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Interval-valued intuitionistic $Q-$fuzzy bi-ideals in ternary semirings

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Abstract
In this paper we introduce the notions of interval-valued Q - fuzzy bi-ideal, interval-valued anti Q - fuzzy bi-ideal and interval-valued intuitionistic Q - fuzzy bi-ideal in ternary semirings and some of the basic properties of these ideals are investigated. We also introduce normal interval-valued intuitionistic Q - fuzzy ideals in ternary semirings.

Keywords
Interval-valued intuitionistic Q - fuzzy ideal, interval-valued Q -fuzzy bi-ideal, interval-valued anti Q - fuzzy bi-ideal, interval-valued intuitionistic Q - fuzzy bi-ideal, normal interval-valued intuitionistic Q - fuzzy ideal, interval-valued intuitionistic Q - fuzzy maximal ideal.

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Contents

1. Introduction

The notion of ternary algebraic system was introduced by Lehmer [13] in 1932. He investigated certain ternary algebraic systems called triplexes. In 1971, Lister [14] characterized additive semigroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring. Dutta and Kar [3] introduced a notion of ternary semirings which is a generalization of ternary rings and semirings, and they studied some properties of ternary semirings [3–9, 12]. The theory of fuzzy sets was first studied by Zadeh [16] in 1965. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory, etc. Interval-valued fuzzy sets were introduced independently by Zadeh [17], Grattan-Guiness [10], Jahn [11], Sambuc [15] in the same year 1975 as a generalization of fuzzy set. An interval-valued fuzzy set is a fuzzy set whose membership function is many-valued and forms an interval in the membership scale. On the other hand, Atanassov [1] introduced the notion of intuitionistic fuzzy sets as an extension of fuzzy set in which not only a membership degree is given, but also a non-membership degree is involved. Atanassov and Gargov [2] introduced the notion of interval-valued intuionistic fuzzy sets which is a common generalization of intuitionistic fuzzy sets and interval-valued fuzzy sets. Osman kazanci etal[18]have introduced the notion of intuitionistic Q- fuzzification of N-subgroups in a near ring . Lekkoksung [19] studied intuitionistic Q-fuzzy K-ideals of semigroups. Balasubramaniyan etal[20] Introduced interval-valued intuitionistic Q- fuzzy ideals in ternary semirings. In this paper we introduce the notions of interval-valued Q-fuzzy bi-ideal, interval-valued anti Q-fuzzy bi-ideal and interval-valued intuitionistic Q-fuzzy bi-ideal in ternary semirings and some of the basic properties of these ideals are investigated. We also introduce normal interval-valued intuitionistic fuzzy ideals in ternary semirings.

2. Preliminaries

In this section, we refer to some elementary aspects of the theory of ternary semirings and interval-valued fuzzy algebraic systems that are necessary for this paper.

Definition 2.1. *A nonempty set S together with a binary operation called, addition + and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if $(S,+)$ is a commutative semigroup satisfying the following conditions:*
(i) $(abc)de = a(bcd)e = ab(cde)$,
(ii) $(a+b)cd = acd + bcd$,
(iii) $a(b+c)d = abd + acd$
and (iv) $ab(c+d) = abc + abd$ for all $a,b,c,d,e \in S$.

Definition 2.2. *Let S be a ternary semiring. If there exists an element $0 \in S$ such that $0 + x = x = x + 0$ and $0xy = x0y = xy0 = 0$ for all $x, y \in S$, then 0 is called the zero element or simply the zero of the ternary semiring S. In this case we say that S is a ternary semiring with zero.*

Throughout this paper S denotes a ternary semiring with zero.

Definition 2.3. *An additive subsemigroup T of S is called a ternary subsemiring of S if $t_1t_2t_3 \in T$ for all $t_1, t_2, t_3 \in T$.*

Definition 2.4. *An additive subsemigroup I of S is called a left [resp. right, lateral] ideal of S if $s_1s_2i \in I$ [resp. $is_1s_2 \in I$, $s_1is_2 \in I$] for all $s_1, s_2 \in S$ and $i \in I$. If I is a left, right and lateral ideal of S, then I is called an ideal of S.*

It is obvious that every ideal of a ternary semiring with zero contains the zero element.

Definition 2.5. *An additive subsemigroup $(B,+)$ of a ternary semiring S is called a bi-ideal of S if $BSBSB \subseteq B$.*

Definition 2.6. *Let S_1 and S_2 be ternary semirings. A mapping $f : S_1 \to S_2$ is said to be a homomorphism if $f(x+y) = f(x) + f(y)$ and $f(xyz) = f(x)f(y)f(z)$ for all $x, y, z \in S_1$.*

Let $f : S_1 \to S_2$ be an onto homomorphism of ternary semirings. Note that if I is an ideal of S_1, then $f(I)$ is an ideal of S_2. If S_1 and S_2 be ternary semirings with zero 0, then $f(0) = 0$.

Definition 2.7. *An interval number on $[0,1]$, denoted by $\widetilde{a}$, is defined as the closed sub interval of $[0,1]$, where $\widetilde{a} = [a^-, a^+]$ satisfying $0 \le a^- \le a^+ \le 1$.*

The set of all interval numbers is denoted by $D[0,1]$. The interval $[a,a]$ is identified with the number $a \in [0,1]$.

Definition 2.8. *Let $\widetilde{a} = [a^-, a^+]$ and $\widetilde{b} = [b^-, b^+]$ be two interval numbers in $D[0,1]$. Then*
i) $\widetilde{a} \le \widetilde{b}$ if and only if $a^- \le b^-$ and $a^+ \le b^+$
ii) $\widetilde{a} + \widetilde{b} = [a^- + b^-, a^+ + b^+]$
iii) If $\widetilde{a} \ge \widetilde{b}$ then $\widetilde{a} - \widetilde{b} = [\, min\{a^- - b^-, a^+ - b^+\},\ max\{a^- - b^-, a^+ - b^+\}\,]$
iv) $inf\widetilde{a_i} = [\bigwedge_{i\in I} a_i{}^-, \bigwedge_{i\in I} a_i{}^+]$, $sup\widetilde{a_i} = [\bigvee_{i\in I} a_i{}^-, \bigvee_{i\in I} a_i{}^+]$ for interval numbers $\widetilde{a_i} = [a_i{}^-, a_i{}^+] \in D[0,1], i \in I$.

Definition 2.9. *Let $\{\widetilde{a_i}\}, i = 1, 2, \cdots, n$ for some $n \in Z^+$ be a finite number of interval numbers, where $\widetilde{a_i} = [a_i{}^-, a_i{}^+]$. Then we define $Max^i\{\widetilde{a_i}\} = [\ max\{a_i{}^-\},\ \ max\{a_i{}^+\}\ ]$ and $Min^i\{\widetilde{a_i}\} = [\ min\{a_i{}^-\},\ min\{a_i{}^+\}\ ]$.*

In this paper we assume that any two interval numbers in $D[0,1]$ are comparable. i.e. for any two interval numbers $\widetilde{a}$ and $\widetilde{b}$ in $D[0,1]$, we have either $\widetilde{a} \le \widetilde{b}$ or $\widetilde{a} > \widetilde{b}$. It is clear that $(D[0,1], \le, \vee, \wedge)$ is a complete lattice with $\widetilde{0} = [0,0]$ as the least element and $\widetilde{1} = [1,1]$ as the greatest element.

