Malaya Journal of Matematik, Vol. S, No. 1, 151-153, 2019
https://doi.org/10.26637/MJM0S01/0035

A note on the singular chromatic number of some product graphs

N. Paramaguru[1,2]

Abstract
A proper coloring of a graph G is an assignment of colors to the vertices of G such that adjacent vertices are assigned distinct colors. The minimum number of colors in a proper coloring of G is the chromatic number $\chi(G)$ of G. For a graph G and a proper coloring $c: V(G) \to \{1,2,\ldots,k\}$ of the vertices of G for some positive integer k, the color code of a vertex v of G (with respect to c) is the ordered pair code(v)=$(c(v), S_v)$, where $S_v = \{c(u) : u \in N(v)\}$. The coloring c is singular if distinct vertices have distinct color codes and the singular chromatic number $\chi_{si}(G)$ of G is the minimum positive integer k for which G has a singular k-coloring. In this paper, we determine the singular chromatic number of Cartesian product of paths.

Keywords
Proper coloring, chromatic number, singular chromatic number.

AMS Subject Classification
05C15, 05C22.

[1] *Department of Mathematics, Government Arts College for Women, Krishnagiri-635002, Tamil Nadu, India.*
[2] *Department of Mathematics, Annamalai University, Annamalai Nagar-608002, Tamil Nadu, India.*
Corresponding author: npguru@gmail.com
Article History: Received **24** March **2019**; Accepted **09** May **2019**
©2019 MJM.

Contents

1. Introduction

For a graph G, let $c: V(G) \to \mathbb{N}$ be a proper vertex coloring of G. For each vertex v of G, let S_v be the set of colors of the neighbors of v, that is, $S_v = \{c(u) : u \in N(v)\}$ where $N(v)$ is the neighborhood of v. The *color code* code(v) of v is then defined as the ordered pair $(c(v), S_v)$. If code(u) $\neq$ code(v) for every two distinct vertices u and v of G, then c is called a *singular coloring* of G. Therefore, a singular coloring of a graph G is a coloring that uses the color of each vertex together with the set of colors of its neighbors to distinguish all vertices in G. If a singular coloring c uses k colors, then c is a *singular k-coloring*. For each positive integer k, let $\mathbb{N}_k = \{1,2,\ldots,k\}$. Thus, we assume that every singular k-coloring uses the colors in $\mathbb{N}_k$. A graph G is singularly k-colorable if G has a singular k-coloring.

The minimum k for which G has a singular k-coloring is called the *singular chromatic number* of G and is denoted by $\chi_{si}(G)$. Since a coloring assigning distinct colors to distinct vertices of a graph G is a singular coloring of G, the singular chromatic number exists for every graph. The concept of singular coloring was introduced by Kolasinski, Lin and Okamoto [3].

In [3], Kolasinski, Lin and Okamoto characterized: Let G be a graph of order n, then $\chi_{si}(G) = n$ if and only if G is a complete multipartite graph or an empty graph; and also proved that: For each $n \geq 3$, let k be the unique positive integer such that $(k-1)\binom{k-1}{2} + 1 \leq n \leq k\binom{k}{2}$, then $\chi_{si}(P_n) = k$, $\chi_{si}(C_n) = k$ if $n \neq k\binom{k}{2} - 1$; $\chi_{si}(C_n) = k+1$ if $n = k\binom{k}{2} - 1$.

The *Cartesian product* $G \square H$ of two graphs G and H has $V(G \square H) = V(G) \times V(H)$, and two vertices (u_1, u_2) and (v_1, v_2) of $G \square H$ are adjacent if and only if either $u_1 = v_1$ and $u_2 v_2 \in E(H)$ or $u_2 = v_2$ and $u_1 v_1 \in E(G)$.

In this paper, we compute the singular chromatic number of Cartesian product of paths.

2. Results

We have the following observations:

Observation 2.1. *[3] For every graph G of order n, $\chi(G) \leq \chi_{si}(G) \leq n$.*

Observation 2.2. *[3] Let G be a graph.*
(a) $\chi_{si}(G) = 1$ if and only if $G = K_1$.
(b) $\chi_{si}(G) = 2$ if and only if $G = \{K_2, K_2 \cup K_1, K_2 \cup 2K_1.\}$

Observation 2.3. *[3] If $\chi_{si}(G) = k$, then each of the following holds.*
(a) G contains at most k isolated vertices.
(b) G contains at most one clique of order k.
(c) Each vertex in G is adjacent to at most $k-1$ end-vertices.

For the path P_n on n vertices, let $V(P_n) = \{1,2,\ldots,n\}$, $E(P_n) = \{\{i,i+1\} : i \in \{1,2,\ldots,n-1\}\}$.

