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Two methods of generalized Ulam-Hyers stability of a quattuorvigintic functional equation in various Banach spaces

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Abstract
In this paper, we introduce and investigate the generalized Ulam - Hyers stability of a quattuorvigintic functional equation in various Banach spaces using two methods.

Keywords
Quattuorvigintic functional equation, generalized Ulam - Hyers stability, Banach space, quasi beta Banach space, fixed point.

AMS Subject Classification
39B52, 39B72, 39B82.

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Contents

1. Introduction

In [41], Ulam proposed the universal Ulam stability problem: When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?. In [18], Hyers gave the first assenting answer to the question of Ulam for additive functional equations on Banach spaces. Hyers result has since then seen many important generalizations, both in terms of the control condition used to define the concept of approximate solution [2, 16, 28, 35]. On the other hand, a fixed point alternative method is very important for the solution of the Ulam problem see [10] and [9].

During the last seven decades, the stability problems of a variety of functional equations in quite a lot of spaces have been broadly investigated by number of mathematicians [3, 5, 8, 12, 15, 17, 22, 27, 32, 34, 36, 39, 42].

Let us introduce the following quattuorvigintic functional equation

$$\begin{aligned} & f(x+12y)-24f(x+11y)+276f(x+10y)-2024f(x+9y) \\ & +10626f(x+8y)-42504f(x+7y)+134596f(x+6y) \\ & -346104f(x+5y)+735471f(x+4y)-1307504f(x+3y) \\ & +1961256f(x+2y)-2496144f(x+y)+2704156f(x) \\ & -2496144f(x-y)+1961256f(x-2y)-1307504f(x-3y) \\ & +735471f(x-4y)-346104f(x-5y)+134596f(x-6y) \\ & -42504f(x-7y)+10626f(x-8y)-2024f(x-9y) \\ & +276f(x-10y)-24f(x-11y)+f(x-12y)=24!f(y) \end{aligned} \tag{1.1}$$

where $\quad 24! = 620448401733239439360000.$

Now, we will recall the fundamental result in fixed point theory.

Theorem 1.1. *[23] (The alternative of fixed point) Suppose that for a complete generalized metric space (X,d) and a strictly contractive mapping $T: X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*
(F_1) $\qquad d(T^n x, T^{n+1}x) = \infty \quad \forall\ n \geq 0,$
or
(F_2) *there exists a natural number n_0 such that:*
$(FPC1)$ $d(T^n x, T^{n+1}x) < \infty$ *for all $n \geq n_0$;*
$(FPC2)$*The sequence $(T^n x)$ is convergent to a fixed point y^* of T*
$(FPC3)$ *y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;*
$(FPC4)$ $d(y^*, y) \leq \frac{1}{1-L}\ d(y, Ty)$ *for all $y \in Y$.*

In Section 2, the solution of the quattuorvigintic functional equation (1.1) is given. In Sections 3 and 4 the generalized Ulam - Hyers stability of a quattuorvigintic functional equation (1.1) in Banach spaces and quasi-beta Banach spaces using Ulam - Hyers and Ulam - Radus methods are respectively established.

2. Solution of (1.1)

In this section, the general solution of the quattuorvigintic functional equation (1.1) is given. For this, let us consider $\mathscr{V}_1$ and $\mathscr{V}_2$ be real vector spaces.

Theorem 2.1. *If $f : \mathscr{V}_1 \to \mathscr{V}_2$ be a mapping satisfying (1.1) for all $x, y \in \mathscr{V}_1$ then f is quattuorvigintic.*

Proof. Replacing (x,y) by $(0,0); (x,x); (x,-x)$ in (1.1), we arrive the set of equations

$$f(0) = 0 \tag{2.1}$$

$$\begin{aligned} &f(13x) - 24f(12x) + 276f(11x) - 2024f(10x)\\ &+ 10626f(9x) - 42504f(8x) + 134596f(7x)\\ &- 346104f(6x) + 735471f(5x) - 1307504f(4x)\\ &+ 1961256f(3x) - 2496144f(2x) + 2704156f(x)\\ &- 2496144f(0) + 1961256f(-x) - 1307504f(-2x)\\ &+ 735471f(-3x) - 346104f(-4x) + 134596f(-5x)\\ &- 42504f(-6x) + 10626f(-7x) - 2024f(-8x)\\ &+ 276f(-9x) - 24f(-10x) + f(-11x)\\ &= (620448401733239439360000)f(x) \end{aligned} \tag{2.2}$$

$$\begin{aligned} &f(-11x) - 24f(-10x) + 276f(-9x) - 2024f(-8x)\\ &+ 10626f(-7x) - 42504f(-6x) + 134596f(-5x)\\ &- 346104f(-4x) + 735471f(-3x) - 1307504f(-2x)\\ &+ 1961256f(-x) - 2496144f(0) + 2704156f(x)\\ &- 2496144f(2x) + 1961256f(3x) - 1307504f(4x)\\ &+ 735471f(5x) - 346104f(6x) + 134596f(7x)\\ &- 42504f(8x) + 10626f(9x) - 2024f(10x)\\ &+ 276f(11x) - 24f(12x) + f(13x)\\ &= (620448401733239439360000)f(-x) \end{aligned} \tag{2.3}$$

for all $x \in \mathscr{V}_1$. Comparing (2.2) and (2.3), we get the evenness of f, that is

$$f(-x) = f(x) \tag{2.4}$$

for all $x \in \mathscr{V}_1$.

Replacing (x,y) by $(12x,x); (11x,x); (10x,x); (9x,x); (8x,x); (7x,x); (6x,x); (5x,x); (4x,x); (3x,x); (2x,x);\ (0,x); (0,2x);$ in (1.1) and using evenness of f, we arrive the following equations

$$\begin{aligned} &f(24x) - 24f(23x) + 276f(22x) - 2024f(21x)\\ &+ 10626f(20x) - 42504f(19x) + 134596f(18x)\\ &- 346104f(17x) + 735471f(16x) - 1307504f(15x)\\ &+ 1961256f(14x) - 2496144f(13x) + 2704156f(12x)\\ &- 2496144f(11x) + 1961256f(10x) - 1307504f(9x)\\ &+ 735471f(8x) - 346104f(7x) + 134596f(6x)\\ &- 42504f(5x) + 10626f(4x) - 2024f(3x) + 276f(2x)\\ &- 620448401733239439360024f(x) = 0 \end{aligned} \tag{2.5}$$

$$\begin{aligned} &f(23x) - 24f(22x) + 276f(21x) - 2024f(20x)\\ &+ 10626f(19x) - 42504f(18x) + 134596f(17x)\\ &- 346104f(16x) + 735471f(15x) - 1307504f(14x)\\ &+ 1961256f(13x) - 2496144f(12x) + 2704156f(11x)\\ &- 2496144f(10x) + 1961256f(9x) - 1307504f(8x)\\ &+ 735471f(7x) - 346104f(6x) + 134596f(5x)\\ &- 42504f(4x) + 10626f(3x) - 2024f(2x)\\ &- 620448401733239439359723f(x) = 0 \end{aligned} \tag{2.6}$$