Definition 2.10. *Let X be a non-empty set. A map $\widetilde{\mu} : X \times Q \to D[0,1]$ is called an interval-valued Q- fuzzy subset of X. The complement of an interval-valued Q- fuzzy subset $\widetilde{\mu}$ of a set X is denoted by $\widetilde{\mu}^c$ and defined as $\widetilde{\mu}^c(x,q) = \widetilde{1} - \widetilde{\mu}(x,q)$, for all $x \in X, q \in Q$.*

Note: We can write $\widetilde{\mu}(x,q) = [\mu^-(x,q), \mu^+(x,q)]$ for all $x \in X, q \in Q$, for any interval-valued Q - fuzzy subset $\widetilde{\mu}$ of a non empty set X, where μ^- and μ^+ are some Q- fuzzy subsets of X.

Definition 2.11. *Let $\widetilde{\mu}$ and $\widetilde{\nu}$ be two interval-valued Q- fuzzy subsets of a non-empty set X. Then $\widetilde{\mu}$ is said to be a subset of $\widetilde{\nu}$, denoted by $\widetilde{\mu} \subseteq \widetilde{\nu}$ if $\widetilde{\mu}(x,q) \le \widetilde{\nu}(x,q)$, i.e., $\mu^-(x,q) \le \nu^-(x,q)$ and $\mu^+(x,q) \le \nu^+(x,q)$, for all $x \in X, q \in Q$ where $\widetilde{\mu}(x,q) = [\mu^-(x,q), \mu^+(x,q)]$ and $\widetilde{\nu}(x,q) = [\nu^-(x,q), \nu^+(x,q)]$.*

Definition 2.12. *An upper level set of an interval-valued Q - fuzzy subset $\widetilde{\mu}$ denoted by $\overline{U}(\widetilde{\mu};\widetilde{t})$ is defined as $\overline{U}(\widetilde{\mu};\widetilde{t}) = \{x \in X/\widetilde{\mu}(x,q) \ge \widetilde{t}\}$ and a lower level set of an interval-valued fuzzy subset $\widetilde{\mu}$ denoted by $\overline{L}(\widetilde{\mu};\widetilde{t})$ is defined as $\overline{L}(\widetilde{\mu};\widetilde{t}) = \{x \in X/\widetilde{\mu}(x,q) \le \widetilde{t}\}$, for all $\widetilde{t} \in D[0,1]$.*

Definition 2.13. *Let I be any subset of a ternary semiring S. The interval-valued characteristic function of I denoted by $\widetilde{\chi}_I$ is defined as*

$$\widetilde{\chi}_I = \begin{cases} \widetilde{1} & if\ \ x \in I \\ \widetilde{0} & otherwise \end{cases}$$

Definition 2.14. *Let $\widetilde{\mu}$,$\widetilde{\nu}$ and $\widetilde{h}$ be any three interval-valued Q - fuzzy subsets of a ternary semiring S. Then $\widetilde{\mu} \cap \widetilde{\nu}$, $\widetilde{\mu} \cup \widetilde{\nu}$, $\widetilde{\mu} + \widetilde{\nu}$, $\widetilde{\mu} \circ \widetilde{\nu} \circ \widetilde{h}$ are interval-valued Q-fuzzy subsets of S defined by for all $x \in S, q \in Q$*

$$(\widetilde{\mu} \cap \widetilde{\nu})(x,q) = Min^i\{\widetilde{\mu}(x,q),\ \widetilde{\nu}(x,q)\}$$
$$(\widetilde{\mu} \cup \widetilde{\nu})(x,q) = Max^i\{\widetilde{\mu}(x,q),\ \widetilde{\nu}(x,q)\}$$

$$(\widetilde{\mu} + \widetilde{\nu})(x,q) = \begin{cases} sup\{Min^i\{\widetilde{\mu}(y,q), \widetilde{\nu}(z,q)\}\} if\ \ x = y + z \\ \widetilde{0} \qquad otherwise \end{cases}$$

$$(\widetilde{\mu} \circ \widetilde{\nu} \circ \widetilde{h})(x,q) = \begin{cases} sup\{Min^i\{\widetilde{\mu}(u,q), \widetilde{\nu}(v,q), \widetilde{h}(w,q)\}\} if\ \ x = uvw, \\ \widetilde{0} \qquad otherwise \end{cases}$$

Definition 2.15. *An interval-valued intuitionistic Q - fuzzy subset (IIQFS for short) defined on non-empty set S as objects of the form*

$A = \{< \widetilde{\mu}_A(x,q), \widetilde{\nu}_A(x,q) > /x \in S, q \in Q\}$, where the function $\widetilde{\mu} : S \times Q \to D[0,1]$ and $\widetilde{\nu} : S \times Q \to D[0,1]$ denote the degree of membership (namely $\widetilde{\mu}_A(x,q)$) and the degree of non-membership (namely $\widetilde{\nu}_A(x,q)$) for each element $x \in S, q \in Q$ to the set A, respectively, and $\widetilde{0} \leq \widetilde{\mu}_A(x,q) + \widetilde{\nu}_A(x,q) \leq \widetilde{1}$ for each $x \in S, q \in Q$.

For the sake of simplicity, we shall use the symbol $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ for the interval-valued intuitionistic Q - fuzzy subset $A = \{< \widetilde{\mu}_A(x,q), \widetilde{\nu}_A(x,q) > /x \in S, q \in Q\}$.

3. Interval-valued intuitionistic Q - fuzzy bi-ideals

Definition 3.1. *An interval-valued intuitionistic Q - fuzzy subset $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ in S is called an interval-valued intuitionistic Q - fuzzy right (left, lateral) ideal of S if*
1. $\widetilde{\mu}_A(x+y,q) \geq Min^i\{\widetilde{\mu}_A(x,q), \widetilde{\mu}_A(y,q)\}$
2. $\widetilde{\mu}_A(xyz,q) \geq \widetilde{\mu}_A(x,q)$
$(\widetilde{\mu}_A(xyz,q) \geq \widetilde{\mu}_A(z,q), \widetilde{\mu}_A(xyz,q) \geq \widetilde{\mu}_A(y,q))$
3. $\widetilde{\nu}_A(x+y,q) \leq Max^i\{\widetilde{\nu}_A(x,q), \widetilde{\nu}_A(y,q)\}$
4. $\widetilde{\nu}_A(xyz,q) \leq \widetilde{\nu}_A(x,q)$
$(\widetilde{\nu}_A(xyz,q) \leq \widetilde{\nu}_A(z,q),\ \widetilde{\nu}_A(xyz,q) \leq \widetilde{\nu}_A(y,q))$ for all $x,y,z \in S$and $q \in Q$.

Definition 3.2. *Let $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ be an interval-valued intuitionistic Q - fuzzy subset of S and let $\widetilde{s}, \widetilde{t} \in D[0,1]$. Then the set $\overline{S}_A^{(\widetilde{s},\widetilde{t})} = \{x \in S/\ \widetilde{\mu}_A(x,q) \geq \widetilde{s}, \widetilde{\nu}_A(x,q) \leq \widetilde{t}\}$ is called a $(\widetilde{s},\widetilde{t})$-level set of $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$.*

The set $\{(\widetilde{s},\widetilde{t}) \in Im(\widetilde{\mu}_A) \times Im(\widetilde{\nu}_A)/\ \widetilde{s}+\widetilde{t} \leq \widetilde{1}\}$ is called image of $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$.
Clearly $\overline{S}_A^{(\widetilde{s},\widetilde{t})} = \overline{U}(\widetilde{\mu}_A;\widetilde{s}) \cap \overline{L}(\widetilde{\nu}_A;\widetilde{t})$, where $\overline{U}(\widetilde{\mu}_A;\widetilde{s})$ and $\overline{L}(\widetilde{\nu}_A;\widetilde{t})$ are upper and lower level subsets of $\widetilde{\mu}_A$ and $\widetilde{\nu}_A$ respectively.

