additive subsemigroup T of S is called a ternary subsemiring of S if $t_1 t_2 t_3 \in T$ for all $t_1, t_2, t_3 \in T$.

Definition 2.4. An additive subsemigroup I of S is called a left [resp. right, lateral] ideal of S if $s_1 s_2 i \in I$ [resp. $i s_1 s_2 \in I, s_1 s_2 i \in I$] for all $s_1, s_2 \in S$ and $i \in I$. If I is a left, right and lateral ideal of S , then I is called an ideal of S .

It is obvious that every ideal of a ternary semiring with zero contains the zero element.

Definition 2.5. Let S_1 and S_2 be ternary semirings. A mapping $f : S_1 \rightarrow S_2$ is said to be a homomorphism if $f(x + y) = f(x) + f(y)$ and $f(xyz) = f(x)f(y)f(z)$ for all $x, y, z \in S_1$.

Let $f : S_1 \rightarrow S_2$ be an onto homomorphism of ternary semirings. Note that if I is an ideal of S_1 , then $f(I)$ is an ideal of S_2 . If S_1 and S_2 be ternary semirings with zero 0 , then $f(0) = 0$.

Let X be a non-empty set. A map $\mu : X \rightarrow [0, 1]$ is called a fuzzy subset in X .

Definition 2.6. Let f and g be any two fuzzy subsets of a nonempty set X . Then $f \cap g, f \cup g, f + g, f \circ g$ are fuzzy subsets of X defined by

$$(f \cap g)(x) = \min\{f(x), g(x)\}$$

$$(f \cup g)(x) = \max\{f(x), g(x)\}$$

$$(f + g)(x) = \begin{cases} \sup\{\min\{f(y), g(z)\}\} & \text{if } x = y + z \\ 0 & \text{otherwise} \end{cases}$$

$$(f \circ g)(x) = \begin{cases} \sup\{\min\{f(u), g(v)\}\} & \text{if } x = uv \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.7. Let $a, b \in [0, 1]$ then $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

Definition 2.8. Let X be a nonempty set and let μ be a fuzzy subset of X . Let $0 \leq t \leq 1$. Then the set $U(\mu; t) = \{x \in X / \mu(x) \geq t\}$ is called a level set of X with respect to μ .

Definition 2.9. Let I be any subset of a ternary semiring S . The characteristic function of I denoted by χ_I is defined as

$$\chi_I(x) = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.10. Let μ be a fuzzy subset of a ternary semiring S . Then μ is called a fuzzy ternary subsemiring of S if

1. $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$
2. $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\}$ for all $x, y, z \in S$.

Definition 2.11. A fuzzy subset μ of a ternary semiring S is called a fuzzy ideal of S if

- (i) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$
- (ii) $\mu(xyz) \geq \mu(x)$
- (iii) $\mu(xyz) \geq \mu(z)$ and
- (iv) $\mu(xyz) \geq \mu(y)$ for all $x, y, z \in S$.

A fuzzy subset μ with conditions (i) and (ii) is called a fuzzy right ideal of S . If μ satisfies (i) and (iii), then it is called a fuzzy left ideal of S . Also if μ satisfies (i) and (iv), then it is called a fuzzy lateral ideal of S . It is clear that μ is a fuzzy ideal of a ternary semiring S if and only if $\mu(xyz) \geq \max\{\mu(x), \mu(y), \mu(z)\}$ for all $x, y, z \in S$.

Definition 2.12. A fuzzy subset μ of a set X of the form

$$\mu(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

is said to be a fuzzy point with support x and value t , and is denoted by x_t . A fuzzy point x_t is said to belong to a fuzzy subset μ , written as $x_t \in \mu$ if $\mu(x) \geq t$. A fuzzy point x_t is said to be quasi-coincident with a fuzzy subset μ , written as $x_t q \mu$ if $\mu(x) + t > 1$. The symbol $x_t \in \vee q \mu$ means that $x_t \in \mu$ or $x_t q \mu$. Similarly, $x_t \in \wedge q \mu$ denotes that $x_t \in \mu$ and $x_t q \mu$. Also, $x_t \notin \mu$ and $x_t \notin \overline{\vee q} \mu$ are respectively mean that $x_t \in \mu$ and $x_t \in \vee q \mu$ do not hold.

3. ($\in, \in \vee q$)- Fuzzy ideals

Definition 3.1. Let μ be a fuzzy subset of S . Then μ is called a ($\in, \in \vee q$)-fuzzy ternary subsemiring of S if for all $x, y, z \in S$ and $t, r, s \in (0, 1]$

1. $x_t \in \mu, y_r \in \mu$ implies $(x + y)_{t \wedge r} \in \vee q \mu$
2. $x_t \in \mu, y_r \in \mu, z_s \in \mu$ implies $(xyz)_{t \wedge r \wedge s} \in \vee q \mu$.