$$\begin{aligned} &f(22x) - 24f(21x) + 276f(20x) - 2024f(19x)\\ &+ 10626f(18x) - 42504f(17x) + 134596f(16x)\\ &- 346104f(15x) + 735471f(14x) - 1307504f(13x)\\ &+ 1961256f(12x) - 2496144f(11x) + 2704156f(10x)\\ &- 2496144f(9x) + 1961256f(8x) - 1307504f(7x)\\ &+ 735471f(6x) - 346104f(5x) + 134596f(4x)\\ &- 42504f(3x) + 10627f(2x)\\ &- 620448401733239439362048f(x) = 0 \end{aligned} \tag{2.7}$$

$$\begin{aligned} &f(21x) - 24f(20x) + 276f(19x) - 2024f(18x)\\ &+ 10626f(17x) - 42504f(16x) + 134596f(15x)\\ &- 346104f(14x) + 735471f(13x) - 1307504f(12x)\\ &+ 1961256f(11x) - 2496144f(10x) + 2704156f(9x)\\ &- 2496144f(8x) + 1961256f(7x) - 1307504f(6x)\\ &+ 735471f(5x) - 346104f(4x) + 134597f(3x)\\ &- 42528f(2x) - 620448401733239439349098f(x) = 0 \end{aligned} \tag{2.8}$$

$$
\begin{aligned}
&f(20x)-24f(19x)+276f(18x)-2024f(17x)\\
&+10626f(16x)-42504f(15x)+134596f(14x)\\
&-346104f(13x)+735471f(12x)-1307504f(11x)\\
&+1961256f(10x)-2496144f(9x)+2704156f(8x)\\
&-2496144f(7x)+1961256f(6x)-1307504f(5x)\\
&+735472f(4x)-346128f(3x)+134872f(2x)\\
&-620448401733239439404528f(x)=0 \qquad (2.9)
\end{aligned}
$$

$$
\begin{aligned}
&f(19x)-24f(18x)+276f(17x)-2024f(16x)\\
&+10626f(15x)-42504f(14x)+134596f(13x)\\
&-346104f(12x)+735471f(11x)-1307504f(10x)\\
&+1961256f(9x)-2496144f(8x)+2704156f(7x)\\
&-2496144f(6x)+1961257f(5x)-1307528f(4x)\\
&+735747f(3x)-348128f(2x)\\
&-620448401733239439214778f(x)=0 \qquad (2.10)
\end{aligned}
$$

$$
\begin{aligned}
&f(18x)-24f(17x)+276f(16x)-2024f(15x)\\
&+10626f(14x)-42504f(13x)+134596f(12x)\\
&-346104f(11x)+735471f(10x)-1307504f(9x)\\
&+1961256f(8x)-2496144f(7x)+2704157f(6x)\\
&-2496168f(5x)+1961532f(4x)\\
&-1309528f(3x)+746097f(2x)\\
&-620448401733239439748608f(x)=0 \qquad (2.11)
\end{aligned}
$$

$$
\begin{aligned}
&f(17x)-24f(16x)+276f(15x)-2024f(14x)\\
&+10626f(13x)-42504f(12x)+134596f(11x)\\
&-346104f(10x)+735471f(9x)-1307504f(8x)\\
&+1961257f(7x)-2496168f(6x)+2704432f(5x)\\
&-2498168f(4x)+1971882f(3x)-1350008f(2x)\\
&-620448401733239438489933f(x)=0 \qquad (2.12)
\end{aligned}
$$

$$
\begin{aligned}
&f(16x)-24f(15x)+276f(14x)-2024f(13x)\\
&+10626f(12x)-42504f(11x)+134596f(10x)\\
&-346104f(9x)+735472f(8x)-1307528f(7x)\\
&+1961532f(6x)-2498168f(5x)+2714782f(4x)\\
&-25388648f(3x)+2095852f(2x)\\
&-620448401733239441013608f(x)=0 \qquad (2.13)
\end{aligned}
$$

$$
\begin{aligned}
&f(15x)-24f(14x)+276f(13x)-2024f(12x)\\
&+10626f(11x)-42504f(10x)+134597f(9x)\\
&-346128f(8x)+735747f(7x)-1309528f(6x)\\
&+1971882f(5x)-2538648f(4x)+2838752f(3x)\\
&-2842248f(2x)-620448401733239436663273f(x)=0 \qquad (2.14)
\end{aligned}
$$

$$
\begin{aligned}
&f(14x)-24f(13x)+276f(12x)-2024f(11x)\\
&+10627f(10x)-42528f(9x)+134872f(8x)\\
&-348128f(7x)+746097f(6x)-1350008f(5x)\\
&+2095852f(4x)-2842248f(3x)+3439627f(2x)\\
&-620448401733239443163648f(x)=0 \qquad (2.15)
\end{aligned}
$$

$$
\begin{aligned}
&f(12x)-24f(11x)+276f(10x)-2024f(9x)\\
&+10626f(8x)-42504f(7x)+134596f(6x)\\
&-346104f(5x)+735471f(4x)-1307504f(3x)\\
&+1961256f(2x)\\
&-310224200866619722176144f(x) \qquad (2.16)
\end{aligned}
$$

$$
\begin{aligned}
&f(24x)-24f(22x)+276f(20x)-2024f(18x)\\
&+10626f(16x)-42504f(14x)+134596f(12x)\\
&-346104f(10x)+735471f(8x)-1307504f(6x)\\
&+1961256f(4x)\\
&-310224200866619722176144f(2x)=0 \qquad (2.17)
\end{aligned}
$$

for all $x \in \mathscr{V}_1$. Subtracting (2.17) from (2.5), we get

$$
\begin{aligned}
&24f(23x)-300f(22x)+2040f(21x)-10350f(20x)\\
&+42504f(19x)-136620f(18x)+346104f(17x)\\
&-724845f(16x)+1307504f(15x)-2003760f(14x)\\
&+2496144f(13x)-2569560f(12x)+2496144f(11x)\\
&-2307630f(10x)+1307504f(9x)+346104f(7x)\\
&-1442100f(6x)+42504f(5x)+1950630f(4x)\\
&+2024f(3x)-310224200866619722176420f(2x)\\
&-620448401733239439360024f(x)=0 \qquad (2.18)
\end{aligned}
$$

for all $x \in \mathscr{V}_1$. Multiplying (2.6) by 24 and subtracting from (2.18), we reach

$$
\begin{aligned}
&276f(22x)-4600f(21x)+38226f(20x)-212520f(19x)\\
&+883476f(18x)-2884200f(17x)+7581651f(16x)\\
&-16343800f(15x)+29376336f(14x)-44574000f(13x)\\
&+57337896f(12x)-62403600f(11x)+57600096f(10x)\\
&-45762640f(9x)+31380096f(8x)-17305200f(7x)\\
&+6864396f(6x)-3187800f(5x)+2970726f(4x)\\
&-253000f(3x)-310224200866619722127844f(2x)\\
&+15511210043330985983993376f(x)=0 \qquad (2.19)
\end{aligned}
$$

for all $x \in \mathscr{V}_1$. Multiplying (2.7) by 276 and subtracting from

(2.19), we obtain

$$\begin{aligned}&2024f(21x)-37950f(20x)+346104f(19x)\\&-2049300f(18x)+8846904f(17x)\\&-29566845f(16x)+79180904f(15x)\\&-173613660f(14x)+316297104f(13x)\\&-483968760f(12x)+626532144f(11x)\\&-688746960f(10x)+643173104f(9x)\\&-509926560f(8x)+343565904f(7x)\\&-196125600f(6x)+92336904f(5x)\\&-34177770f(4x)+11478104f(3x)\\&-3102242008666197250608896f(2x)\\&-1867549689217050712479186 24f(x)=0\end{aligned}\tag{2.20}$$