Definition 3.3. *An interval-valued Q - fuzzy subset $\widetilde{\mu}$ of a ternary semiring S is said to be an interval-valued Q - fuzzy bi-ideal of S if*
1. $\widetilde{\mu}_A(x+y,q) \geq Min^i\{\widetilde{\mu}_A(x,q), \widetilde{\mu}_A(y,q)\}$
2. $\widetilde{\mu}_A(xs_1ys_2z,q) \geq Min^i\{\widetilde{\mu}_A(x,q), \widetilde{\mu}_A(y,q), \widetilde{\mu}_A(z,q)\}$ for all $x,s_1,y,s_2,z \in S$ and $q \in Q$.

Definition 3.4. *An interval-valued Q - fuzzy subset $\widetilde{\mu}$ of a ternary semiring S is said to be an interval-valued anti Q - fuzzy bi-ideal of S if*
1. $\widetilde{\mu}_A(x+y,q) \leq Max^i\{\widetilde{\mu}_A(x,q), \widetilde{\mu}_A(y,q)\}$
2. $\widetilde{\mu}_A(xs_1ys_2z,q) \leq Max^i\{\widetilde{\mu}_A(x,q), \widetilde{\mu}_A(y,q), \widetilde{\mu}_A(z,q)\}$ for all $x,s_1,y,s_2,z \in S$ and $q \in Q$.

Definition 3.5. *An interval-valued intuitionistic Q - fuzzy subset $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ in S is called an interval-valued intuitionistic Q - fuzzy bi-ideal of S if*
1. $\widetilde{\mu}_A(x+y,q) \geq Min^i\{\widetilde{\mu}_A(x,q), \widetilde{\mu}_A(y,q)\}$
2. $\widetilde{\mu}_A(xs_1ys_2z,q) \geq Min^i\{\widetilde{\mu}_A(x,q), \widetilde{\mu}_A(y,q), \widetilde{\mu}_A(z,q)\}$
3. $\widetilde{\nu}_A(x+y,q) \leq Max^i\{\widetilde{\nu}_A(x,q), \widetilde{\nu}_A(y,q)\}$
4. $\widetilde{\nu}_A(xs_1ys_2z,q) \leq Max^i\{\widetilde{\nu}_A(x,q), \widetilde{\nu}_A(y,q), \widetilde{\nu}_A(z,q)\}$ for all $x,s_1,y,s_2,z \in S$ and $q \in Q$.

Example 3.6. *Consider*

$$S = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & h \end{pmatrix} : a,b,c,d,e,h \in Z_0^- \right\}.$$

Then S is a ternary semiring with respect to matrix addition and matrix multiplication. Let

$$B = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & j \\ 0 & 0 & 0 \end{pmatrix} : i,j \in Z_0^- \right\}$$

Let the interval-valued Q - fuzzy subset $\widetilde{\mu}_A$ and $\widetilde{\nu}_A$ of S be defined by

$$\widetilde{\mu}_A(x,q) = \begin{cases} [0.6,0.8] & if\ \ x \in B \\ [0.1,0.3] & otherwise \end{cases}$$

$$\widetilde{\nu}_A(x,q) = \begin{cases} [0.1,0.2] & if\ \ x \in B \\ [0.4,0.6] & otherwise \end{cases}$$

Then $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ is an interval-valued intuitionistic Q - fuzzy bi-ideal of S, but not an interval-valued intuitionistic Q - fuzzy ideal of S. Since $\widetilde{\mu}_A(ssb,q) = [0.1,0.3] < \widetilde{\mu}_A(b,q)$; $\widetilde{\mu}_A(sbs,q) = [0.1,0.3] < \widetilde{\mu}_A(b,q)$; $\widetilde{\mu}_A(bss,q) = [0.1,0.3] < \widetilde{\mu}_A(b,q)$; $\widetilde{\nu}_A(ssb,q) = [0.4,0.6] > \widetilde{\nu}_A(b,q)$; $\widetilde{\nu}_A(sbs,q) = [0.4,0.6] > \widetilde{\nu}_A(b,q)$ and $\widetilde{\nu}_A(bss,q) = [0.4,0.6] > \widetilde{\nu}_A(b,q)$, where

$$s = \begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Theorem 3.7. *If an interval-valued Q-fuzzy subset $\widetilde{\mu}$ is an interval-valued Q - fuzzy bi-ideal of a ternary semiring S if and only if $\widetilde{\mu}^c$ is an interval-valued anti Q - fuzzy bi-ideal of S.*

Proof. Let $\widetilde{\mu}$ be an interval-valued Q - fuzzy bi-ideal of a ternary semiring S. Let $x,y,z \in S, q \in Q$. Then
$\widetilde{\mu}(x+y,q) \geq Min^i\{\widetilde{\mu}(x,q), \widetilde{\mu}(y,q)\}$
$\Rightarrow -\widetilde{\mu}(x+y,q) \leq -Min^i\{\widetilde{\mu}(x,q), \widetilde{\mu}(y,q)\}$
$\Rightarrow \widetilde{1} - \widetilde{\mu}(x+y,q) \leq \widetilde{1} - Min^i\{\widetilde{\mu}(x,q), \widetilde{\mu}(y,q)\}$
$\Rightarrow \widetilde{1} - \widetilde{\mu}(x+y,q) \leq Max^i\{\widetilde{1} - \widetilde{\mu}(x,q), \widetilde{1} - \widetilde{\mu}(y,q)\}$
$\Rightarrow \widetilde{\mu}^c(x+y,q) \leq Max^i\{\widetilde{\mu}^c(x,q), \widetilde{\mu}^c(y,q)\}$ and $\widetilde{\mu}(xs_1ys_2z,q) \geq Min^i\{\widetilde{\mu}(x,q), \widetilde{\mu}(y,q), \widetilde{\mu}(z,q)\}$
$\Rightarrow -\widetilde{\mu}(xs_1ys_2z,q) \leq -Min^i\{\widetilde{\mu}(x,q), \widetilde{\mu}(y,q), \widetilde{\mu}(z,q)\}$
$\Rightarrow \widetilde{1} - \widetilde{\mu}(xs_1ys_2z,q) \leq \widetilde{1} - Min^i\{\widetilde{\mu}(x,q), \widetilde{\mu}(y,q), \widetilde{\mu}(z,q)\}$
$\Rightarrow \widetilde{1} - \widetilde{\mu}(xs_1ys_2z,q) \leq Max^i\{\widetilde{1} - \widetilde{\mu}(x,q), \widetilde{1} - \widetilde{\mu}(y,q), \widetilde{1} - \widetilde{\mu}(z,q)\}$
$\Rightarrow \widetilde{\mu}^c(xs_1ys_2z,q) \leq Max^i\{\widetilde{\mu}^c(x,q), \widetilde{\mu}^c(y,q), \widetilde{\mu}^c(z,q)\}$.
Thus $\widetilde{\mu}^c$ is an interval-valued anti Q - fuzzy bi-ideal of S. By similar argument, we can prove the converse part. □

Theorem 3.8. *Every interval-valued intuitionistic Q - fuzzy ideal of S is an interval-valued intuitionistic Q - fuzzy bi-ideal of S.*

Proof. Let $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ be an interval-valued intuitionistic Q - fuzzy ideal of S. Then
$\widetilde{\mu}(xs_1ys_2z,q) \geq Min^i\{\widetilde{\mu}(x,q), \widetilde{\mu}(s_1ys_2,q), \widetilde{\mu}(z,q)\}$
$\geq Min^i\{\widetilde{\mu}(x,q), \widetilde{\mu}(y,q), \widetilde{\mu}(z,q)\}$
and $\widetilde{\nu}(xs_1ys_2z,q) \leq Max^i\{\widetilde{\nu}(x,q), \widetilde{\nu}(s_1ys_2,q), \widetilde{\nu}(z,q)\}$
$\leq Max^i\{\widetilde{\nu}(x,q), \widetilde{\nu}(y,q), \widetilde{\nu}(z,q)\}$.
Thus A is an interval-valued intuitionistic Q - fuzzy bi-ideal of S. □