Lemma 3.2. Let μ be a fuzzy subset of S . Then μ is a ($\in, \in \vee q$)-fuzzy ternary subsemiring of S if and only if

- 1a. $\mu(x + y) \geq \mu(x) \wedge \mu(y) \wedge 0.5$
- 2a. $\mu(xyz) \geq \mu(x) \wedge \mu(y) \wedge \mu(z) \wedge 0.5$ for all $x, y, z \in S$.

Proof: 1. $\Rightarrow$ 1a: Let us assume that (1a) is not valid. Then there exist $x, y \in S$ such that $\mu(x + y) < \mu(x) \wedge \mu(y) \wedge 0.5$. We consider the following two cases:

i) $\mu(x) \wedge \mu(y) < 0.5$ and ii) $\mu(x) \wedge \mu(y) \geq 0.5$.

case i): We have $\mu(x + y) < \mu(x) \wedge \mu(y)$. Choose t such that $\mu(x + y) < t < \mu(x) \wedge \mu(y)$. Then $x_t \in \mu, y_t \in \mu$. Since $\mu(x + y) < t < \mu(x) \wedge \mu(y)$, then $(x + y)_t \notin \mu$ and $(x + y)_t \notin q \mu$.

$y) < t < 0.5$, then $\mu(x+y) + t < 1$. We have $(x+y)_t \in \nabla \bar{q} \mu$, which is a contradiction.

case ii): We have $\mu(x+y) < 0.5 \leq \mu(x) \wedge \mu(y)$. Then $x_{0.5}, y_{0.5} \in \mu$, but $(x+y)_{0.5} \in \nabla \bar{q} \mu$, which is a contradiction. Therefore (1a) holds. Similarly we get (2. $\Rightarrow$ 2a :).

1a. $\Rightarrow$ 1 : Let $x, y \in S$ and $t, r \in (0, 1]$ be such that $x_t \in \mu$ and $y_r \in \mu$. Then $\mu(x) \geq t$ and $\mu(y) \geq r$. Now we have $\mu(x+y) \geq \mu(x) \wedge \mu(y) \wedge 0.5 \geq t \wedge r \wedge 0.5$. If $t \wedge r \leq 0.5$, we have $\mu(x+y) \geq t \wedge r$ implies $(x+y)_{t \wedge r} \in \mu$. If $t \wedge r > 0.5$ implies $\mu(x+y) + \{t \wedge r\} > 1$. Hence $(x+y)_{t \wedge r} \in \nabla \bar{q} \mu$. Thus (1) holds. Similarly we get (2a. $\Rightarrow$ 2 :).

Theorem 3.3. Let μ be a fuzzy subset of a ternary semiring S . Then μ is a $(\in, \in \nabla q)$ -fuzzy ternary subsemiring of S if and only if $U(\mu; t)$ is a ternary subsemiring of S for all $t \in (0, 0.5]$ whenever non-empty.

Proof: Let μ be a $(\in, \in \nabla q)$ -fuzzy ternary subsemiring of S and $t \in (0, 0.5]$. Let $x, y, z \in U(\mu; t)$ then $\mu(x) \geq t$, $\mu(y) \geq t$, $\mu(z) \geq t$. Now $\mu(x+y) \geq \mu(x) \wedge \mu(y) \wedge 0.5 \geq t \wedge t \wedge 0.5 = t$. Thus $\mu(x+y) \geq t$. Hence $x+y \in U(\mu; t)$. Similarly $xyz \in U(\mu; t)$. Hence $U(\mu; t)$ is a ternary subsemiring of S .

Conversely, let $U(\mu; t)$ be a ternary subsemiring of S for all $t \in (0, 0.5]$. If there exist $x, y, z \in S$ such that $\mu(x+y) < t = \mu(x) \wedge \mu(y) \wedge 0.5$ then $t \in (0, 0.5]$ and $x, y \in U(\mu; t)$ with $x+y \notin U(\mu; t)$ this contradicts to $U(\mu; t)$ is a ternary subsemiring. Hence $\mu(x+y) \geq \mu(x) \wedge \mu(y) \wedge 0.5$. Similarly we have $\mu(xyz) \geq \mu(x) \wedge \mu(y) \wedge \mu(z) \wedge 0.5$. Therefore μ is a $(\in, \in \nabla q)$ -fuzzy ternary subsemiring of S .

Definition 3.4. Let μ be a fuzzy subset of S . Then μ is called a $(\in, \in \nabla q)$ -fuzzy right (resp. left, lateral) ideal of S if for all $x, y, z \in S$ and $t, r \in (0, 1]$

1. $x_t \in \mu$, $y_r \in \mu$ implies $(x+y)_{t \wedge r} \in \nabla \bar{q} \mu$
2. $x_t \in \mu$, $y, z \in S$ implies $(xyz)_t \in \nabla \bar{q} \mu$ (resp. $(yzx)_t \in \nabla \bar{q} \mu$, $(yxz)_t \in \nabla \bar{q} \mu$).