for all $x\in\mathscr{V}_1$. Multiplying (2.8) by 20247 and subtracting from (2.20), we have

$$\begin{aligned}&10626f(20x)-212520f(19x)+2047276f(18x)\\&-12660120f(17x)+56461251f(16x)\\&-193241400f(15x)+526900836f(14x)\\&-1172296200f(13x)+2162419336f(12x)\\&-3343050000f(11x)+4363448496f(10x)\\&-4830038640f(9x)+4542268896f(8x)\\&-3626016240f(7x)+2450262496f(6x)\\&-1396256400f(5x)+666336726f(4x)\\&-260946224f(3x)\\&+310224200866619638984224f(2x)\\&-144254253402997781696490492976f(x)=0\end{aligned}\tag{2.21}$$

for all $x\in\mathscr{V}_1$. Multiplying (2.9) by 10626 and subtracting from (2.21), we arrive

$$\begin{aligned}&42504f(19x)-88550f(18x)+8846904f(17x)\\&-56450625f(16x)+258406104f(15x)\\&-903316260f(14x)+2505404904f(13x)\\&-5652695510f(12x)+10550487504f(11x)\\&-16476857760f(10x)+21693987504f(9x)\\&-24192092760f(8x)+22898009904f(7x)\\&-18390043760f(6x)+12497281104f(5x)\\&-7148788746f(4x)+3417009904f(3x)\\&-310224200866621072134096f(2x)\\&-803542725084471839796030075 04f(x)=0\end{aligned}\tag{2.22}$$

for all $x\in\mathscr{V}_1$. Multiplying (2.10) by 42504 and subtracting from (2.22), we get

$$\begin{aligned}&134596f(18x)-2884200f(17x)+29577471f(16x)\\&-193241400f(15x)+903273756f(14x)\\&-3215463480f(13x)+9058108906f(12x)\\&-20709971880f(11x)+390997292256f(10x)\\&-61667237520f(9x)+81904011816f(8x)\\&-92039436720f(7x)+87706060816f(6x)\\&-70863986424f(5x)+48426381366f(4x)\\&-27855180584f(3x)\\&+310224200866606275301584f(2x)\\&-344069661181167931039879316 16f(x)=0\end{aligned}\tag{2.23}$$

for all $x\in\mathscr{V}_1$. Multiplying (2.11) by 134596 and subtracting from (2.23), we obtain

$$\begin{aligned}&346104f(17x)-7571025f(16x)+79180904f(15x)\\&-526943340f(14x)+2505404904f(13x)\\&-9057974310f(12x)+25874242104f(11x)\\&-59894162460f(10x)+114317570864f(9x)\\&-182073200760f(8x)+243931561104f(7x)\\&-276262654756f(6x)+265110241704f(5x)\\&-215587979706f(4x)+148402050104f(3x)\\&-310224200866706696973396f(2x)\\&-117916839197803888736391573984f(x)=0\end{aligned}\tag{2.24}$$

for all $x\in\mathscr{V}_1$. Multiplying (2.12) by 346104 and subtracting from (2.24), we have

$$\begin{aligned}&735471f(16x)-16343800f(15x)+173571156f(14x)\\&-1172296200f(13x)+5652830106f(12x)\\&-20709971880f(11x)+59893816356f(10x)\\&-140231884120f(9x)+270459163656f(8x)\\&-434867331624f(7x)+587671074716f(6x)\\&-670904491224f(5x)+649037957766f(4x)\\&-534074197624f(3x)\\&+310224200866239453804564f(2x)\\&-332656512831284991355511345016f(x)=0\end{aligned}\tag{2.25}$$

for all $x\in\mathscr{V}_1$. Multiplying (2.13) by 735471 and subtracting from (2.25), we arrive

$$\begin{aligned}&1307504f(15x)-29418840f(14x)+316297104f(13x)\\&-2162284740f(12x)+10550487504f(11x)\\&-39097638360f(10x)+114317570864f(9x)\\&-270459163656f(8x)+526781594064f(7x)\\&-854978826856f(6x)+1166425625904f(5x)\\&-1347605474556f(4x)+1333027785584f(3x)\\&-310224200867780892170856f(2x)\\&+788978319302432336277230634384f(x)=0\end{aligned} \tag{2.26}$$

for all $x\in\mathscr{V}_1$. Multiplying (2.14) by 1307504 and subtracting from (2.26), we get

$$\begin{aligned}&1961256f(14x)-44574000f(13x)\\&+484103356f(12x)-3343050000f(11x)\\&+16476511656f(10x)-61668545024f(9x)\\&+182104580856f(8x)-435210551424f(7x)\\&+857234271256f(6x)-1411817976624f(5x)\\&+1971686940036f(4x)-2378651809424f(3x)\\&+310224200864064641541864f(2x)\\&+1600217086362249832672206734976f(x)=0\end{aligned} \tag{2.27}$$

for all $x\in\mathscr{V}_1$. Multiplying (2.15) by 1961256, using evenness of f and subtracting from (2.27), we obtain

$$\begin{aligned}&2496144f(13x)-57203300f(12x)\\&+626532144f(11x)-4365755856f(10x)\\&+21739750144f(9x)-82413938376f(8x)\\&+247557577344f(7x)-606052946576f(6x)\\&+1235893313424f(5x)-2138815370076f(4x)\\&+3195724134064f(3x)\\&-310224200870810630633376f(2x)\\&+2817075236695197609001357035686 4f(x)=0\end{aligned} \tag{2.28}$$

for all $x\in\mathscr{V}_1$. Multiplying (2.2) by 2496144 and subtracting from (2.28), we have

$$\begin{aligned}&2704156f(12x)-64899744f(11x)\\&+746347056f(10x)-5473211744f(9x)\\&+28734361656f(8x)-114937446624f(7x)\\&+363968580976f(6x)-935919208224f(5x)\\&+1988828317476f(4x)-3535694786624f(3x)\\&-310224200861316177500064f(2x)\\&+4365803792247991305489858025536f(x)\end{aligned} \tag{2.29}$$

for all $x\in\mathscr{V}_1$. Multiplying (2.16) by 2704156 and subtracting from (2.29), we arrive

$$\begin{aligned}&-310224200866619719680000f(2x)\\&+5204698426366666226930810880000f(x)\end{aligned} \tag{2.30}$$

for all $x\in\mathscr{V}_1$. We achieve from (2.29) that

$$\begin{aligned}f(2x)&=\left(\frac{5204698426366666226930810880000}{310224200866619719680000}\right)f(x)\\&=16777216\ f(x)\\&=2^{24}f(x)\end{aligned} \tag{2.31}$$

for all $x\in\mathscr{V}_1$. □

3. Stability Analysis: Banach Space

In this section, we prove the the generalized Ulam - Hyers stability of a quattuorvigintic functional equation (1.1) in Banach spaces via two different methods. To provide the stability results, let us assume a function $f:\Gamma_1\to\Gamma_2$ by

$$\begin{aligned}Df_{24}(x,y)=&f(x+12y)-24f(x+11y)+276f(x+10y)\\&-2024f(x+9y)+10626f(x+8y)\\&-42504f(x+7y)+134596f(x+6y)\\&-346104f(x+5y)+735471f(x+4y)\\&-1307504f(x+3y)+1961256f(x+2y)\\&-2496144f(x+y)+2704156f(x)\\&-2496144f(x-y)+1961256f(x-2y)\\&-1307504f(x-3y)+735471f(x-4y)\\&-346104f(x-5y)+134596f(x-6y)\\&-42504f(x-7y)+10626f(x-8y)\\&-2024f(x-9y)+276f(x-10y)\\&-24f(x-11y)+f(x-12y)-24!f(y)\end{aligned}$$

where $24! = 620448401733239439360000$ for all $x,y\in\Gamma_1$, where Γ_1 and Γ_2 are normed space and Banach space respectively.