Let $c((i,j))$ denotes the color assigned to the vertex (v_i, v_j) and $code((i,j))$ denotes the color code of the vertex (v_i, v_j).

The following figures shows that the singular colorings of $P_3 \square P_3$, $P_3 \square P_4$ and $P_3 \square P_5$.

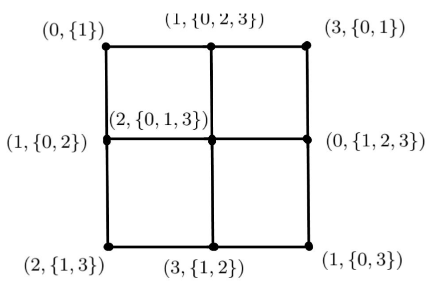

Figure 1. The singular 4-coloring of $P_3 \square P_3$

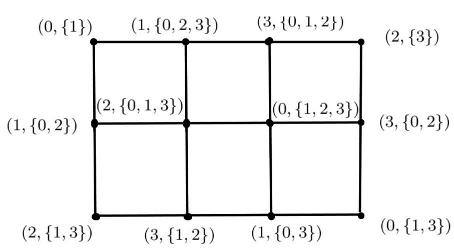

Figure 2. The singular 4-coloring of $P_3 \square P_4$

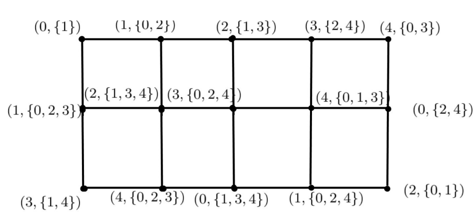

Figure 3. The singular 5-coloring of $P_3 \square P_5$

Theorem 2.4. *For $n \geq 6$, $\chi_{si}(P_3 \square P_n) \leq n-1$.*

Proof. Define $c : V(P_3 \square P_n) \rightarrow \{0,1,2,\ldots,n-2\}$ as follows: $c((1,j)) \equiv (j-1)(\mod (n-1))$, $c((2,j)) \equiv j(\mod (n-1))$ and $c((3,j)) \equiv (j+2)(\mod (n-1))$, where $j \in \{1,2,3,\ldots,n\}$.

Then, $code((1,1)) = (0,\{1\})$;
$code((1,j)) = ((j-1)(\mod (n-1)), \{(j-2)(\mod (n-1)), j(\mod (n-1))\})$ if $j \in \{2,3,4,\ldots,n\}$;
$code((2,j)) = (j(\mod (n-1)), \{(j-1)(\mod (n-1)), (j+1)(\mod (n-1)), (j+2)(\mod (n-1))\})$ if $j \in \{1,2,3,\ldots,n-1\}$;
$code((2,n)) = (1,\{0,3\})$;
$code((3,j)) = ((j+2)(\mod (n-1)), \{j(\mod (n-1)), (j+1)(\mod (n-1)), (j+3)(\mod (n-1))\})$ if $j \in \{2,3,4,\ldots,n-1\}$;
$code((3,1)) = (3,\{1,4\})$;
$code((3,n)) = (3,\{1,2\})$. Hence, $\chi_{si}(P_3 \square P_n) \leq n-1$.

Theorem 2.5. *For $m \geq 3$, $n \geq 3$, and $m \leq n$, $\chi_{si}(P_m \square P_n) \leq n+1$.*