Lemma 3.5. Let μ be a fuzzy subset of S . Then μ is a $(\in, \in \nabla q)$ -fuzzy right (resp. left, lateral) ideal of S if and only if

- 1a. $\mu(x+y) \geq \mu(x) \wedge \mu(y) \wedge 0.5$
- 2a. $\mu(xyz) \geq \mu(x) \wedge 0.5$ (resp. $\mu(yzx) \geq \mu(x) \wedge 0.5$, $\mu(yxz) \geq \mu(x) \wedge 0.5$) for all $x, y, z \in S$.

Proof: By Lemma 3.2, we have $1 \Leftrightarrow 1a$.

2 $\Rightarrow$ 2a : Let us assume that (2a) is not valid. Then there exist $x, y, z \in S$ such that $\mu(xyz) < \mu(x) \wedge 0.5$. If $\mu(x) < 0.5$, we have $\mu(xyz) < \mu(x)$. Choose t such that $\mu(xyz) < t < \mu(x)$. Then $x_t \in \mu$, but $(xyz)_t \in \nabla \bar{q} \mu$, which contradicts (2).

If $\mu(x) \geq 0.5$ we have $\mu(xyz) < 0.5 \leq \mu(x)$. This implies that $x_{0.5} \in \mu$, but $(xyz)_{0.5} \in \nabla \bar{q} \mu$, which contradicts (2). Therefore (2a) holds.

2a $\Rightarrow$ 2 : Let $x_t \in \mu$ and $y, z \in S$ and $t \in (0, 1]$ be such that $\mu(x) \geq t$. We have $\mu(xyz) \geq \mu(x) \wedge 0.5 \geq t \wedge 0.5$ which implies that $\mu(xyz) \geq t$ or $\mu(xyz) \geq 0.5$ according to $t \leq 0.5$ or $t > 0.5$. Hence $(xyz)_t \in \nabla \bar{q} \mu$. Thus (2) holds.

Theorem 3.6. Let I be a right (resp. left, lateral) ideal of S . Then for every $t \in (0, 0.5]$ there exists a $(\in, \in \nabla q)$ -fuzzy right (resp. left, lateral) ideal μ of S such that $U(\mu; t) = I$.

Proof: Let μ be a fuzzy subset of S defined by

$$\mu(x) = \begin{cases} t & \text{if } x \in I \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in S$, where $t \in (0, 0.5]$. Obviously $U(\mu; t) = I$. Assume that $\mu(x+y) < \mu(x) \wedge \mu(y) \wedge 0.5$ for some $x, y \in S$. Since $|Im\mu| = 2$, it follows that $\mu(x+y) = 0$ and $\mu(x) \wedge \mu(y) \wedge 0.5 = t$. Hence $\mu(x) = t = \mu(y)$ and so $x, y \in I$. Since I is a right ideal of S , then $x+y \in I$ and so $\mu(x+y) = t$ which is a contradiction. Therefore $\mu(x+y) \geq \mu(x) \wedge \mu(y) \wedge 0.5$. Assume that $\mu(xyz) < \mu(x) \wedge 0.5$ for some $x, y, z \in S$. Then $\mu(xyz) = 0$ and $\mu(x) \wedge 0.5 = t$. Hence $\mu(x) = t$ and so $x \in I$. Since I is a right ideal of S , then $xyz \in I$ and so $\mu(xyz) = t$ which is a contradiction. Therefore $\mu(xyz) \geq \mu(x) \wedge 0.5$. Hence μ is a $(\in, \in \nabla q)$ -fuzzy right ideal of S .

Theorem 3.7. Let μ be a fuzzy subset of a ternary semiring S . Then μ is a $(\in, \in \nabla q)$ -fuzzy right (resp. left, lateral) ideal of S if and only if $U(\mu; t)$ is a right (resp. left, lateral) ideal of S for all $t \in (0, 0.5]$ whenever nonempty.

Proof: Let μ be a $(\in, \in \nabla q)$ -fuzzy right ideal of S and $t \in (0, 0.5]$. By Theorem 3.3, $U(\mu; t)$ is a ternary subsemiring. Let $x \in U(\mu; t)$ and $y, z \in S$. Then $\mu(xyz) \geq \mu(x) \wedge 0.5 \geq t \wedge 0.5 = t$. Thus $\mu(xyz) \geq t$, then $xyz \in U(\mu; t)$. Hence $U(\mu; t)$ is a right ideal of S . Conversely, let $U(\mu; t)$ be a right ideal for all $t \in (0, 0.5]$. By Theorem 3.3, μ is a $(\in, \in \nabla q)$ -fuzzy ternary subsemiring of S . If there exist $x, y, z \in S$ such that $\mu(xyz) < t = \mu(x) \wedge 0.5$, then $x \in U(\mu; t)$ and $y, z \in S$ with $xyz \notin U(\mu; t)$. This contradicts that $U(\mu; t)$ is a right ideal. Hence $\mu(xyz) \geq \mu(x) \wedge 0.5$. Therefore μ is a $(\in, \in \nabla q)$ -fuzzy right ideal of S .