3.1 Ulam - Hyers Method

Theorem 3.1. *Let $f:\Gamma_1\to\Gamma_2$ be a function fulfilling the functional inequality*

$$\|Df_{24}(x,y)\|\le\mathscr{V}(x,y) \tag{3.1}$$

where $\mathscr{V}:\Gamma_1^2\to[0,\infty)$ is a function satisfying the condition

$$\lim_{s\to\infty}\frac{\mathscr{V}(2^{ps}x,2^{ps}y)}{2^{24ps}}=0 \tag{3.2}$$

for all $x,y\in\Gamma_1$ with $p=\pm1$. Then there exists a unique quattuorvigintic function $\mathscr{Q}_{24}:\Gamma_1\to\Gamma_2$ by

$$\mathscr{Q}_{24}(x)=\lim_{s\to\infty}\frac{f(2^{ps}x)}{2^{24ps}} \tag{3.3}$$

which satisfies (1.1) and

$$\|f(x)-\mathscr{Q}_{24}(x)\| \leq \frac{1}{2^{24}} \sum_{r=\frac{1-p}{2}}^{\infty} \frac{\mathscr{V}_{24}^{D}(2^{pr}x, 2^{pr}x)}{2^{24pr}} \tag{3.4}$$

for all $x \in \Gamma_1$. The function $\mathscr{V}_{24}^{D}(2^{pr}x, 2^{pr}x)$ defined by

$$\begin{aligned}\mathscr{V}_{24}^{D}(x,x) = {} & \mathscr{V}(0, 2\cdot 2^{pr}x) + \mathscr{V}(12\cdot 2^{pr}x, 2^{pr}x) \\ & + 24\ \mathscr{V}(11\cdot 2^{pr}x, 2^{pr}x) \\ & + 276\ \mathscr{V}(10\cdot 2^{pr}x, 2^{pr}x) \\ & + 20247\ \mathscr{V}(9\cdot 2^{pr}x, 2^{pr}x) \\ & + 10626\ \mathscr{V}(8\cdot 2^{pr}x, 2^{pr}x) \\ & + 42504\ \mathscr{V}(7\cdot 2^{pr}x, 2^{pr}x) \\ & + 134596\ \mathscr{V}(6\cdot 2^{pr}x, 2^{pr}x) \\ & + 346104\ \mathscr{V}(5\cdot 2^{pr}x, 2^{pr}x) \\ & + 735471\ \mathscr{V}(4\cdot 2^{pr}x, 2^{pr}x) \\ & + 1307504\ \mathscr{V}(3\cdot 2^{pr}x, 2^{pr}x) \\ & + 1961256\ \mathscr{V}(2\cdot 2^{pr}x, 2^{pr}x) \\ & + 2496144\ \mathscr{V}(2^{pr}x, 2^{pr}x) \\ & + 2704156\mathscr{V}(0, 2^{pr}x)\end{aligned} \tag{3.5}$$

for all $x \in \Gamma_1$.

Proof. Replacing (x,y) by $(12x,x);(11x,x);(10x,x);(9x,x);(8x,x);(7x,x);(6x,x);(5x,x);(4x,x);(3x,x);(2x,x);(0,x);(0,2x)$ in (3.1) and using evenness of f, and one can achieve from Theorem 2.1, that

$$\begin{aligned}& = \left\|f(2x) - 16777216\ f(x)\right\| \\ & = \left\|f(2x) - 2^{24}f(x)\right\| \\ & \leq \mathscr{V}(0,2x) + \mathscr{V}(12x,x) + 24\mathscr{V}(11x,x) \\ & \quad + 276\mathscr{V}(10x,x) + 20247\mathscr{V}(9x,x) \\ & \quad + 10626\mathscr{V}(8x,x) + 42504\mathscr{V}(7x,x) \\ & \quad + 134596\mathscr{V}(6x,x) + 346104\mathscr{V}(5x,x) \\ & \quad + 735471\mathscr{V}(4x,x) + 1307504\mathscr{V}(3x,x) \\ & \quad + 1961256\mathscr{V}(2x,x) + 2496144\mathscr{V}(x,x) \\ & \quad + 2704156\mathscr{V}(0,x)\end{aligned} \tag{3.6}$$

for all $x \in \Gamma_1$. Define

$$\begin{aligned}\mathscr{V}_{24}^{D}(x,x) = {} & \mathscr{V}(0,2x) + \mathscr{V}(12x,x) + 24\mathscr{V}(11x,x) \\ & + 276\mathscr{V}(10x,x) + 20247\mathscr{V}(9x,x) \\ & + 10626\mathscr{V}(8x,x) + 42504\mathscr{V}(7x,x) \\ & + 134596\mathscr{V}(6x,x) + 346104\mathscr{V}(5x,x) \\ & + 735471\mathscr{V}(4x,x) + 1307504\mathscr{V}(3x,x) \\ & + 1961256\mathscr{V}(2x,x) + 2496144\mathscr{V}(x,x) \\ & + 2704156\mathscr{V}(0,x)\end{aligned} \tag{3.7}$$

for all $x \in \Gamma_1$. Using (3.7) in (3.6), we arrive

$$\left\|f(2x) - 2^{24}f(x)\right\| \leq \mathscr{V}_{24}^{D}(x,x) \tag{3.8}$$

for all $x \in \Gamma_1$. It follows from (3.8) that

$$\left\|\frac{f(2x)}{2^{24}} - f(x)\right\| \leq \frac{\mathscr{V}_{24}^{D}(x,x)}{2^{24}} \tag{3.9}$$

for all $x \in \Gamma_1$. Now substitute x by $2x$ and dividing by 2^{24} in (3.9), we get

$$\left\|\frac{f(2^2x)}{2^{48}} - \frac{f(2x)}{2^{24}}\right\| \leq \frac{\mathscr{V}_{24}^{D}(2x,2x)}{2^{48}} \tag{3.10}$$

for all $x \in \Gamma_1$. From (3.9) and (3.10), we have

$$\left\|\frac{f(2^2x)}{2^{48}} - f(x)\right\| = \frac{1}{2^{24}}\left[\mathscr{V}_{24}^{D}(x,x) + \frac{\mathscr{V}_{24}^{D}(2x,2x)}{2^{24}}\right] \tag{3.11}$$

for all $x \in \Gamma_1$. Generalizing for a positive integer s, we obtain

$$\left\|\frac{f(2^sx)}{2^{24s}} - f(x)\right\| \leq \frac{1}{2^{24}} \sum_{r=0}^{s-1} \frac{\mathscr{V}_{24}^{D}(2^rx, 2^rx)}{2^{24r}} \tag{3.12}$$