Theorem 3.9. *An IIQFS $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ in S is an interval-valued intuitionistic Q - fuzzy bi-ideal of S if and only if any level set $\overline{S}_A^{(\widetilde{s},\widetilde{t})}$ is a bi-ideal of S for $\widetilde{s},\widetilde{t} \in D[0,1]$ whenever nonempty.*

Proof. Let $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ be an interval-valued intuitionistic Q - fuzzy bi-ideal of S. Let $x,y,z \in \overline{S}_A^{(\widetilde{s},\widetilde{t})}$ and $u,v \in S$.
Then $\widetilde{\mu}_A(x+y,q) \geq Min^i\{\widetilde{\mu}_A(x,q), \widetilde{\mu}_A(y,q)\} \geq \widetilde{s}$ and $\widetilde{\nu}_A(x+y,q) \leq Max^i\{\widetilde{\nu}_A(x,q), \widetilde{\nu}_A(y,q)\} \leq \widetilde{t}$. So $x+y \in \overline{S}_A^{(\widetilde{s},\widetilde{t})}$.
Again $\widetilde{\mu}_A(xuyvz,q) \geq Min^i\{\widetilde{\mu}_A(x,q), \widetilde{\mu}_A(y,q), \widetilde{\mu}_A(z,q)\} \geq \widetilde{s}$
and $\widetilde{\nu}_A(xuyvz,q) \leq Max^i\{\widetilde{\nu}_A(x,q), \widetilde{\nu}_A(y,q), \widetilde{\nu}_A(z,q)\} \leq \widetilde{t}$
which implies $xuyvz \in \overline{S}_A^{(\widetilde{s},\widetilde{t})}$. Hence $\overline{S}_A^{(\widetilde{s},\widetilde{t})}$ is a bi-ideal.
Conversely let $\overline{S}_A^{(\widetilde{s},\widetilde{t})}$ be a bi-ideal of S, for any $\widetilde{s},\widetilde{t} \in D[0,1]$ with $\widetilde{s}+\widetilde{t} \leq \widetilde{1}$.
Let $x,y \in S$ such that $\widetilde{\mu}_A(x,q) = \widetilde{\alpha}_1, \widetilde{\mu}_A(y,q) = \widetilde{\alpha}_2$
and $\widetilde{\nu}_A(x,q) = \widetilde{\beta}_1, \widetilde{\nu}_A(y,q) = \widetilde{\beta}_2$ where $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\beta}_1, \widetilde{\beta}_2 \in D[0,1]$.
Then $\widetilde{\alpha}_1 + \widetilde{\beta}_1 \leq \widetilde{1}, \widetilde{\alpha}_2 + \widetilde{\beta}_2 \leq \widetilde{1}$.
Let $\widetilde{\alpha} = Min^i\{\widetilde{\alpha}_1, \widetilde{\alpha}_2\}$ and $\widetilde{\beta} = Max^i\{\widetilde{\beta}_1, \widetilde{\beta}_2\}$ then $x,y \in S_A^{(\widetilde{\alpha},\widetilde{\beta})}$.
Since $S_A^{(\widetilde{\alpha},\widetilde{\beta})}$ be a bi-ideal of S then $x+y \in S_A^{(\widetilde{\alpha},\widetilde{\beta})}$
that means $\widetilde{\mu}_A(x+y,q) \geq \widetilde{\alpha} = Min^i\{\widetilde{\mu}_A(x,q), \widetilde{\mu}_A(y,q)\}$, $\widetilde{\nu}_A(x+y,q) \leq \widetilde{\beta} = Max^i\{\widetilde{\nu}_A(x,q), \widetilde{\nu}_A(y,q)\}$.
Similarly we prove
$\widetilde{\mu}_A(xuyvz,q) \geq Min^i\{\widetilde{\mu}_A(x,q), \widetilde{\mu}_A(y,q), \widetilde{\mu}_A(z,q)\}$
and $\widetilde{\nu}_A(xuyvz,q) \leq Max^i\{\widetilde{\nu}_A(x,q), \widetilde{\nu}_A(y,q), \widetilde{\nu}_A(z,q)\}$.
Therefore A is an interval-valued intuitionistic Q - fuzzy bi-ideal. □

Corollary 3.10. *An IIQFS $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ in S is an interval-valued intuitionistic Q - fuzzy bi-ideal of S if and only if for every $\widetilde{s},\widetilde{t} \in D[0,1]$ such that $\widetilde{s}+\widetilde{t} \leq \widetilde{1}$ all non-empty $\overline{U}(\widetilde{\mu}_A;\widetilde{s})$ and $\overline{L}(\widetilde{\nu}_A;\widetilde{t})$ are bi-ideals of S.*

Theorem 3.11. *Let I be a non-empty subset of a ternary semiring S. Then an IIQFS $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ defined by*

$$\widetilde{\mu}_A(x,q) = \begin{cases} \widetilde{s}_2 & if\ \forall x \in I \\ \widetilde{s}_1 & otherwise \end{cases}$$

$$\widetilde{\nu}_A(x,q) = \begin{cases} \widetilde{t}_2 & if\ \forall x \in I \\ \widetilde{t}_1 & otherwise \end{cases}$$

where $\widetilde{0} \leq \widetilde{s}_1 < \widetilde{s}_2 \leq \widetilde{1}$, $\widetilde{0} \leq \widetilde{t}_2 < \widetilde{t}_1 \leq \widetilde{1}$ and $\widetilde{s}_i + \widetilde{t}_i \leq \widetilde{1}$ for each $i = 1,2$ is an interval-valued intuitionistic Q - fuzzy bi-ideal of S if and only if I is a bi-ideal of S.

Proof. Let I be a bi-ideal of S. Let $x,y,z,u,v \in S, q \in Q$.
If $x,y,z \in I$, then $x+y, xuyvz \in I, q \in Q$.
Then $\widetilde{\mu}_A(x+y,q) = \widetilde{s}_2 \geq Min^i\{\widetilde{\mu}_A(x,q), \widetilde{\mu}_A(y,q)\}$,
$\widetilde{\nu}_A(x+y,q) = \widetilde{t}_2 \leq Max^i\{\widetilde{\nu}_A(x,q), \widetilde{\nu}_A(y,q)\}$,
$\widetilde{\mu}_A(xuyvz,q) = \widetilde{s}_2 \geq Min^i\{\widetilde{\mu}_A(x,q), \widetilde{\mu}_A(y,q), \widetilde{\mu}_A(z,q)\}$
and $\widetilde{\nu}_A(xuyvz,q) = \widetilde{t}_2 \leq Max^i\{\widetilde{\nu}_A(x,q), \widetilde{\nu}_A(y,q)\widetilde{\nu}_A(z,q)\}$.
If either x or y or $z \notin I$, then also
$\widetilde{\mu}_A(x+y,q) \geq \widetilde{s}_1 = Min^i\{\widetilde{\mu}_A(x,q), \widetilde{\mu}_A(y,q)\}$,
$\widetilde{\nu}_A(x+y,q) \leq \widetilde{t}_1 = Max^i\{\widetilde{\nu}_A(x,q), \widetilde{\nu}_A(y,q)\}$,
$\widetilde{\mu}_A(xuyvz,q) \geq \widetilde{s}_1 = Min^i\{\widetilde{\mu}_A(x,q), \widetilde{\mu}_A(y,q), \widetilde{\mu}_A(z,q)\}$
and $\widetilde{\nu}_A(xuyvz,q) \leq \widetilde{t}_1 = Max^i\{\widetilde{\nu}_A(x,q), \widetilde{\nu}_A(y,q), \widetilde{\nu}_A(z,q)\}$.
Hence $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ is an interval-valued intuitionistic Q - fuzzy bi-ideal of S.
Conversely, let $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ is an interval-valued intuitionistic Q - fuzzy bi-ideal of S. Then $\overline{S}_A^{(\widetilde{s}_2,\widetilde{t}_2)} = I$. So, by Theorem 3.9, I must be a bi-ideal of S. □