Theorem 3.8. A non empty subset I of S is a right (resp. left, lateral) ideal of S if and only if χ_I is a $(\in, \in \nabla q)$ -fuzzy right (resp. left, lateral) ideal of S .

Proof: Let I be a right ideal of S then by theorem 3.6, χ_I is a $(\in, \in \nabla q)$ -fuzzy right ideal of S . Conversely, assume that χ_I is a $(\in, \in \nabla q)$ -fuzzy right ideal of a ternary semiring S . Let $x, y \in I$. Then $\chi_I(x+y) \geq \chi_I(x) \wedge \chi_I(y) \wedge 0.5 = 0.5$ which implies $\chi_I(x+y) = 1$. Thus $x+y \in I$. Similarly if $x \in I$; $y, z \in S$ we have $xyz \in I$. Hence I is a right ideal of S .

Theorem 3.9. Every fuzzy right (resp. left, lateral) ideal of S is a $(\in, \in \nabla q)$ -fuzzy right (resp. left, lateral) ideal of S .

Proof: Straight forward.

The converse part of the above theorem need not be true.

Example 3.10. Let S be a ternary semiring consists of non-positive integers. Let

$$\mu(x) = \begin{cases} 0.7 & \text{if } x = -4 \\ 0.5 & \text{if } x \in \langle -2 \rangle \text{ and } x \neq -4 \\ 0.1 & \text{otherwise} \end{cases}$$

Clearly μ is a $(\in, \in \nabla q)$ -fuzzy right ideal. But μ is not a fuzzy right ideal, since $\mu(-64) = -4 \cdot -4 \cdot -4 < \mu(-4)$.

In the following theorem, we give a condition for a ($\in, \in \vee q$)-fuzzy right (resp. left, lateral) ideal of S to be a fuzzy right (resp. left, lateral) ideal of S .

Theorem 3.11. *Let μ be a ($\in, \in \vee q$)-fuzzy right (resp. left, lateral) ideal of S such that $\mu(x) < 0.5$ for all $x \in S$. Then μ is a fuzzy right (resp. left, lateral) ideal of S .*

Proof: Let $x, y, z \in S$. Since μ is a ($\in, \in \vee q$)-fuzzy right ideal of S , we have $\mu(x+y) \geq \mu(x) \wedge \mu(y) \wedge 0.5 = \mu(x) \wedge \mu(y)$ and $\mu(xyz) \geq \mu(x) \wedge 0.5 = \mu(x)$. Thus μ is a fuzzy right ideal of S .

Definition 3.12. *For any fuzzy subset μ of S and $t \in (0, 1]$ we consider two subsets: $Q(\mu; t) = \{x \in S/x_q \mu\}$ and $[\mu]_t = \{x \in S/x_t \in \vee q \mu\}$. Obviously, $[\mu]_t = U(\mu; t) \cup Q(\mu; t)$. We call $[\mu]_t$ as a $\in \vee q$ -level set and $Q(\mu; t)$ a q -level set of μ .*

Lemma 3.13. *Let μ be a fuzzy subset of S . If $t \in (0, 0.5]$ then $[\mu]_t = U(\mu; t)$.*

Proof: Let $t \in (0, 0.5]$. Clearly $U(\mu; t) \subseteq [\mu]_t$. Let $x \in [\mu]_t$. If $x \notin U(\mu; t)$, then $\mu(x) < t$ and so $\mu(x) + t \leq t + t = 2t \leq 1$. This implies that $x \notin Q(\mu; t)$. Thus $x \notin U(\mu; t) \cup Q(\mu; t) = [\mu]_t$. This leads to a contradiction and so $x \in U(\mu; t)$. Thus $[\mu]_t \subseteq U(\mu; t)$. Therefore $[\mu]_t = U(\mu; t)$.