for all $x \in \Gamma_1$. To prove the convergence of the sequence

$$\left\{\frac{f(2^sx)}{2^{24s}}\right\},$$

replacing x by 2^tx and dividing by 2^{24t} in (3.12), for any $s,t > 0$, we get that the sequence $\left\{\frac{f(2^sx)}{2^{24s}}\right\}$ is a Cauchy in Γ_2 and so it converges. Therefore, we see that a mapping $\mathscr{Q}_{24}(x) : \Gamma_1 \to \Gamma_2$ defined by

$$\mathscr{Q}_{24}(x) = \lim_{s\to\infty} \frac{f(2^sx)}{2^{24s}}$$

for all $x \in \Gamma_1$. Letting $s \to \infty$ in (3.12), we see that (3.4) holds for all $x \in \Gamma_1$. In order to show that $\mathscr{Q}_{24}$ satisfies (1.1), replacing (x,y) by $(2^sx, 2^sy)$ and dividing by 2^{24s} in (3.1), we have

$$\begin{aligned}\|\mathscr{Q}_{24}(x,y)\| & = \lim_{s\to\infty} \frac{1}{2^{24s}} \|Df_{24}(2^sx, 2^sy)\| \\ & \leq \lim_{s\to\infty} \frac{1}{2^{24s}} \mathscr{V}(2^sx, 2^sy)\end{aligned}$$

for all $x,y \in \Gamma_1$. Taking the limit as s approaches to infinity in above inequality, we find that the mapping $\mathscr{Q}_{24}$ is a quattuorvigintic mapping satisfying the functional equation (1.1) for all $x,y \in \Gamma_1$.

To prove that $\mathscr{Q}_{24}$ is unique, we assume now that there is $\mathscr{Q}'_{24}$ as another quattuorvigintic mapping satisfying (1.1) and the inequality (3.4). Then it follows easily that

$$\mathscr{Q}_{24}(2^sx) = 2^{24s}\mathscr{Q}_{24}, \qquad \mathscr{Q}'_{24}(2^sx) = 2^{24s}\mathscr{Q}'_{24}(x)$$

for all $x \in \Gamma_1$ and all $s \in \mathbb{N}$. Thus

$$\begin{aligned}& \left\|\mathscr{Q}_{24}(x) - \mathscr{Q}'_{24}(x)\right\| \\ & = \frac{1}{2^{24s}} \left\|\mathscr{Q}_{24}(2^sx) - \mathscr{Q}'_{24}(2^sx)\right\| \\ & \leq \frac{1}{2^{24s}} \left\{\|\mathscr{Q}_{24}(2^sx) - f(2^sx)\| + \left\|f(2^sx) - \mathscr{Q}'_{24}(2^sx)\right\|\right\} \\ & \leq \frac{1}{2^{24}} \sum_{r=0}^{\infty} \frac{\mathscr{V}(2^{r+s}x, 2^{r+s}x)}{2^{24(r+s)}}\end{aligned}$$

for all $x \in \Gamma_1$. Therefore, as $s \to \infty$ in the above inequality, one establishes

$$\mathscr{Q}_{24}(x) - \mathscr{Q}'_{24}(x) = 0$$

for all $x \in \Gamma_1$, completing the proof of the claimed uniqueness of $\mathscr{Q}_{24}$. Hence the theorem holds for $p = 1$.

Case (ii): Assume $p = -1$.

Now replacing x by $\frac{x}{2}$ in (3.8), we get

$$\left\| f(x) - 2^{24} f\left(\frac{x}{2}\right) \right\| \leq \mathscr{V}^D_{24}\left(\frac{x}{2}, \frac{x}{2}\right) \tag{3.13}$$

for all $x \in \Gamma_1$. The rest of the proof is similar to that of case $p = 1$. Hence for $p = -1$ also the theorem holds. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.1 concerning the stabilities of (1.1).

Corollary 3.2. *Let $f : \Gamma_1 \to \Gamma_2$ be a mapping. If there exist real numbers ϕ and ψ such that*

$$\|Df_{24}(x,y)\| \leq \begin{cases} \phi, & \\ \phi\left\{||x||^{\psi} + ||y||^{\psi}\right\}, & \psi \neq 24; \\ \phi||x||^{\psi}||y||^{\psi}, & \psi \neq 12; \end{cases} \tag{3.14}$$

for all $x, y \in \Gamma_1$, then there exists a unique quattuorvigintic function $\mathscr{Q}_{24} : \Gamma_1 \to \Gamma_2$ such that

$$\|f(x) - \mathscr{Q}_{24}(x)\| \leq \begin{cases} \dfrac{\phi_C}{|2^{24}-1|}, \\ \dfrac{\phi_S||z||^{\psi}}{|2^{24}-2^{\psi}|}, \\ \dfrac{\phi_P||z||^{2\psi}}{|2^{24}-2^{2\psi}|}, \end{cases} \tag{3.15}$$

where

$$\begin{aligned} \phi_C &= 9758910\phi, \\ \phi_S &= \Big\{\Big(12^{\psi} + 24\cdot 11^{\psi} + 276\cdot 10^{\psi} \\ &\quad +20247\cdot 9^{\psi} + 10626\cdot 8^{\psi} \\ &\quad +42504\cdot 7^{\psi} + 134596\cdot 6^{\psi} \\ &\quad +346104\cdot 5^{\psi} + 735471\cdot 4^{\psi} \\ &\quad +1307504\cdot 3^{\psi} + 1961256\cdot 2^{\psi}\Big) \\ &\qquad +12255054\Big\}, \\ \phi_P &= \Big(12^{\psi} + 24\cdot 11^{\psi} + 276\cdot 10^{\psi} \\ &\quad +20247\cdot 9^{\psi} + 10626\cdot 8^{\psi} \\ &\quad +42504\cdot 7^{\psi} + 134596\cdot 6^{\psi} \\ &\quad +346104\cdot 5^{\psi} + 735471\cdot 4^{\psi} \\ &\quad +1307504\cdot 3^{\psi} + 1961256\cdot 2^{\psi} + 2496144\Big), \end{aligned}$$

for all $x \in \Gamma_1$.

3.2 Ulam - Radus Method

Theorem 3.3. *Let $f : \Gamma_1 \to \Gamma_2$ be a function fulfilling th functional inequality*

$$\|Df_{24}(x,y)\| \leq \mathscr{V}(x,y) \tag{3.16}$$

where $\mathscr{V} : \Gamma_1^2 \to [0,\infty)$ is a function satisfying the condition

$$\lim_{s\to\infty} \frac{1}{d_i^{24s}} \mathscr{V}(d_i^s x, d_i^s y) = 0 \tag{3.17}$$

where

$$d_i = \begin{cases} 2 & if \quad i = 0, \\ \frac{1}{2} & if \quad i = 1 \end{cases} \tag{3.18}$$

for all $x, y \in \Gamma_1$. Assume that there exists $L = L(i)$ such that the function

$$\mathscr{V}^F_{24}(x,x) = \mathscr{V}^D_{24}\left(\frac{x}{2}, \frac{x}{2}\right) \tag{3.19}$$

where $\mathscr{V}^D_{24}(x,x)$ is defined in (3.5) with the property

$$\frac{1}{d_i^{24}} \mathscr{V}^F_{24}(d_i x, d_i x) = L \;\; \mathscr{V}^F_{24}(x,x) \tag{3.20}$$

for all $x \in \Gamma_1$. Then there exists a unique quattuorvigintic function $\mathscr{Q}_{24} : \Gamma_1 \to \Gamma_2$ by

$$\mathscr{Q}_{24}(x) = \lim_{s\to\infty} \frac{f(d_i^s x)}{d_i^{24s}} \tag{3.21}$$

which satisfies (1.1) and

$$\|f(x) - \mathscr{Q}_{24}(x)\| \leq \left(\frac{L^{1-i}}{1-L}\right) \mathscr{V}^F_{24}(x,x) \tag{3.22}$$

for all $x \in \Gamma_1$.