Theorem 3.12. *Let $(\widetilde{\mu}_i, \widetilde{\nu}_i)_{i\in I}$ be a family of interval-valued intuitionistic Q - fuzzy bi-ideals of S then $(\cap\widetilde{\mu}_i, \cup\widetilde{\nu}_i)$ is also an interval-valued intuitionistic Q - fuzzy bi-ideal of S.*

Proof. Let $\widetilde{\mu} = \bigcap_{i\in I} \widetilde{\mu}_i$ and $\widetilde{\nu} = \bigcup_{i\in I} \widetilde{\nu}_i$. For any $x,y,z \in S, q \in Q$,
1. $\widetilde{\mu}(x+y,q) = \bigcap_{i\in I} \widetilde{\mu}_i(x+y,q) \geq \bigcap_{i\in I} Min^i\{\widetilde{\mu}_i(x,q), \widetilde{\mu}_i(y,q)\}$
$= Min^i\{\bigcap_{i\in I} \widetilde{\mu}_i(x,q), \bigcap_{i\in I} \widetilde{\mu}_i(y,q)\} = Min^i\{\widetilde{\mu}(x,q), \widetilde{\mu}(y,q)\}$.
2. $\widetilde{\mu}(xs_1ys_2z,q) = \bigcap_{i\in I} \widetilde{\mu}_i(xs_1ys_2z,q)$
$\geq \bigcap_{i\in I} Min^i\{\widetilde{\mu}_i(x,q), \widetilde{\mu}_i(y,q), \widetilde{\mu}_i(z,q)\}$
$= Min^i\{\bigcap_{i\in I} \widetilde{\mu}_i(x,q), \bigcap_{i\in I} \widetilde{\mu}_i(y,q), \bigcap_{i\in I} \widetilde{\mu}_i(z,q)\}$
$= Min^i\{\widetilde{\mu}(x,q), \widetilde{\mu}(y,q), \widetilde{\mu}(z,q)\}$
3. $\widetilde{\nu}(x+y,q) = \bigcup_{i\in I} \widetilde{\nu}_i(x+y,q) \leq \bigcup_{i\in I} Max^i\{\widetilde{\nu}_i(x,q), \widetilde{\nu}_i(y,q)\}$
$= Max^i\{\bigcup_{i\in I} \widetilde{\nu}_i(x,q), \bigcup_{i\in I} \widetilde{\nu}_i(y,q)\} = Max^i\{\widetilde{\nu}(x,q), \widetilde{\nu}(y,q)\}$
4. $\widetilde{\nu}(xs_1ys_2z,q) = \bigcup_{i\in I} \widetilde{\nu}_i(xs_1ys_2z,q)$
$\leq \bigcup_{i\in I} Max^i\{\widetilde{\nu}_i(x,q), \widetilde{\nu}_i(y,q), \widetilde{\nu}_i(z,q)\}$
$= Max^i\{\bigcup_{i\in I} \widetilde{\nu}_i(x,q), \bigcup_{i\in I} \widetilde{\nu}_i(y,q), \bigcup_{i\in I} \widetilde{\nu}_i(z,q)\}$
$= Max^i\{\widetilde{\nu}(x,q), \widetilde{\nu}(y,q), \widetilde{\nu}(z,q)\}$.
Therefore $(\cap\widetilde{\mu}_i, \cup\widetilde{\nu}_i)$ is an interval-valued intuitionistic Q - fuzzy bi-ideal of S. □

Theorem 3.13. *An IIQFS $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ in S is an interval-valued intuitionistic Q - fuzzy bi-ideal of S if and only if the interval-valued fuzzy subsets $\widetilde{\mu}_A$ and $\widetilde{\nu}_A^c$ are interval-valued Q - fuzzy bi-ideals of S.*

Proof. If $A = (\widetilde{\mu}_A, \widetilde{\nu}_A)$ is an interval-valued intuitionistic Q - fuzzy bi-ideal of S, then clearly $\widetilde{\mu}_A$ is an interval-valued Q-fuzzy bi-ideal of S. For all $x,y,z,s_1,s_2 \in S, q \in Q$,
$\widetilde{\nu}_A^c(x+y,q) = \widetilde{1} - \widetilde{\nu}_A(x+y,q) \geq \widetilde{1} - Max^i\{\widetilde{\nu}_A(x,q), \widetilde{\nu}_A(y,q)\}$
$= Min^i\{\widetilde{1} - \widetilde{\nu}_A(x,q), \widetilde{1} - \widetilde{\nu}_A(y,q)\}$
$= Min^i\{\widetilde{\nu}_A^c(x,q), \widetilde{\nu}_A^c(y,q)\}$ and
$\widetilde{\nu}_A^c(xs_1ys_2z,q) = \widetilde{1} - \widetilde{\nu}_A(xs_1ys_2z,q)$

$\geq \widetilde{1}-Max^i\{\widetilde{\nu}_A(x,q),\widetilde{\nu}_A(y,q),\widetilde{\nu}_A(z,q)\}$
$=Min^i\{\widetilde{1}-\widetilde{\nu}_A(x,q),\widetilde{1}-\widetilde{\nu}_A(y,q),\widetilde{1}-\widetilde{\nu}_A(z,q)\}$
$=Min^i\{\widetilde{\nu}^c_A(x,q),\widetilde{\nu}^c_A(y,q),\widetilde{\nu}^c_A(z,q)\}$.
Thus $\widetilde{\nu}^c_A$ is an interval-valued Q - fuzzy bi-ideal of S.
Conversely assume that $\widetilde{\mu}_A$ and $\widetilde{\nu}^c_A$ are interval-valued Q - fuzzy bi-ideals of S, then clearly the conditions (1) and (2) of definition 3.5 are satisfied.
Now for all $x,y,z,s_1,s_2\in S$,
$\widetilde{1}-\widetilde{\nu}_A(x+y,q)=\widetilde{\nu}^c_A(x+y,q)\geq Min^i\{\widetilde{\nu}^c_A(x,q),\widetilde{\nu}^c_A(y,q)\}$
$=Min^i\{\widetilde{1}-\widetilde{\nu}_A(x,q),\widetilde{1}-\widetilde{\nu}_A(y,q)\}=$
$\widetilde{1}-Max^i\{\widetilde{\nu}_A(x,q),\widetilde{\nu}_A(y,q)\}$
which implies $-\widetilde{\nu}_A(x+y,q)\geq -Max^i\{\widetilde{\nu}_A(x,q),\widetilde{\nu}_A(y,q)\}$
implies $\widetilde{\nu}_A(x+y,q)\leq Max^i\{\widetilde{\nu}_A(x,q),\widetilde{\nu}_A(y,q)\}$ and
$\widetilde{1}-\widetilde{\nu}_A(xs_1ys_2z,q)=\widetilde{\nu}^c_A(xs_1ys_2z,q)$
$\geq Min^i\{\widetilde{\nu}^c_A(x,q),\widetilde{\nu}^c_A(y,q),\widetilde{\nu}^c_A(z,q)\}$
$=Min^i\{\widetilde{1}-\widetilde{\nu}_A(x,q),\widetilde{1}-\widetilde{\nu}_A(y,q),\widetilde{1}-\widetilde{\nu}_A(z,q)\}$
$=\widetilde{1}-Max^i\{\widetilde{\nu}_A(x,q),\widetilde{\nu}_A(y,q),\widetilde{\nu}_A(z,q)\}$ which implies
$-\widetilde{\nu}_A(xs_1ys_2z,q)\geq -Max^i\{\widetilde{\nu}_A(x,q),\widetilde{\nu}_A(y,q),\widetilde{\nu}_A(z,q)\}$
implies $\widetilde{\nu}_A(xs_1ys_2z,q)\leq Max^i\{\widetilde{\nu}_A(x,q),\widetilde{\nu}_A(y,q),\widetilde{\nu}_A(z,q)\}$. □