Theorem 3.14. *A fuzzy subset μ of S is a ($\in, \in \vee q$)-fuzzy right (resp. left, lateral) ideal of S if and only if $[\mu]_t (\neq \emptyset)$ is a right (resp. left, lateral) ideal of S for all $t \in (0, 1]$.*

Proof: Assume that μ is a ($\in, \in \vee q$)-fuzzy right ideal of S and let $t \in (0, 1]$ be such that $[\mu]_t \neq \emptyset$. Let $x, y \in [\mu]_t$ then $\mu(x) \geq t$ or $\mu(x) + t > 1$ and $\mu(y) \geq t$ or $\mu(y) + t > 1$. We can consider four cases:

- i) $\mu(x) \geq t$ and $\mu(y) \geq t$
- ii) $\mu(x) \geq t$ and $\mu(y) + t > 1$
- iii) $\mu(x) + t > 1$ and $\mu(y) \geq t$
- iv) $\mu(x) + t > 1$ and $\mu(y) + t > 1$.

Consider case (i): $\mu(x) \geq t$ and $\mu(y) \geq t$. This implies that $\mu(x+y) \geq \mu(x) \wedge \mu(y) \wedge 0.5 \geq t \wedge 0.5$. If $t > 0.5$, then $\mu(x+y) \geq 0.5$ and so $\mu(x+y) + t > 1$ that is, $(x+y)_t q \mu$. If $t \leq 0.5$, then $\mu(x+y) \geq t$ and thus $(x+y)_t \in \mu$. Therefore $(x+y)_t \in \vee q \mu$ that is $x+y \in [\mu]_t$.

case (ii) $\mu(x) \geq t$ and $\mu(y) + t > 1$. If $t > 0.5$, then $\mu(x+y) \geq \mu(x) \wedge \mu(y) \wedge 0.5 > t \wedge (1-t) \wedge 0.5 = 1-t$ that is $\mu(x+y) + t > 1$ and thus $(x+y)_t q \mu$. If $t \leq 0.5$, then $\mu(x+y) \geq \mu(x) \wedge \mu(y) \wedge 0.5 > t \wedge (1-t) \wedge 0.5 = t$ that is $(x+y)_t \in \mu$ and thus $(x+y)_t \in \vee q \mu$. This means that $x+y \in [\mu]_t$.

Similarly, we can prove the result for the case (iii).

Next we consider the case (iv): $\mu(x) + t > 1$ and $\mu(y) + t > 1$. If $t > 0.5$, then $\mu(x+y) \geq \mu(x) \wedge \mu(y) \wedge 0.5 > (1-t) \wedge (1-t) \wedge 0.5 = 1-t$. So, $\mu(x+y) + t > 1$ that is $(x+y)_t q \mu$. If $t \leq 0.5$, then $\mu(x+y) \geq \mu(x) \wedge \mu(y) \wedge 0.5 > (1-t) \wedge (1-t) \wedge 0.5 = 0.5 \geq t$ that is $(x+y)_t \in \mu$ and hence $(x+y)_t \in \vee q \mu$. This means that $x+y \in [\mu]_t$.

Consequently, $[\mu]_t$ is a ternary subsemiring of $(S, +)$. Let $x \in [\mu]_t$ and $y, z \in S$ then $\mu(x) \geq t$ or $\mu(x) + t > 1$. Assume

that $\mu(x) \geq t$. Since μ is a ($\in, \in \vee q$)-fuzzy right ideal of S , we have $\mu(xyz) \geq \mu(x) \wedge 0.5 \geq t \wedge 0.5$. If $t > 0.5$, then $\mu(xyz) \geq 0.5$ which implies $\mu(xyz) + t > 0.5 + 0.5 = 1$. So, $\mu(xyz) + t > 1$ and thus $(xyz)_t q \mu$. If $t \leq 0.5$, then $\mu(xyz) \geq t$. Hence $(xyz)_t \in \mu$. Let us assume that $\mu(x) + t > 1$. Since μ is a ($\in, \in \vee q$)-fuzzy right ideal of S , we have $\mu(xyz) \geq \mu(x) \wedge 0.5 > (1-t) \wedge 0.5$. If $t > 0.5$, then $\mu(xyz) > 1-t$. So, $(xyz)_t q \mu$. If $t \leq 0.5$, then $\mu(xyz) > 0.5 \geq t$. Thus $(xyz)_t \in \mu$. This implies that $(xyz)_t \in \vee q \mu$, that is $xyz \in [\mu]_t$. Therefore $[\mu]_t$ is a right ideal of S .

