A note on the singular chromatic number of some product graphs

N. Paramaguru¹,²

Abstract
A proper coloring of a graph $G$ is an assignment of colors to the vertices of $G$ such that adjacent vertices are assigned distinct colors. The minimum number of colors in a proper coloring of $G$ is the chromatic number $\chi(G)$ of $G$. For a graph $G$ and a proper coloring $c : V(G) \rightarrow \{1, 2, \ldots, k\}$ of the vertices of $G$ for some positive integer $k$, the color code of a vertex $v$ of $G$ (with respect to $c$) is the ordered pair $\text{code}(v) = (c(v), S_v)$, where $S_v = \{c(u) : u \in N(v)\}$. The coloring $c$ is singular if distinct vertices have distinct color codes and the singular chromatic number $\chi_s(G)$ of $G$ is the minimum positive integer $k$ for which $G$ has a singular $k$-coloring. In this paper, we determine the singular chromatic number of Cartesian product of paths.

Keywords
Proper coloring, chromatic number, singular chromatic number.

AMS Subject Classification
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¹Department of Mathematics, Government Arts College for Women, Krishnagiri-635002, Tamil Nadu, India.
²Department of Mathematics, Annamalai University, Annamalai Nagar-608002, Tamil Nadu, India.

Corresponding author: npguru@gmail.com

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1. Introduction

For a graph $G$, let $c : V(G) \rightarrow \mathbb{N}$ be a proper vertex coloring of $G$. For each vertex $v$ of $G$, let $S_v$ be the set of colors of the neighbors of $v$, that is, $S_v = \{c(u) : u \in N(v)\}$ where $N(v)$ is the neighborhood of $v$. The color code $\text{code}(v)$ of $v$ is then defined as the ordered pair $(c(v), S_v)$. If $\text{code}(u) \neq \text{code}(v)$ for every two distinct vertices $u$ and $v$ of $G$, then $c$ is called a singular coloring of $G$. Therefore, a singular coloring of a graph $G$ is a coloring that uses the color of each vertex together with the set of colors of its neighbors to distinguish all vertices in $G$. If a singular coloring $c$ uses $k$ colors, then $c$ is a singular $k$-coloring. For each positive integer $k$, let $\mathbb{N}_k = \{1, 2, \ldots, k\}$. Thus, we assume that every singular $k$-coloring uses the colors in $\mathbb{N}_k$. A graph $G$ is singularly $k$-colorable if $G$ has a singular $k$-coloring.

The minimum $k$ for which $G$ has a singular $k$-coloring is called the singular chromatic number of $G$ and is denoted by $\chi_s(G)$. Since a coloring assigning distinct colors to distinct vertices of a graph $G$ is a singular coloring of $G$, the singular chromatic number exists for every graph. The concept of singular coloring was introduced by Kolasinski, Lin and Okamoto [3].

In [3], Kolasinski, Lin and Okamoto characterized: Let $G$ be a graph of order $n$, then $\chi_s(G) = n$ if and only if $G$ is a complete multipartite graph or an empty graph; and also proved that: For each $n \geq 3$, let $k$ be the unique positive integer such that $(k-1)(\frac{k-1}{2}) + 1 \leq n \leq \frac{k(k-1)}{2}$, then $\chi_s(P_n) = k$; $\chi_s(C_n) = k$ if $n \neq k(\frac{k}{2}) - 1$; $\chi_s(C_n) = k + 1$ if $n = k(\frac{k}{2}) - 1$.

The Cartesian product $G \square H$ of two graphs $G$ and $H$ has $V(G \square H) = V(G) \times V(H)$, and two vertices $(u_1, u_2)$ and $(v_1, v_2)$ of $G \square H$ are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_2 = v_2$ and $u_1v_1 \in E(G)$.

In this paper, we compute the singular chromatic number of Cartesian product of paths.

2. Results

We have the following observations:

Observation 2.1. [3] For every graph $G$ of order $n$, $\chi_s(G) = \chi_s(G) \leq \chi_s(G) \leq n$. 

Define \( \chi_u(G) = 1 \) if and only if \( G = K_1 \).

(b) \( \chi_u(G) = 2 \) if and only if \( G = \{K_2, K_2 \cup K_1, K_2 \cup 2K_1\} \).

**Observation 2.2.** [3] Let \( G \) be a graph.
(a) \( \chi_u(G) = 1 \) if and only if \( G = K_1 \).
(b) \( \chi_u(G) = 2 \) if and only if \( G = \{K_2, K_2 \cup K_1, K_2 \cup 2K_1\} \).