Proof.
Let us define $c : V(P_m \square P_n) \rightarrow \{0,1,2,\ldots,n\}$ as follows:
$c((i,j)) = j-1$ if $(i,j) \in (\{1\} \times \{1,2,3,\ldots,n\})$;
$c((i,j)) = j$ if $(i,j) \in (\{2\} \times \{1,2,3,\ldots,n\})$;
$c((i,j)) = (i+j-1) \equiv 0 \mod (n+1)$ if $(i,j) \in (\{3,7,11,\ldots\} \times \{1,2,3,\ldots,n\})$;
$c((i,j)) = (i+j) \equiv 0 \mod (n+1)$ if $(i,j) \in (\{4,8,12,\ldots\} \times \{1,2,3,\ldots,n\})$;
$c((i,j)) = (\lfloor \frac{i}{2} \rfloor + j - 1) \equiv 0 \mod (n+1)$ if $(i,j) \in (\{5,9,13,\ldots\} \times \{1,2,3,\ldots,n\})$;
$c((i,j)) = (\frac{i}{2} + j) \equiv 0 \mod (n+1)$ if $(i,j) \in (\{6,10,14,\ldots\} \times \{1,2,3,\ldots,n\})$;
then the color code is:
$code((1,1)) = (0,\{1\})$;
$code((i,j)) = (j-1, \ \{j-2, \ j\})$ if $(i,j) \in (\{1\} \times \{2,3,\ldots,n\})$;
$code((i,j)) = (j, \{j-1, j+1, (j+2) \equiv 0 \mod (n+1)\})$ if $(i,j) \in (\{2\} \times \{1,2,3,\ldots,n-1\})$;
$code((2,n)) = (n, \{1, n-1\})$;
$code((i,j)) = (j+2 \equiv 0 \mod (n+1), \ \{j \equiv 0 \mod (n+1), \ (j+3) \equiv 0 \mod (n+1), \ (j+4) \equiv 0 \mod (n+1), (j+1) \equiv 0 \mod (n+1)\})$ if $(i,j) \in (\{3\} \times \{2,3,4,\ldots,n-1\})$;
$code((3,1)) = (3,\{1,4,5\})$;
$code((3,n)) = (1,\{0,3,n\})$;
$code((i,j)) = ((i+j) \equiv 0 \mod (n+1), \ \{(i+j-2) \equiv 0 \mod (n+1), \ (i+j+1) \equiv 0 \mod (n+1), \ (\frac{i}{2}+j-1) \equiv 0 \mod (n+1), \ (i+j-1) \equiv 0 \mod (n+1)\})$ if $(i,j) \in (\{4,8,12,\ldots, \neq m\} \times \{2,3,4,\ldots,n-1\})$;
$code((i,j)) = ((i+j) \equiv 0 \mod (n+1), \ \{(i+j-2) \equiv 0 \mod \ (i+j+1) \equiv 0 \mod (n+1), \ (\frac{i}{2}) \equiv 0 \mod (n+1)\})$ if $(i,j) \in (\{4,8,12,\cdots \neq m\} \times \{1\})$;
$code((i,j)) = ((i-1) \equiv 0 \mod (n+1), \ \{(i-3) \equiv 0 \mod \ (\frac{i}{2} - 2) \equiv 0 \mod (n+1), \ (i-2) \equiv 0 \mod (n+1)\})$ if $(i,j) \in (\{4,8,12,\cdots \neq m\} \times \{n\})$;
$code((i,j)) = ((\lfloor \frac{i}{2} \rfloor + j - 1) \equiv 0 \mod (n+1), \ \{(i+j-1) \equiv 0 \mod (n+1), \ (\lfloor \frac{i}{2} \rfloor + j - 2) \equiv 0 \mod (n+1), \ (\lfloor \frac{i}{2} \rfloor + j) \equiv 0 \mod (n+1), \ (\lfloor \frac{i}{2} \rfloor + j + 1) \equiv 0 \mod (n+1)\})$ if $(i,j) \in (\{5,9,13,\cdots \neq m\} \times \{2,3,4,\ldots,n-1\})$;
$code((i,j)) = ((\lfloor \frac{i}{2} \rfloor) \equiv 0 \mod (n+1), \ \{(i) \equiv 0 \mod (n+1), \ (\lfloor \frac{i}{2} \rfloor + j) \equiv 0 \mod (n+1), \ (\lfloor \frac{i}{2} \rfloor + j + 1) \equiv 0 \mod (n+1)\})$ if $(i,j) \in (\{5,9,13,\ldots\} \times \{1\})$;
$code((i,j)) = ((\lfloor \frac{i}{2} \rfloor + j - 1) \equiv 0 \mod (n+1), \ \{(i+j-$

$1) \equiv 0 \mod (n+1)$, $(\lfloor \frac{i}{2} \rfloor + j+1) \equiv 0 \mod (n+1)$, $(\lfloor \frac{i}{2} \rfloor + j-2) \equiv 0 \mod (n+1), \})$ if $(i,j) \in (\{5,9,13,\cdots \neq m\} \times \{n\})$;
$code((i,j)) = ((\frac{i}{2}+j) \equiv 0 \mod (n+1), \ \{(\frac{i}{2}+j-2) \equiv 0 \mod (n+1), \ (\frac{i}{2}+j+1) \equiv 0 \mod (n+1), \ (i+j) \equiv 0 \mod (n+1), \ (\frac{i}{2}+j-1) \equiv 0 \mod (n+1), \})$ if $(i,j) \in (\{6,10,14,\cdots \neq m\} \times \{2,3,4,\ldots,n-1\})$;
$code((i,j)) = ((\frac{i}{2}+j) \equiv 0 \mod (n+1), \ \{(\frac{i}{2}+j-2) \equiv 0 \mod (n+1), \ (\frac{i}{2}+j+1) \equiv 0 \mod (n+1), \ (i+j) \equiv 0 \mod (n+1), \})$ if $(i,j) \in (\{6,10,14,\cdots \neq m\} \times \{1\})$;
$code((i,j)) = ((\frac{i}{2}+j) \equiv 0 \mod (n+1), \ \{(\frac{i}{2}+j-2) \equiv 0 \mod (n+1), \ (\frac{i}{2}+j-1) \equiv 0 \mod (n+1), \ (i+j) \equiv 0 \mod (n+1), \})$ if $(i,j) \in (\{6,10,14,\cdots \neq m\} \times \{1\})$;
$code((i,j)) = ((i+j-1) \equiv 0 \mod (n+1), \ \{(\lfloor \frac{i}{2} \rfloor + j) \equiv 0 \mod (n+1), \ (i+j) \equiv 0 \mod (n+1), \ (i+j+1) \equiv 0 \mod (n+1), \ (i+j-2) \equiv 0 \mod (n+1)\})$ if $(i,j) \in (\{7,11,15\cdots \neq m\} \times \{2,3,4,\ldots,n-1\})$;
$code((i,j)) = (i, \ \{\lfloor \frac{i}{2} \rfloor +1, \ i+1, \ i+2\})$ if $(i,j) \in (\{7,11,15,\cdots \neq m\} \times \{1\})$;
$code((i,j)) = ((i+j-1) \equiv 0 \mod (n+1), \ \{(\lfloor \frac{i}{2} \rfloor + j) \equiv 0 \mod (n+1), \ (i+j-2) \equiv 0 \mod (n+1), \ (i+j+1) \equiv 0 \mod (n+1)\})$ if $(i,j) \in (\{7,11,15,\cdots \neq m\} \times \{n\})$;