Conversely, assume that $[\mu]_t$ is a right ideal of S for all $t \in (0, 1]$. Suppose that $\mu(x+y) < \mu(x) \wedge \mu(y) \wedge 0.5$. Choose t such that $\mu(x+y) < t < \mu(x) \wedge \mu(y) \wedge 0.5$. Then $0 < t \leq 0.5$ and $x, y \in U(\mu; t) \subseteq [\mu]_t$. Since $[\mu]_t$ is a right ideal of S , then $x+y \in [\mu]_t$ and we have $\mu(x+y) \geq t$ or $\mu(x+y) + t > 1$, which is a contradiction. Thus $\mu(x+y) \geq \mu(x) \wedge \mu(y) \wedge 0.5$, for all $x, y \in S$. Now let $x, y, z \in S$ be such that $\mu(xyz) < t < \mu(x) \wedge 0.5$. Then $0 < t \leq 0.5$ and $x \in U(\mu; t) \subseteq [\mu]_t$. Since $[\mu]_t$ is a right ideal of S , then $xyz \in [\mu]_t$ and so $\mu(xyz) \geq t$ or $\mu(xyz) + t > 1$, which is a contradiction to our assumption. Hence $\mu(xyz) \geq \mu(x) \wedge 0.5$ for all $x, y, z \in S$. Therefore μ is a ($\in, \in \vee q$)-fuzzy right ideal of S .

Theorem 3.15. *If μ is a ($\in, \in \vee q$)-fuzzy right (resp. left, lateral) ideal of S then the set $Q(\mu; t) (\neq \emptyset)$ is a right (resp. left, lateral) ideal of S for all $t \in (0.5, 1]$.*

Proof: Assume that μ is a ($\in, \in \vee q$)-fuzzy right ideal of S and let $t \in (0.5, 1]$ be such that $Q(\mu; t) \neq \emptyset$. Let $x, y \in Q(\mu; t)$ then $\mu(x) + t > 1$ and $\mu(y) + t > 1$ which implies $\mu(x) > 1-t < 0.5$ and $\mu(y) > 1-t < 0.5$. Since μ is a ($\in, \in \vee q$)-fuzzy right ideal we have $\mu(x+y) \geq \mu(x) \wedge \mu(y) \wedge 0.5 > 1-t$. This implies that $x+y \in Q(\mu; t)$. Similarly, let $x \in Q(\mu; t)$ and $y, z \in S$ then $xyz \in Q(\mu; t)$. Therefore $Q(\mu; t)$ is a right ideal of S .

Theorem 3.16. *Let $f : S_1 \rightarrow S_2$ be an onto homomorphism of ternary semirings S_1 and S_2 . Let μ be a fuzzy subset of S_2 . Then μ is a ($\in, \in \vee q$)-fuzzy right (resp. left, lateral) ideal of S_2 if and only if $f^{-1}(\mu)$ is a ($\in, \in \vee q$)-fuzzy right (resp. left, lateral) ideal of S_1 where $[f^{-1}(\mu)](x) = \mu(f(x))$ for all $x \in S_1$.*

Proof: Let μ be a ($\in, \in \vee q$)-fuzzy right ideal of S_2 . Let $x_1, x_2, x_3 \in S_1$. Now $f^{-1}(\mu)(x_1 + x_2) = \mu(f(x_1 + x_2)) = \mu(f(x_1) + f(x_2)) \geq \mu(f(x_1)) \wedge \mu(f(x_2)) \wedge 0.5 = f^{-1}(\mu)(x_1) \wedge f^{-1}(\mu)(x_2) \wedge 0.5$.

Now $f^{-1}(\mu)(x_1 x_2 x_3) = \mu(f(x_1 x_2 x_3)) = \mu(f(x_1) f(x_2) f(x_3)) \geq \mu(f(x_1)) \wedge 0.5 = f^{-1}(\mu)(x_1) \wedge 0.5$. Thus $f^{-1}(\mu)$ is a ($\in, \in \vee q$)-fuzzy right ideal of S_1 . Conversely let $f^{-1}(\mu)$ is a ($\in, \in \vee q$)-fuzzy right ideal of S_1 . Let $y_1, y_2, y_3 \in S_2$ such that $f(x_1) = y_1$, $f(x_2) = y_2$ and $f(x_3) = y_3$ where $x_1, x_2, x_3 \in S_1$. $\mu(y_1 + y_2) = \mu(f(x_1) + f(x_2)) = \mu(f(x_1 + x_2)) = f^{-1}(\mu)(x_1 + x_2) \geq f^{-1}(\mu)(x_1) \wedge f^{-1}(\mu)(x_2) \wedge 0.5 = \mu(f(x_1)) \wedge \mu(f(x_2)) \wedge 0.5 = \mu(y_1) \wedge \mu(y_2) \wedge 0.5$. Similarly we have $\mu(y_1 y_2 y_3) \geq \mu(y_1) \wedge 0.5$. Thus μ is a ($\in, \in \vee q$)-fuzzy right ideal of S_2 .

Theorem 3.17. Let $f : S_1 \rightarrow S_2$ be an onto homomorphism of ternary semirings S_1 and S_2 and let μ be a fuzzy subset of S_1 . If μ is a ($\in, \in \vee q$)-fuzzy right (resp. left, lateral) ideal of S_1 then $f(\mu)$ is a ($\in, \in \vee q$)-fuzzy right (resp. left, lateral) ideal of S_2 where $f(\mu)(y) = \sup\{\mu(x)/f(x) = y\}$.