Proof. Assume a set $\mathscr{S} = \{q/q : \Gamma_1 \to \Gamma_2,\ q(0) = 0\}$ and introduce the generalized metric in the set as follows

$$\inf\{\rho \in (0,\infty) : \|q_1(x) - q_2(x)\| \leq \phi \;\, \mathscr{V}^F_{24}(x,x), x \in \Gamma_1\}. \tag{3.23}$$

It is easy to see that (3.23) is complete with respect to the defined metric. Define $J : \mathscr{S} \to \mathscr{S}$ by

$$Jq(x) = \frac{1}{d_i^{24}} q(d_i x),$$

for all $x \in \Gamma_1$. Now, from (3.23) and $q_1, q_2 \in \mathscr{S}$, we arrive $d(Jq_1, Jq_2) \leq L\phi$ (See Theorem [23]). This implies J is a strictly contractive mapping on $\mathscr{S}$ with Lipschitz constant L.

It follows from (3.23), (3.9) and (3.20) for the case $i = 0$, we reach

$$\begin{aligned} &\left\| \frac{f(2x)}{2^{24}} - f(x) \right\| \leq \frac{1}{2^{24}} \mathscr{V}^D_{24}(x,x) \\ &\Rightarrow \|Jf(x) - f(x)\| \leq L^{1-i} \;\, \mathscr{V}^F_{24}(x,x), x \in \Gamma_1. \end{aligned} \tag{3.24}$$

Again, it follows from (3.23), (3.13) and (3.20) for the case $i=1$, we get

$$\left\|f(x)-2^{24}f\left(\frac{x}{2}\right)\right\| \le \mathscr{V}_{24}^{D}\left(\frac{x}{2},\frac{x}{2}\right)$$
$$\Rightarrow \|f(x)-Jf(x)\| \le L^{1-i}\mathscr{V}_{24}^{F}(x,x), x\in\Gamma_1. \tag{3.25}$$

Thus, from (3.24), (3.25) and (3.23), we arrive

$$\inf\left\{L^{1-i}\in(0,\infty): \begin{array}{l}\left\|f(x)-Jf(x)\right\| \\ \le L^{1-i}\mathscr{V}_{24}^{F}(x,x),\end{array} x\in\Gamma_1\right\}. \tag{3.26}$$

Hence property (FPC1) of Theorem 1.1 holds. It follows from property (FPC2) of Theorem 1.1 that there exists a fixed point $\mathscr{Q}_{24}$ of J in $\mathscr{S}$ such that

$$\mathscr{Q}_{24}(x) = \lim_{s\to\infty}\frac{1}{d_i^{24s}}f(d_i^s x) \tag{3.27}$$

for all $x\in\Gamma_1$. In order to show that $\mathscr{Q}_{24}$ satisfies (1.1), replacing (x,y) by $(d_i^s x, d_i^s y)$ and dividing by d_i^{24s} in (3.16), we have

$$\|\mathscr{Q}_{24}(x,y)\| = \lim_{s\to\infty}\frac{1}{d_i^{24s}}\|Df_{24}(d_i^s x,d_i^s y)\|$$
$$\le \lim_{s\to\infty}\frac{1}{d_i^{24s}}\mathscr{V}(d_i^s x,d_i^s y)=0$$

for all $x,y\in\Gamma_1$ i.e., $\mathscr{Q}_{24}$ satisfies the functional equation (1.1). By property (FPC3) of Theorem 1.1, $\mathscr{Q}_{24}$ is the unique fixed point of J in the set

$$\Delta = \{\mathscr{Q}_{24}\in\mathscr{S}: d(f,\mathscr{Q}_{24})<\infty\},$$

such that

$$\inf\left\{\phi\in(0,\infty): \|f(x)-\mathscr{Q}_{24}(x)\| \le \phi\mathscr{V}_{24}^{F}(x,x), x\in\Gamma_1\right\}.$$

Finally by property (FPC4) of Theorem 1.1, we obtain

$$\|f(x)-\mathscr{Q}_{24}(x)\| \le \|f(x)-Jf(x)\| \le \frac{L^{1-i}}{1-L},$$

which yields

$$\inf\left\{\frac{L^{1-i}}{1-L}\in(0,\infty): \begin{array}{l}\|f(x)-\mathscr{Q}_{24}(x)\| \\ \le \left(\frac{L^{1-i}}{1-L}\right)\mathscr{V}_{24}^{F}(x,x)\end{array}, x\in\Gamma_1\right\}.$$

This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.3 concerning the stability of (1.1).

Corollary 3.4. *Let $f:\Gamma_1\to\Gamma_2$ be a mapping. If there exist real numbers a and ψ satisfying (3.14) for all $x,y\in\Gamma_1$, then there exists a unique quattuorvigintic function $\mathscr{Q}_{24}:\Gamma_1\to\Gamma_2$ such that (3.15) holds for all $x\in\Gamma_1$.*

Proof. If we define

$$\mathscr{V}(x,y) = \begin{cases}\phi, \\ \phi\left\{||x||^{\psi}+||y||^{\psi}\right\}, \\ \phi||x||^{\psi}||y||^{\psi},\end{cases} \tag{3.28}$$

and replacing (x,y) by $(d_i^n x, d_i^n y)$ in (3.28) and divided by d_i^n, one can see that (3.17) holds for all $x,y\in\Gamma_1$. But from (3.19), (3.28) and (3.7), we have