Finally, we have
Case(1). $m \equiv 0 \mod 4$.
$code((m,1)) = (0, \ \{1, \ m-1\})$;
$code((m,n)) = (n-1, \ \{n-3, n-2\})$;
$code((i,j)) = (j-1, \ \{(j-2) \equiv 0 \mod (n+1), \ j \equiv 0 \mod (n+1), \ (i+j-2) \equiv 0 \mod (n+1)\})$ if $(i,j) \in (\{m\} \times \{2,3,\ldots,n-1\})$;

Case(2). $m \equiv 1 \mod 4$.
$code((m,1)) = (\lfloor \frac{m}{2} \rfloor, \ \{n, \ \lfloor \frac{m}{2} \rfloor +1\})$;
$code((m,n)) = (\lfloor \frac{m}{2} \rfloor -2, \ \{\lfloor \frac{m}{2} \rfloor -1, n-2\})$;
$code((i,j)) = ((\lfloor \frac{i}{2} \rfloor + j-1) \equiv 0 \mod (n+1), \ \{(m+j-1) \equiv 0 \mod (n+1), \ (\lfloor \frac{i}{2} \rfloor + j \equiv 0 \mod (n+1), \ (\lfloor \frac{i}{2} \rfloor + j-2) \equiv 0 \mod (n+1)\})$ if $(i,j) \in (\{m\} \times \{2,3,\ldots,n-1\})$;

Case(3). $m \equiv 2 \mod 4$.
$code((m,1)) = (\frac{m}{2}+1, \ \{\frac{m}{2}-1, \ \frac{m}{2}+2\})$;
$code((m,n)) = (\frac{m}{2}-1, \ \{\frac{m}{2}-3, \ \frac{m}{2}-2\})$;
$code((i,j)) = ((\frac{m}{2}+j) \equiv 0 \mod (n+1), \ \{(\frac{m}{2}+j-2) \equiv 0 \mod (n+1), \ (\frac{m}{2}+j+1) \equiv 0 \mod (n+1), \ \frac{m}{2}+j-1) \equiv 0 \mod (n+1)\})$ if $(i,j) \in (\{m\} \times \{2,3,\ldots,n-1\})$;

Case(4). $m \equiv 3 \mod 4$.
$code((m,1)) = (m, \ \{\lfloor \frac{m}{2} \rfloor +1, \ (m+1) \equiv 0 \mod (n+1)\})$;
$code((m,n)) = (m-2, \ \{\lfloor \frac{m}{2} \rfloor -1, \ m-3\})$;
$code((i,j)) = ((m+j-1) \equiv 0 \mod (n+1), \ \{(\lfloor \frac{m}{2} \rfloor + j) \equiv 0 \mod (n+1), \ (m+j) \equiv 0 \mod (n+1), \ (m+j-2) \equiv 0 \mod (n+1)\})$ if $(i,j) \in (\{m\} \times \{2,3,\ldots,n-1\})$.
Hence, $\chi_{si}(P_m \square P_n) \leq n+1$.

References

[1] R. Balakrishnan and K. Ranganathan, A Textbook of Graph Theory, Springer-Verlag, New York, 2000.

[2] G. Chartrand and P. Zhang, *Chromatic Graph Theory,* Chapman Hall/CRC press, Boca Raton (2009).

[3] K. Kolasinski, J. Lin and F. Okamoto, The singular Chromatic Number of a Graph, Ars Combinatoria, 118 (2015) 13-31.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