Proof: Let us assume that μ is a ($\in, \in \vee q$)-fuzzy right ideal of S_1 . Let $y_1, y_2, y_3 \in S_2$ then there exist $x_1, x_2, x_3 \in S_1$ such that $f(x_1) = y_1$, $f(x_2) = y_2$ and $f(x_3) = y_3$. Now $f(\mu)(y_1 + y_2) = \sup_{x \in S_1} \{\mu(x)/f(x) = y_1 + y_2\}$
 $\geq \sup_{x_1, x_2 \in S_1} \{\mu(x_1 + x_2)/f(x_1 + x_2) = y_1 + y_2\}$
 $= \sup_{x_1, x_2 \in S_1} \{\mu(x_1 + x_2)/f(x_1) + f(x_2) = y_1 + y_2\}$
 $\geq \sup\{\mu(x_1) \wedge \mu(x_2) \wedge 0.5 / f(x_1) = y_1, f(x_2) = y_2\}$
 $= \sup\{\mu(x_1)/f(x_1) = y_1\} \wedge \sup\{\mu(x_2)/f(x_2) = y_2\} \wedge 0.5$
 $= f(\mu)(y_1) \wedge f(\mu)(y_2) \wedge 0.5$.
 Now $f(\mu)(y_1 y_2 y_3) \geq \sup\{\mu(x_1 x_2 x_3)/f(x_1 x_2 x_3) = y_1 y_2 y_3\}$
 $= \sup\{\mu(x_1 x_2 x_3)/f(x_1) f(x_2) f(x_3) = y_1 y_2 y_3\}$
 $\geq \sup\{\mu(x_1) \wedge 0.5 / f(x_1) = y_1\}$
 $= \sup\{\mu(x_1)/f(x_1) = y_1\} \wedge 0.5$
 $= f(\mu)(y_1) \wedge 0.5$.

Thus $f(\mu)$ is a ($\in, \in \vee q$)-fuzzy right ideal of S_2 .

The following example shows that converse of the above theorem need not be true.

Example 3.18. Let Z_0^- and Z_6^- be the ternary semirings of negative integers and negative integer modulo 6 respectively. The mapping f defined by $f : Z_0^- \rightarrow Z_6^-$, $f(x) = k \pmod{6}$ where $k \equiv x \pmod{6}$, $-5 \leq k \leq 0$ is a homomorphism and f is onto. Let

$$\mu(x) = \begin{cases} 0.6 & \text{if } x = -18 \\ 0.3 & \text{if } x \in \langle -9 \rangle \text{ and } x \neq -18 \\ 0.4 & \text{if } x \in \langle -3 \rangle \text{ and } x \notin \langle -9 \rangle \\ 0.2 & \text{otherwise} \end{cases}$$

Then

$$f(\mu)(x) = \begin{cases} 0.6 & \text{if } x = 0 \\ 0.4 & \text{if } x = -3 \\ 0.2 & \text{otherwise} \end{cases}$$

Clearly $f(\mu)$ is a ($\in, \in \vee q$)-fuzzy right ideal of Z_6^- but μ is not a ($\in, \in \vee q$)-fuzzy right ideal of Z_0^- , since $\mu(-18 + (-18)) < \mu(-18) \wedge \mu(-18) \wedge 0.5$.

Theorem 3.19. Every ($\in \vee q, \in \vee q$)-fuzzy right (resp. left, lateral) ideal of S is a ($\in, \in \vee q$)-fuzzy right (resp. left, lateral) ideal of S .

Proof: Let μ be a ($\in \vee q, \in \vee q$)-fuzzy right ideal of S . Suppose that $x, y \in S$ and $t, r \in (0, 0.5]$ such that $x_t \in \mu$ and $y_r \in \mu$. Then $x_t \in \vee q \mu$ and $y_r \in \vee q \mu$. By hypothesis, $(x + y)_{t \wedge r} \in \vee q \mu$. Now $x, y, z \in S$ and $t \in (0, 0.5]$ such that $x_t \in \mu$. Then $x_t \in \vee q \mu$ so by hypothesis, $(xyz)_t \in \vee q \mu$.

Theorem 3.20. A fuzzy subset μ of S is a ($\in, \in$)-fuzzy right (resp. left, lateral) ideal of S if and only if it is a fuzzy right (resp. left, lateral) ideal of S .