$$\begin{aligned}
\mathscr{V}_{24}^{F}(x,x) &= \mathscr{V}_{24}^{D}(x,x)\left(\frac{x}{2},\frac{x}{2}\right) \\
&= \mathscr{V}\left(0,2\frac{x}{2}\right)+\mathscr{V}\left(12\frac{x}{2},\frac{x}{2}\right)+24\mathscr{V}\left(11\frac{x}{2},\frac{x}{2}\right) \\
&\quad +276\mathscr{V}\left(10\frac{x}{2},\frac{x}{2}\right)+20247\mathscr{V}\left(9\frac{x}{2},\frac{x}{2}\right) \\
&\quad +10626\mathscr{V}\left(8\frac{x}{2},\frac{x}{2}\right)+42504\mathscr{V}\left(7\frac{x}{2},\frac{x}{2}\right) \\
&\quad +134596\mathscr{V}\left(6\frac{x}{2},\frac{x}{2}\right)+346104\mathscr{V}\left(5\frac{x}{2},\frac{x}{2}\right) \\
&\quad +735471\mathscr{V}\left(4\frac{x}{2},\frac{x}{2}\right)+1307504\mathscr{V}\left(3\frac{x}{2},\frac{x}{2}\right) \\
&\quad +1961256\mathscr{V}\left(2\frac{x}{2},\frac{x}{2}\right)+2496144\mathscr{V}\left(\frac{x}{2},\frac{x}{2}\right) \\
&\quad +2704156\mathscr{V}\left(0,\frac{x}{2}\right) \\
&= \begin{cases} 9758910\,\phi, \\ \dfrac{\phi\,||x||^{\psi}}{2^{\psi}}\Big\{\Big(12^{\psi}+24\cdot 11^{\psi}+276\cdot 10^{\psi} \\ \quad +20247\cdot 9^{\psi}+10626\cdot 8^{\psi} \\ \quad +42504\cdot 7^{\psi}+134596\cdot 6^{\psi} \\ \quad +346104\cdot 5^{\psi}+735471\cdot 4^{\psi} \\ \quad +1307504\cdot 3^{\psi}+1961256\cdot 2^{\psi}\Big) \\ \quad +12255054\Big\}, \\ \dfrac{\phi\,||x||^{2\psi}}{2^{\psi}}\Big(12^{\psi}+24\cdot 11^{\psi}+276\cdot 10^{\psi} \\ \quad +20247\cdot 9^{\psi}+10626\cdot 8^{\psi} \\ \quad +42504\cdot 7^{\psi}+134596\cdot 6^{\psi} \\ \quad +346104\cdot 5^{\psi}+735471\cdot 4^{\psi} \\ \quad +1307504\cdot 3^{\psi}+1961256\cdot 2^{\psi} \\ \quad +2496144\Big), \end{cases} \\
&= \begin{cases}\phi_C, \\ \dfrac{\phi_S}{2^{\psi}}, \\ \dfrac{\phi_P}{2^{2\psi}},\end{cases}
\end{aligned} \tag{3.29}$$

for all $x\in\Gamma_1$. Now, similarly by (3.20), (3.28) and (3.7), we prove

$$\frac{1}{d_i^{24}}\mathscr{V}_{24}^{F}(d_i x, d_i x) = \begin{cases} d_i^{-24}\,\phi_C, \\ d_i^{\psi-24}\phi_S, \\ d_i^{2\psi-24}\phi_P,\end{cases}$$

for all $x\in\Gamma_1$. Thus, the inequality (3.22) holds for

(i). $L = d_i^{-24}$ if $i=0$ and $L=\frac{1}{d_i^{-24}}$ if $i=1$, (ii). $L = d_i^{\psi-24}$ for $\psi < 24$ if $i = 0$ and $L = \frac{1}{d_i^{\psi-24}}$ for $\psi > 24$ if $i = 1$,

(iii). $L = d_i^{2\psi-24}$ for $2\psi > 24$ if $i = 0$ and $L = \frac{1}{d_i^{2\psi-24}}$ for $2\psi > 24$ if $i = 1$.

Now, from (3.22), we prove the following cases for condition (i).

$$\begin{array}{ll} L = d_i^{-24} = 2^{-24}, i = 0 & L = \frac{1}{d_i^{-24}} = 2^{24}, i = 1 \\ \| f(x) - \mathscr{Q}_{24}(x) \| & \| f(x) - \mathscr{Q}_{24}(x) \| \\ \leq \left(\frac{L^{1-i}}{1-L}\right) \mathscr{V}_{24}^F(x,x) & \leq \left(\frac{L^{1-i}}{1-L}\right) \mathscr{V}_{24}^F(x,x) \\ = \left(\frac{(2^{-24})^{1-0}}{1-2^{-24}}\right) \phi_C & = \left(\frac{(2^{24})^{1-1}}{1-2^{24}}\right) \phi_C \\ = \left(\frac{2^{-24}}{1-2^{-24}}\right) \phi_C & = \left(\frac{1}{1-2^{24}}\right) \phi_C \\ = \left(\frac{1}{2^{24}-1}\right) \phi_C & = \left(\frac{1}{1-2^{24}}\right) \phi_C \end{array}$$

Also, from (3.22), we prove the following cases for condition (ii).

$$\begin{array}{ll} L = d_i^{\psi-24} = 2^{\psi-24}, i = 0 & L = \frac{1}{d_i^{\psi-24}} = \frac{1}{2^{\psi-24}}, i = 1 \\ \| f(x) - \mathscr{Q}_{24}(x) \| & \| f(x) - \mathscr{Q}_{24}(x) \| \\ \leq \left(\frac{L^{1-i}}{1-L}\right) \mathscr{V}_{24}^F(x,x) & \leq \left(\frac{L^{1-i}}{1-L}\right) \mathscr{V}_{24}^F(x,x) \\ = \left(\frac{(2^{\psi-24})^{1-0}}{1-2^{\psi-24}}\right) \phi_S & = \left(\frac{(2^{24-b})^{1-1}}{1-2^{24-b}}\right) \phi_S \\ = \left(\frac{2^{\psi-24}}{1-2^{\psi-24}}\right) \phi_S & = \left(\frac{1}{1-2^{24-b}}\right) \phi_S \\ = \left(\frac{2^{\psi}}{2^{24}-2^{\psi}}\right) \phi_S & = \left(\frac{2^{\psi}}{2^{\psi}-2^{24}}\right) \phi_S \end{array}$$

Finally, from (3.22), we prove the following cases for condition (iii).

$$\begin{array}{ll} L = d_i^{2\psi-24} & L = \frac{1}{d_i^{2\psi-24}} \\ \quad = 2^{2\psi-24}, i = 0 & \quad = 2^{24-2b}, i = 1 \\ \| f(x) - \mathscr{Q}_{24}(x) \| & \| f(x) - \mathscr{Q}_{24}(x) \| \\ \leq \left(\frac{L^{1-i}}{1-L}\right) \mathscr{V}_{24}^F(x,x) & \leq \left(\frac{L^{1-i}}{1-L}\right) \mathscr{V}_{24}^F(x,x) \\ = \left(\frac{(2^{2\psi-24})^{1-0}}{1-2^{2\psi-24}}\right) \phi_P & = \left(\frac{(2^{24-2b})^{1-1}}{1-2^{24-2b}}\right) \phi_P \\ = \left(\frac{2^{2\psi-24}}{1-6^{2b-24}}\right) \phi_P & = \left(\frac{1}{1-2^{24-2b}}\right) \phi_P \\ = \left(\frac{2^{2\psi}}{2^{24}-2^{2\psi}}\right) \phi_P & = \left(\frac{2^{2\psi}}{2^{2\psi}-2^{24}}\right) \phi_P \end{array}$$

Hence the proof is complete. □

4. Stability Results : Quasi Beta Banach Spaces

In this section, we prove the the generalized Ulam - Hyers stability of a quattuorvigintic functional equation (1.1) in Quasi beta Banach spaces via two different methods. To provide the stability results, let us assume a function $f : \Gamma_1 \to \Gamma_2$ by

$$\begin{aligned} Df_{24}(x,y) =& f(x+12y) - 24f(x+11y) + 276f(x+10y) \\ & - 2024f(x+9y) + 10626f(x+8y) \\ & - 42504f(x+7y) + 134596f(x+6y) \\ & - 346104f(x+5y) + 735471f(x+4y) \\ & - 1307504f(x+3y) + 1961256f(x+2y) \\ & - 2496144f(x+y) + 2704156f(x) \\ & - 2496144f(x-y) + 1961256f(x-2y) \\ & - 1307504f(x-3y) + 735471f(x-4y) \\ & - 346104f(x-5y) + 134596f(x-6y) \\ & - 42504f(x-7y) + 10626f(x-8y) \\ & - 2024f(x-9y) + 276f(x-10y) \\ & - 24f(x-11y) + f(x-12y) - 24!f(y) \end{aligned}$$

where $24! = 620448401733239439360000$ for all $x, y \in \Gamma_1$, where Γ_1 and Γ_2 are quasi normed space and quasi beta Banach space respectively.