Corollary 3.14. *If an IIQFS $A=(\widetilde{\mu}_A,\widetilde{\nu}_A)$ in S is an interval-valued intuitionistic Q - fuzzy bi-ideal of S if and only if IIQFS $A_1=(\widetilde{\mu}_A,\widetilde{\mu}^c_A)$ and IIQFS $A_2=(\widetilde{\nu}^c_A,\widetilde{\nu}_A)$ are interval-valued intuitionistic Q - fuzzy bi-ideals of S.*

Proof. It is straightforward by Theorem 3.7 and Theorem 3.13. □

Theorem 3.15. *Let S_1,S_2 be ternary semirings and let $\Phi: S_1\to S_2$ be a homomorphism. If $B=(\widetilde{\mu}_B,\widetilde{\nu}_B)$ is an interval-valued intuitionistic Q - fuzzy bi-ideal of S_2, then $\Phi^{-1}(B)=(\Phi^{-1}(\widetilde{\mu}_B),\Phi^{-1}(\widetilde{\nu}_B))$ where $\Phi^{-1}(\widetilde{\mu}_B)(x,q)=\widetilde{\mu}_B(\Phi(x,q))$ and $\Phi^{-1}(\widetilde{\nu}_B)(x,q)=\widetilde{\nu}_B(\Phi(x,q))$ for all $x\in S_1,q\in Q$, is an interval-valued intuitionistic Q - fuzzy bi-ideal of S_1.*

Proof. Assume $B=(\widetilde{\mu}_B,\widetilde{\nu}_B)$ is an interval-valued intuitionistic Q - fuzzy bi-ideal of S_2, and let $x,y,z,u,v\in S_1,q\in Q$. Then
1. $\Phi^{-1}(\widetilde{\mu}_B)(x+y,q)=\widetilde{\mu}_B(\Phi(x+y,q))$
$=\widetilde{\mu}_B(\Phi(x,q)+\Phi(y,q))$
$\geq Min^i\{\widetilde{\mu}_B(\Phi(x,q)),\widetilde{\mu}_B(\Phi(y,q))\}$
$=Min^i\{\Phi^{-1}(\widetilde{\mu}_B)(x,q),\Phi^{-1}(\widetilde{\mu}_B)(y,q)\}$.
2. $\Phi^{-1}(\widetilde{\mu}_B)(xuyvz,q)=\widetilde{\mu}_B(\Phi(xuyvz,q))$
$=\widetilde{\mu}_B(\Phi(x,q)\Phi(u,q)\Phi(y,q)\Phi(v,q)\Phi(z,q))$
$\geq Min^i\{\widetilde{\mu}_B(\Phi(x,q)),\widetilde{\mu}_B(\Phi(y,q)),\widetilde{\mu}_B(\Phi(z,q))\}$
$=Min^i\{\Phi^{-1}(\widetilde{\mu}_B)(x,q),\Phi^{-1}(\widetilde{\mu}_B)(y,q),\Phi^{-1}(\widetilde{\mu}_B)(z,q)\}$.
3. $\Phi^{-1}(\widetilde{\nu}_B)(x+y,q)=\widetilde{\nu}_B(\Phi(x+y,q))$
$=\widetilde{\nu}_B(\Phi(x,q)+\Phi(y,q))$
$\leq Max^i\{\widetilde{\nu}_B(\Phi(x,q)),\widetilde{\nu}_B(\Phi(y,q))\}$
$=Max^i\{\Phi^{-1}(\widetilde{\nu}_B)(x,q),\Phi^{-1}(\widetilde{\nu}_B)(y,q)\}$.
4. $\Phi^{-1}(\widetilde{\nu}_B)(xuyvz,q)=\widetilde{\nu}_B(\Phi(xuyvz,q))$
$=\widetilde{\nu}_B(\Phi(x,q)\Phi(u,q)\Phi(y,q)\Phi(v,q)\Phi(z,q))$
$\leq Max^i\{\widetilde{\nu}_B(\Phi(x,q)),\widetilde{\nu}_B(\Phi(y,q)),\widetilde{\nu}_B(\Phi(z,q))\}$
$=Max^i\{\Phi^{-1}(\widetilde{\nu}_B)(x,q),\Phi^{-1}(\widetilde{\nu}_B)(y,q),\Phi^{-1}(\widetilde{\nu}_B)(z,q)\}$.
Therefore $\Phi^{-1}(B)=(\Phi^{-1}(\widetilde{\mu}_B),\Phi^{-1}(\widetilde{\nu}_B))$ is an interval-valued intuitionistic Q - fuzzy bi-ideal of S_1. □

4. Normal interval-valued intuitionistic Q - fuzzy right ideals

Definition 4.1. *An interval-valued intuitionistic Q - fuzzy right (left, lateral) ideal $A=(\widetilde{\mu}_A,\widetilde{\nu}_A)$ of a ternary semiring S is said to be normal if $A(0,q)=(\widetilde{1},\widetilde{0})$ that means $\widetilde{\mu}_A(0,q)=\widetilde{1}$, $\widetilde{\nu}_A(0,q)=\widetilde{0}$. Denote by NIIQFRI(S) (NIIQFLI(S), NIIQFMI(S)) the set of all normal interval-valued intuitionistic Q - fuzzy right (left, lateral) ideals of S.*
Note that NIIQFRI(S) (NIIQFLI(S), NIIQFMI(S)) is a poset under set inclusion.

Example 4.2. *Consider the ternary semiring $S=Z^-_0$, the set of all non positive integers with usual addition and ternary multiplication. Let the interval-valued Q - fuzzy subset $\widetilde{\mu}_A$ and $\widetilde{\nu}_A$ of S be defined by*

$$\widetilde{\mu}_A(x,q)=\begin{cases}\widetilde{1} & if\ \ x=0\\ [0.5,0.6] & if\ \ x\in\langle -2\rangle\setminus\{0\}\\ [0.2,0.3] & otherwise\end{cases}$$

$$\widetilde{\nu}_A(x,q)=\begin{cases}\widetilde{0} & if\ \ x=0\\ [0.1,0.2] & if\ \ x\in\langle -2\rangle\setminus\{0\}\\ [0.4,0.7] & otherwise\end{cases}$$

Then $A=(\widetilde{\mu}_A,\widetilde{\nu}_A)$ is a normal interval-valued intuitionistic Q - fuzzy ideal of S.