**Observation 2.3.** [3] If \( \chi_u(G) = k \), then each of the following holds.
(a) \( G \) contains at most \( k \) isolated vertices.
(b) \( G \) contains at most one clique of order \( k \).
(c) Each vertex in \( G \) is adjacent to at most \( k - 1 \) end-vertices.

For the path \( P_n \) on \( n \) vertices, let \( V(P_n) = \{1, 2, \ldots, n\} \),
\[ E(P_n) = \{(i, i + 1) : i \in \{1, 2, \ldots, n - 1\}\}. \]

Let \( c((i, j)) \) denotes the color assigned to the vertex \((v_i, v_j)\) and \( \text{code}((i, j)) \) denotes the color code of the vertex \((v_i, v_j)\).

The following figures shows that the singular colorings of \( P_3 \pent P_3 \), \( P_3 \pent P_3 \) and \( P_3 \pent P_3 \).

**Theorem 2.5.** For \( m \geq 3 \), \( n \geq 3 \), and \( m \leq n \), \( \chi_u(P_m \pent P_n) \leq n + 1 \).

**Proof.**
Let us define \( c : V(P_m \pent P_n) \to \{0, 1, 2, \ldots, n\} \) as follows:
\[ c((i, j)) = j - 1 \text{ if } (i, j) \in \{(1) \times \{1, 2, 3, \ldots, n\}\}; \]
\[ c((i, j)) = j \text{ if } (i, j) \in \{(2) \times \{1, 2, 3, \ldots, n\}\}; \]
\[ c((i, j)) = i + j - 1 \equiv 0 \mod (n + 1) \text{ if } (i, j) \in \{(3, 7, 11, \ldots) \times \{1, 2, 3, \ldots, n\}\}; \]
\[ c((i, j)) = i + j \equiv 0 \mod (n + 1) \text{ if } (i, j) \in \{(4, 8, 12, \ldots) \times \{1, 2, 3, \ldots, n\}\}; \]
\[ c((i, j)) = (i + j - 1) \equiv 0 \mod (n + 1) \text{ if } (i, j) \in \{(5, 9, 13, \ldots) \times \{1, 2, 3, \ldots, n\}\}; \]
\[ c((i, j)) = (i + j) \equiv 0 \mod (n + 1) \text{ if } (i, j) \in \{(6, 10, 14, \ldots) \times \{1, 2, 3, \ldots, n\}\}; \]

then the color code is:
\[ \text{code}(c(1, 1)) = (0, \{1\}); \]
\[ \text{code}(c(1, 2)) = (1, \{2, 3\}); \]
\[ \text{code}(c(1, 3)) = (2, \{1, 3\}); \]
\[ \text{code}(c(2, 3)) = (2, \{1, 2\}); \]
\[ \text{code}(c(3, 3)) = (3, \{1, 2\}); \]
\[ \text{code}(c(3, 1)) = (3, \{1\}); \]

**Theorem 2.4.** For \( n \geq 6 \), \( \chi_u(P_3 \pent P_n) \leq n - 1 \).

**Proof.** Define \( c : V(P_3 \pent P_n) \to \{0, 1, 2, \ldots, n - 2\} \) as follows:
\[ \text{code}(c(1, j)) = (j - 1) \mod (n - 1), \{j \mod (n - 1)\} \]
\[ \text{code}(c(2, j)) = (j - 1) \mod (n - 1), \{(j - 1) \mod (n - 1)\}; \]
\[ \text{code}(c(3, j)) = (j - 1) \mod (n - 1), \{(j - 1) \mod (n - 1)\}; \]

Then, \( \text{code}(c(1, 1)) = (0, \{1\}); \)
1 \equiv 0 \mod (n+1), \left(\left\lfloor \frac{j}{2} \right\rfloor + j + 1 \right) \equiv 0 \mod (n+1), \left(\left\lfloor \frac{j}{2} \right\rfloor + j - 2 \right) \equiv 0 \mod (n+1), \left\{\left\lfloor \frac{j}{2} \right\rfloor + j \right\}\} \text{ if } (i,j) \in \{(5,9,13,\cdots \neq m) \times \{n\})$

code(i,j) = (\left\lfloor \frac{j}{2} \right\rfloor + j) \equiv 0 \mod (n+1), \{(\left\lfloor \frac{j}{2} \right\rfloor + j - 2) \equiv 0 \mod (n+1), \left\{\left\lfloor \frac{j}{2} \right\rfloor + j \right\}\} \text{ if } (i,j) \in \{(6,10,14,\cdots \neq m) \times \{2,3,4,\ldots,n-1\})$