Proof: Assume that μ is a fuzzy right ideal of S . Let $x, y \in S$ and $t, r \in (0, 1]$ be such that $x_t, y_r \in \mu$. Then $\mu(x) \geq t$ and $\mu(y) \geq r$. Since μ is a fuzzy right ideal of S , we have $\mu(x + y) \geq \mu(x) \wedge \mu(y) \geq t \wedge r$, it follows that $(x + y)_{t \wedge r} \in \mu$. Now let $x, y, z \in S$ and $t \in (0, 1]$ be such that $x_t \in \mu$ and so $\mu(x) \geq t$. Since μ is a fuzzy right ideal of S , we have $\mu(xyz) \geq \mu(x) \geq t$. Thus $(xyz)_t \in \mu$. Therefore μ is a ($\in, \in$)-fuzzy right ideal of S . Conversely, assume that μ is a ($\in, \in$)-fuzzy right ideal of S . On the contrary assume that there exist $x, y \in S$ such that $\mu(x + y) < \mu(x) \wedge \mu(y)$. Choose t such that $\mu(x + y) < t < \mu(x) \wedge \mu(y)$. Then $x_t, y_t \in \mu$ and $(x + y)_t \notin \mu$. This is a contradiction to our assumption that μ is a ($\in, \in$)-fuzzy right ideal of S . Thus $\mu(x + y) \geq \mu(x) \wedge \mu(y)$. Let us assume that $\mu(xyz) < \mu(x)$ for some $x, y, z \in S$. Then there exist t such that $\mu(xyz) < t < \mu(x)$. This implies that $x_t \in \mu$ but $(xyz)_t \notin \mu$. This contradicts our hypothesis. Hence $\mu(xyz) \geq \mu(x)$. Therefore μ is a fuzzy right ideal of S .

References

- [1] Bhakat, S.K. and Das, P., On the definition of a fuzzy subgroup, Fuzzy Sets and Systems, V51, no.2, 235-241, 1992.
- [2] Bhakat, S.K. and Das, P., ($\in, \in \vee q$)-fuzzy subgroup, Fuzzy Sets and Systems, V80, no.3, 359-368, 1996.
- [3] Dutta, T.K., Kar, S., On regular ternary semirings, Advances in Algebra, Proceedings of the ICM Satellite Conference in Algebra and Related Topics, World Scientific, pp. 343-355, 2003.
- [4] Dutta, T.K., Kar, S., On the Jacobson radical of a ternary semiring, SEA Bull. Math., V28, N1, pp. 1-13, 2004.
- [5] Dutta, T.K., Kar, S., A note on the Jacobson radicals of ternary semirings, SEA Bull. Math., V29, N2, pp. 321-331, 2005.
- [6] Dutta, T.K., Kar, S., Two types of Jacobson radicals of ternary semirings, SEA Bull. Math., V29, N4, pp. 677-687, 2005.
- [7] Dutta, T.K., Kar, S., On prime ideals and prime radical of ternary semirings, Bull. Cal. Math. Soc., V97, N5, pp. 445-454, 2005.
- [8] Dutta, T.K., Kar, S., On semiprime ideals and irreducible ideals of ternary semirings, Bull. Cal. Math. Soc., V97, N5, pp. 467-476, 2005.
- [9] Dutta, T.K., Kar, S., A note on regular ternary semirings, Kyungpook Math. J., V46, N3, pp. 357-365, 2006.
- [10] Kar, S., On quasi-ideals and bi-ideals in ternary semirings, Int. J. Math. Math. Sci., V2005, N18, pp. 3015-3023, 2005.
- [11] Kavikumar, J., Khamis, A.B., Fuzzy ideals and fuzzy quasi-ideals in ternary semirings, IAENG Int. J. Appl. Math., V37, N2, pp. 102-106, 2007.
- [12] Kavikumar, J., Khamis, A.B., Jun, Y.B., Fuzzy bi-ideals

- in ternary semirings, Int. J. Comp. Math. Sci., V3, N4, pp. 164-168, 2009.
- [13] Krishnaswamy, D., Anitha, T., Fuzzy prime ideals in ternary semiring, Annals of Fuzzy Mathematics and Informatics, 7(3), 755-763, 2014.
- [14] Lehmer, D.H., A ternary analogue of abelian groups, Amer. J. Math., V59, pp. 329-338, 1932.
- [15] Lister, W.G., Ternary rings, Trans. Amer. Math. Soc., V154, pp. 37-55, 1971.
- [16] Murali, V., Fuzzy points of equivalent fuzzy subsets, Information Sciences, V158, 277-288, 2004.
- [17] Pu, P.M. and Liu, Y.M. Fuzzy topology I, neighbourhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl. V76, 571- 599, 1980.
- [18] Rosenfeld, A., Fuzzy groups, J. Math. Anal. Appl. V35, 512-517, 1971.
- [19] Yuan, X., Zhang, C. and Ren, Y., Generalized fuzzy groups and many-valued implications, Fuzzy Sets and Systems V138, 205-211, 2003.
- [20] Zadeh, L.A., Fuzzy sets, Inform. and Control., V8, pp. 338-353, 1965.

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