Theorem 4.3. *Given an interval-valued intuitionistic Q - fuzzy right (left, lateral) ideal $A=(\widetilde{\mu}_A,\widetilde{\nu}_A)$ of a ternary semiring S. Let $\widetilde{\mu}^+_A(x,q)=\widetilde{\mu}_A(x,q)+\widetilde{1}-\widetilde{\mu}_A(0,q)$ and $\widetilde{\nu}^+_A(x,q)=\widetilde{\nu}_A(x,q)-\widetilde{\nu}_A(0,q)$, for all $x\in S,q\in Q$. Then $A^+=(\widetilde{\mu}^+_A,\widetilde{\nu}^+_A)$ is a normal interval-valued intuitionistic Q - fuzzy right (left, lateral) ideal containing $A=(\widetilde{\mu}_A,\widetilde{\nu}_A)$ of S.*

Proof. For any $x,y,z\in S,q\in Q$
1. $\widetilde{\mu}^+_A(x+y,q)=\widetilde{\mu}_A(x+y,q)+\widetilde{1}-\widetilde{\mu}_A(0,q)$
$\geq Min^i\{\widetilde{\mu}_A(x,q),\widetilde{\mu}_A(y,q)\}+\widetilde{1}-\widetilde{\mu}_A(0,q)$
$=Min^i\{\widetilde{\mu}_A(x,q)+\widetilde{1}-\widetilde{\mu}_A(0,q),\widetilde{\mu}_A(y,q)+\widetilde{1}-\widetilde{\mu}_A(0,q)\}$
$=Min^i\{\widetilde{\mu}^+_A(x,q),\widetilde{\mu}^+_A(y,q)\}$
2. $\widetilde{\mu}^+_A(xyz,q)=\widetilde{\mu}_A(xyz,q)+\widetilde{1}-\widetilde{\mu}_A(0,q)$
$\geq\widetilde{\mu}_A(x,q)+\widetilde{1}-\widetilde{\mu}_A(0,q)=\widetilde{\mu}^+_A(x,q)$
3. $\widetilde{\nu}^+_A(x+y,q)=\widetilde{\nu}_A(x+y,q)-\widetilde{\nu}_A(0,q)$
$\leq Max^i\{\widetilde{\nu}_A(x,q),\widetilde{\nu}_A(y,q)\}-\widetilde{\nu}_A(0,q)$
$=Max^i\{\widetilde{\nu}_A(x,q)-\widetilde{\nu}_A(0,q),\widetilde{\nu}_A(y,q)-\widetilde{\nu}_A(0,q)\}$
$=Max^i\{\widetilde{\nu}^+_A(x,q),\widetilde{\nu}^+_A(y,q)\}$
4. $\widetilde{\nu}^+_A(xyz,q)=\widetilde{\nu}_A(xyz,q)-\widetilde{\nu}_A(0,q)$
$\leq\widetilde{\nu}_A(x,q)-\widetilde{\nu}_A(0,q)=\widetilde{\nu}^+_A(x,q)$.
Hence A^+ is an interval-valued intuitionistic Q - fuzzy right ideal of S. Again we have $\widetilde{\mu}^+_A(0,q)=\widetilde{\mu}_A(0,q)+\widetilde{1}-\widetilde{\mu}_A(0,q)=\widetilde{1}$ and $\widetilde{\nu}^+_A(0,q)=\widetilde{\nu}_A(0,q)-\widetilde{\nu}_A(0,q)=\widetilde{0}$. Hence A^+ is a normal interval-valued intuitionistic Q - fuzzy right ideal of S and by definition $A\subseteq A^+$. □

Corollary 4.4. *Let A and A^+ be as in the Theorem 4.3. A is a normal interval-valued intuitionistic Q - fuzzy right ideal of S if and only if $A^+=A$.*

Remark 4.5. *If $A=(\widetilde{\mu}_A,\widetilde{\nu}_A)$ is an interval-valued intuitionistic Q - fuzzy right (left, lateral) ideal of S, then $(A^+)^+=A^+$. In particular, if A is normal, then $(A^+)^+=A^+=A$.*

Theorem 4.6. *Let $A=(\widetilde{\mu}_A,\widetilde{\nu}_A)$ be an interval-valued intuitionistic Q - fuzzy right (left, lateral) ideal of a ternary semiring S and let $\Phi : D[0,1]\to D[0,1]$ be an increasing function. Then an IIQFS $A_\Phi=((\widetilde{\mu}_A)_\Phi,(\widetilde{\nu}_A)_\Phi)$ where $(\widetilde{\mu}_A)_\Phi(x,q)=\Phi(\widetilde{\mu}_A(x,q))$ and $(\widetilde{\nu}_A)_\Phi(x,q)=\Phi(\widetilde{\nu}_A(x,q))$ for all $x\in S, q\in Q$ is an interval-valued intuitionistic Q - fuzzy right (left, lateral) ideal of S. Moreover, if $\Phi(\widetilde{\mu}_A(0,q))=\widetilde{1}$ and $\Phi(\widetilde{\nu}_A(0,q))=\widetilde{0}$, then A_Φ is normal.*

Proof. Let $x,y,z\in S, q\in Q$
1. $(\widetilde{\mu}_A)_\Phi(x+y,q)=\Phi(\widetilde{\mu}_A(x+y,q))$
$\geq \Phi(Min^i\{\widetilde{\mu}_A(x,q),\widetilde{\mu}_A(y,q)\})$
$=Min^i\{\Phi(\widetilde{\mu}_A(x,q)),\Phi(\widetilde{\mu}_A(y,q))\}$
$=Min^i\{(\widetilde{\mu}_A)_\Phi(x,q),(\widetilde{\mu}_A)_\Phi(y,q)\}$
2. $(\widetilde{\mu}_A)_\Phi(xyz,q)=\Phi(\widetilde{\mu}_A(xyz,q))$
$\geq \Phi(\widetilde{\mu}_A(x,q))=(\widetilde{\mu}_A)_\Phi(x,q)$
3. $(\widetilde{\nu}_A)_\Phi(x+y,q)=\Phi(\widetilde{\nu}_A(x+y,q))$
$\leq \Phi(Max^i\{\widetilde{\nu}_A(x,q),\widetilde{\nu}_A(y,q)\})$
$=Max^i\{\Phi(\widetilde{\nu}_A(x,q)),\Phi(\widetilde{\nu}_A(y,q))\}$
$=Max^i\{(\widetilde{\nu}_A)_\Phi(x,q),(\widetilde{\nu}_A)_\Phi(y,q)\}$
4. $(\widetilde{\nu}_A)_\Phi(xyz,q)=\Phi(\widetilde{\nu}_A(xyz,q))$
$\leq \Phi(\widetilde{\nu}_A(x,q))=(\widetilde{\nu}_A)_\Phi(x,q)$.
Hence A_Φ is an interval-valued intuitionistic Q - fuzzy right ideal of S. If $\Phi(\widetilde{\mu}_A(0,q))=\widetilde{1}, \Phi(\widetilde{\nu}_A(0,q))=\widetilde{0}$ then $(\widetilde{\mu}_A)_\Phi(0,q)=\widetilde{1}$ and $(\widetilde{\nu}_A)_\Phi(0,q)=\widetilde{0}$ and hence $A_\Phi=((\widetilde{\mu}_A)_\Phi,(\widetilde{\nu}_A)_\Phi)$ is a normal interval-valued intuitionistic Q - fuzzy right ideal of S. □

Definition 4.7. *An interval-valued intuitionistic Q - fuzzy ideal $A=(\widetilde{\mu}_A,\widetilde{\nu}_A)$ of a ternary semiring S is said to be an interval-valued intuitionistic Q - fuzzy maximal ideal if it satisfies:*
i) A is non-constant.
ii) A^+ is a maximal element of NIIQFI(S), where NIIQFI(S) denote the set of all normal interval-valued intuitionistic Q - fuzzy ideal of S.