code(i,j) = ((i+j) \equiv 0 \mod (n+1), \{(i+j-2) \equiv 0 \mod (n+1), \left\{\left\lfloor \frac{j}{2} \right\rfloor + j \right\}\} \text{ if } (i,j) \in \{(6,10,14,\cdots \neq m) \times \{1\})$

code(i,j) = ((i+j-1) \equiv 0 \mod (n+1), \{(i+j+1) \equiv 0 \mod (n+1), \left\{\left\lfloor \frac{j}{2} \right\rfloor + j \right\}\} \text{ if } (i,j) \in \{(6,10,14,\cdots \neq m) \times \{1\})$

code(i,j) = ((i+j) \equiv 0 \mod (n+1), \{(i+j-2) \equiv 0 \mod (n+1), \left\{\left\lfloor \frac{j}{2} \right\rfloor + j \right\}\} \text{ if } (i,j) \in \{(6,10,14,\cdots \neq m) \times \{1\})$

code(i,j) = (i, \left\lfloor \frac{j}{2} \right\rfloor + 1, i+1, i+2) \text{ if } (i,j) \in \{(7,11,15,\cdots \neq m) \times \{1\})$

code(i,j) = (i, \left\lfloor \frac{j}{2} \right\rfloor + 1, i+1, i+2) \text{ if } (i,j) \in \{(7,11,15,\cdots \neq m) \times \{1\})$

Finally, we have
Case(1). \ m \equiv 0 \mod 4,
\text{code}(m,1) = (0, \{1, m-1\});
\text{code}(m,m) = (n-1, \{n-3, n-2\});
\text{code}(i,j) = (j-1, \{(j-2) \equiv 0 \mod (n+1), j \equiv 0 \mod (n+1), (i+j-2) \equiv 0 \mod (n+1)\}) \text{ if } (i,j) \in \{(m) \times \{2,3,\ldots,n-1\})$

Case(2). \ m \equiv 1 \mod 4,
\text{code}(m,1) = (\left\lfloor \frac{m}{2} \right\rfloor, \{n, \left\lfloor \frac{m}{2} \right\rfloor + 1\});$
\text{code}(m,m) = (\left\lfloor \frac{m}{2} \right\rfloor - 2, \{\left\lfloor \frac{m}{2} \right\rfloor - 1, n-2\});$
\text{code}(i,j) = ((i+j-1) \equiv 0 \mod (n+1), \{(m-j-1) \equiv 0 \mod (n+1), \left\lfloor \frac{m}{2} \right\rfloor + j \equiv 0 \mod (n+1), \left\lfloor \frac{m}{2} \right\rfloor + j-2 \equiv 0 \mod (n+1)\}) \text{ if } (i,j) \in \{(m) \times \{2,3,\ldots,n-1\})$

Case(3). \ m \equiv 2 \mod 4,
\text{code}(m,1) = (m+1, \{m, m-1, m+2\});$
\text{code}(m,m) = (m-1, \{m-3, m-2\});$
\text{code}(i,j) = ((\left\lfloor \frac{m}{2} \right\rfloor + j) \equiv 0 \mod (n+1), \{(\left\lfloor \frac{m}{2} \right\rfloor + j-2) \equiv 0 \mod (n+1), \left\lfloor \frac{m}{2} \right\rfloor + j+1 \equiv 0 \mod (n+1), \left\lfloor \frac{m}{2} \right\rfloor + j-1 \equiv 0 \mod (n+1)\}) \text{ if } (i,j) \in \{(m) \times \{2,3,\ldots,n-1\})$

Case(4). \ m \equiv 3 \mod 4,
\text{code}(m,1) = (m, \{m, m+1\} \equiv 0 \mod (n+1));$
\text{code}(m,m) = (m-2, \{m-1, m-3\});$
\text{code}(i,j) = ((m+j-1) \equiv 0 \mod (n+1), \{(\left\lfloor \frac{m}{2} \right\rfloor + j) \equiv 0 \mod (n+1), (m+j) \equiv 0 \mod (n+1), (m+j-2) \equiv 0 \mod (n+1)\}) \text{ if } (i,j) \in \{(m) \times \{2,3,\ldots,n-1\})$

Hence, $\chi_{st}(P_m \square P_n) \leq n+1$.

References

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