Example 4.8. *Consider the ternary semiring $S=Z_0^-$, the set of all non positive integers with usual addition and ternary multiplication. Let the interval-valued Q - fuzzy subset $\widetilde{\mu}_A$ and $\widetilde{\nu}_A$ of S be defined by*

$$\widetilde{\mu}_A(x,q)=\begin{cases}\widetilde{1} & if\ \ x\in\langle -2\rangle\\ \widetilde{0} & otherwise\end{cases}$$

$$\widetilde{\nu}_A(x,q)=\begin{cases}\widetilde{0} & if\ \ x\in\langle -2\rangle\\ \widetilde{1} & otherwise\end{cases}$$

Then $A=(\widetilde{\mu}_A,\widetilde{\nu}_A)$ is an interval-valued intuitionistic Q - fuzzy maximal ideal of S.

Theorem 4.9. *Let $A=(\widetilde{\mu}_A,\widetilde{\nu}_A)\in NIIQFRI(S)$ be non-constant such that it is maximal in the poset of NIIFRI(S) under set inclusion. Then both $\widetilde{\mu}_A$ and $\widetilde{\nu}_A$ takes only the values $(\widetilde{1},\widetilde{0})$ and $(\widetilde{0},\widetilde{1})$ respectively.*

Proof. Since A is normal interval-valued intuitionistic Q - fuzzy right ideal, so $A(0,q)=(\widetilde{1},\widetilde{0})$.
Let $x_0\ (\neq 0)\in S, , q\in Q$ be arbitrary with $\widetilde{\mu}_A(x_0,q)\neq\widetilde{1}$.
We claim that $\widetilde{\mu}_A(x_0,q)=\widetilde{0}$. If not then there exists an element $c\in S$ such that $\widetilde{0}<\widetilde{\mu}_A(c,q)<\widetilde{1}$.
Let $A_c=(\widetilde{\sigma}_A,\widetilde{\eta}_A)$ be an interval-valued intuitionistic Q - fuzzy subset of S defined by
$\widetilde{\sigma}_A(x,q)=\frac{1}{2}[\widetilde{\mu}_A(x,q)+\widetilde{\mu}_A(c,q)]$, $\widetilde{\eta}_A(x,q)=\frac{1}{2}[\widetilde{\nu}_A(x,q)+\widetilde{\nu}_A(c,q)]$.
Clearly A_c is well-defined.
Now, $\widetilde{\sigma}_A(0,q)=\frac{1}{2}[\widetilde{\mu}_A(0,q)+\widetilde{\mu}_A(c,q)]\geq\frac{1}{2}[\widetilde{\mu}_A(x,q)+\widetilde{\mu}_A(c,q)]$
$=\widetilde{\sigma}_A(x,q)$, $\widetilde{\eta}_A(0,q)$
$=\frac{1}{2}[\widetilde{\nu}_A(0,q)+\widetilde{\nu}_A(c,q)]$
$\leq\frac{1}{2}[\widetilde{\nu}_A(x,q)+\widetilde{\nu}_A(c,q)]=\widetilde{\eta}_A(x,q)$ for any $x\in S, q\in Q$.
Again, for any $x,y,z\in S, q\in Q$,
1. $\widetilde{\sigma}_A(x+y,q)=\frac{1}{2}[\widetilde{\mu}_A(x+y,q)+\widetilde{\mu}_A(c,q)]$
$\geq\frac{1}{2}[Min^i\{\widetilde{\mu}_A(x,q),\widetilde{\mu}_A(y,q)\}+\widetilde{\mu}_A(c,q)]$
$=Min^i\{\frac{1}{2}[\widetilde{\mu}_A(x,q)+\widetilde{\mu}_A(c,q)],\frac{1}{2}[\widetilde{\mu}_A(y,q)+\widetilde{\mu}_A(c,q)]\}$
$=Min^i\{\widetilde{\sigma}_A(x,q),\widetilde{\sigma}_A(y,q)\}$
2. $\widetilde{\sigma}_A(xyz,q)=\frac{1}{2}[\widetilde{\mu}_A(xyz,q)+\widetilde{\mu}_A(c,q)]$
$\geq\frac{1}{2}[\widetilde{\mu}_A(x,q)+\widetilde{\mu}_A(c,q)]=\widetilde{\sigma}_A(x,q)$
3. $\widetilde{\eta}_A(x+y,q)=\frac{1}{2}[\widetilde{\nu}_A(x+y,q)+\widetilde{\nu}_A(c,q)]$
$\leq\frac{1}{2}[Max^i\{\widetilde{\nu}_A(x,q),\widetilde{\nu}_A(y,q)\}+\widetilde{\nu}_A(c,q)]$
$=Max^i\{\frac{1}{2}[\widetilde{\nu}_A(x,q)+\widetilde{\nu}_A(c,q)],\frac{1}{2}[\widetilde{\nu}_A(y,q)+\widetilde{\nu}_A(c,q)]\}$
$=Max^i\{\widetilde{\eta}_A(x,q),\widetilde{\eta}_A(y,q)\}$
4. $\widetilde{\eta}_A(xyz,q)=\frac{1}{2}[\widetilde{\nu}_A(xyz,q)+\widetilde{\nu}_A(c,q)]$
$\leq\frac{1}{2}[\widetilde{\nu}_A(x,q)+\widetilde{\nu}_A(c,q)]=\widetilde{\eta}_A(x,q)$.
Hence A_c is an interval-valued intuitionistic Q - fuzzy right ideal of S.
Define $A_c^+=(\widetilde{\sigma}_A^+,\widetilde{\eta}_A^+)$. Then by Theorem 4.3, A_c^+ is a normal interval-valued intuitionistic Q - fuzzy right ideal of S, where $\widetilde{\sigma}_A^+(x,q)=\widetilde{\sigma}_A(x,q)+\widetilde{1}-\widetilde{\sigma}_A(0,q)=\frac{1}{2}[\widetilde{\mu}_A(x,q)+\widetilde{\mu}_A(c,q)]+\widetilde{1}-\frac{1}{2}[\widetilde{\mu}_A(0,q)+\widetilde{\mu}_A(c,q)]=\frac{1}{2}[1+\widetilde{\mu}_A(x,q)]$
and $\widetilde{\eta}_A^+(x,q)=\widetilde{\eta}_A(x,q)-\widetilde{\eta}_A(0,q)=\frac{1}{2}[\widetilde{\nu}_A(x,q)+\widetilde{\nu}_A(c,q)]-\frac{1}{2}[\widetilde{\nu}_A(0,q)+\widetilde{\nu}_A(c,q)]=\frac{1}{2}[\widetilde{\nu}_A(x,q)]$. Clearly $A\subseteq A_c^+$. Since $\widetilde{\sigma}_A^+(x,q)=\frac{1}{2}[1+\widetilde{\mu}_A(x,q)]>\widetilde{\mu}_A(x,q)$ and $\widetilde{\eta}_A^+(x,q)=\frac{1}{2}[\widetilde{\nu}_A(x,q)]\leq\widetilde{\nu}_A(x,q)$, A is a proper subset of A_c^+. Again since $\widetilde{\sigma}_A^+(c,q)=\frac{1}{2}[1+\widetilde{\mu}_A(c,q)]<\widetilde{1}=\widetilde{\sigma}_A^+(0,q)$. Hence A_c^+ is non-constant and A is not a maximal element of NIIQFRI(S). This is a contradiction.
Therefore $\widetilde{\mu}_A$ takes only two values $\widetilde{1}$ and $\widetilde{0}$. Hence $\widetilde{\nu}_A$ takes the values $\widetilde{0}$ and $\widetilde{1}$.
This completes the proof. □

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