4.1 Basic Definitions and Notations

In this section, we give some basic facts concerning quasi-β-Normed spaces.

We fix a real number β with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

Definition 4.1. *Let X be a linear space over $\mathbb{K}$. A quasi-β-norm $\| \cdot \|$ is a real-valued function on X satisfying the following:*

$(QB1)$ *$\| x \| \geq 0$ for all $x \in X$ and $\| x \| = 0$ if and only if $x = 0$.*

$(QB2)$ *$\| \rho x \| = | \rho |^{\beta} . \| x \|$ for all $\rho \in \mathbb{K}$ and all $x \in X$.*

$(QB3)$ *There is a constant $K \geq 1$ such that $\| x + y \| \leq K (\| x \| + \| y \|)$ for all $x, y \in X$.*

*The pair $(X, \| \cdot \|)$ is called **quasi-β-normed space** if $\| \cdot \|$ is a quasi-β-norm on X. The smallest possible K is called the modulus of concavity of $\| \cdot \|$.*

Definition 4.2. *A **quasi-β-Banach space** is a complete quasi-β-normed space.*

4.2 Ulam - Hyers Method

Theorem 4.3. *Let $f : \Gamma_1 \to \Gamma_2$ be a function fulfilling th functional inequality*

$$\|Df_{24}(x,y)\| \leq \mathscr{V}(x,y) \tag{4.1}$$

where $\mathscr{V} : \Gamma_1^2 \to [0, \infty)$ is a function satisfying the condition

$$\lim_{s\to\infty} \frac{\mathscr{V}(2^{ps}x, 2^{ps}y)}{2^{24ps}} = 0 \tag{4.2}$$

for all $x, y \in \Gamma_1$ with $p = \pm 1$. Then there exists a unique quattuorvigintic function $\mathscr{Q}_{24} : \Gamma_1 \to \Gamma_2$ by

$$\mathscr{Q}_{24}(x) = \lim_{s\to\infty} \frac{f(2^{ps}x)}{2^{24ps}} \tag{4.3}$$

which satisfies (1.1) and

$$\|f(x)-\mathscr{Q}_{24}(x)\| \leq \frac{K^{s-1}}{2^{24\beta}} \sum_{r=\frac{1-p}{2}}^{\infty} \frac{\mathscr{V}_{24}^{D}(2^{pr}x, 2^{pr}x)}{2^{24pr}} \quad (4.4)$$

for all $x \in \Gamma_1$. *The function* $\mathscr{V}_{24}^{D}(2^{pr}x, 2^{pr}x)$ *defined by*

$$\begin{aligned}\mathscr{V}_{24}^{D}((2^{pr}x, 2^{pr}x) = {} & K^{13}\mathscr{V}(0, 2\cdot 2^{pr}x) + K^{13}\mathscr{V}(12\cdot 2^{pr}x, 2^{pr}x)\\ & + K^{12}24\mathscr{V}(11\cdot 2^{pr}x, 2^{pr}x)\\ & + K^{11}276\mathscr{V}(10\cdot 2^{pr}x, 2^{pr}x)\\ & + K^{10}20247\mathscr{V}(9\cdot 2^{pr}x, 2^{pr}x)\\ & + K^{9}10626\mathscr{V}(8\cdot 2^{pr}x, 2^{pr}x)\\ & + K^{8}42504\mathscr{V}(7\cdot 2^{pr}x, 2^{pr}x)\\ & + K^{7}134596\mathscr{V}(6\cdot 2^{pr}x, 2^{pr}x)\\ & + K^{6}346104\mathscr{V}(5\cdot 2^{pr}x, 2^{pr}x)\\ & + K^{5}735471\mathscr{V}(4\cdot 2^{pr}x, 2^{pr}x)\\ & + K^{4}1307504\mathscr{V}(3\cdot 2^{pr}x, 2^{pr}x)\\ & + K^{3}1961256\mathscr{V}(2\cdot 2^{pr}x, 2^{pr}x)\\ & + K^{2}2496144\mathscr{V}(2^{pr}x, 2^{pr}x)\\ & + K\ 2704156\mathscr{V}(0, 2^{pr}x) \qquad (4.5)\end{aligned}$$

for all $x \in \Gamma_1$.

Proof. Replacing (x,y) by $(x,x); (12x,x); (11x,x); (10x,x); (9x,x); (8x,x); (7x,x); (6x,x); (5x,x); (4x,x); (3x,x); (2x,x); (0,x); (0,2x)$; in (4.1) and using evenness of f,

$$\begin{aligned}&\Big\| f(13x) - 24f(12x) + 277f(11x) - 2048f(10x)\\ &\quad + 10902f(9x) - 44528f(8x) + 145222f(7x)\\ &\quad - 388608f(6x) + 870067f(5x) - 1653608f(4x)\\ &\quad + 2696727f(3x) - 3803648f(2x)\\ &\quad - 620448401733239434694588f(x)\Big\| \leq \mathscr{V}(x,x)\end{aligned} \quad (4.6)$$

$$\begin{aligned}&\Big\| f(24x) - 24f(23x) + 276f(22x) - 2024f(21x)\\ &\quad + 10626f(20x) - 42504f(19x)\\ &\quad + 134596f(18x) - 346104f(17x)\\ &\quad + 735471f(16x) - 1307504f(15x)\\ &\quad + 1961256f(14x) - 2496144f(13x)\\ &\quad + 2704156f(12x) - 2496144f(11x)\\ &\quad + 1961256f(10x) - 1307504f(9x)\\ &\quad + 735471f(8x) - 346104f(7x)\\ &\quad + 134596f(6x) - 42504f(5x)\\ &\quad + 10626f(4x) - 2024f(3x) + 276f(2x)\\ &\quad - 620448401733239439360024f(x)\Big\| \leq \mathscr{V}(12x,x)\end{aligned} \quad (4.7)$$

$$\begin{aligned}&\Big\| f(23x) - 24f(22x) + 276f(21x) - 2024f(20x)\\ &\quad + 10626f(19x) - 42504f(18x)\\ &\quad + 134596f(17x) - 346104f(16x)\\ &\quad + 735471f(15x) - 1307504f(14x)\\ &\quad + 1961256f(13x) - 2496144f(12x)\\ &\quad + 2704156f(11x) - 2496144f(10x)\\ &\quad + 1961256f(9x) - 1307504f(8x)\\ &\quad + 735471f(7x) - 346104f(6x)\\ &\quad + 134596f(5x) - 42504f(4x)\\ &\quad + 10626f(3x) - 2024f(2x)\\ &\quad - 620448401733239439359723f(x)\Big\| \leq \mathscr{V}(11x,x)\end{aligned} \quad (4.8)$$

$$\begin{aligned}&\Big\| f(22x) - 24f(21x) + 276f(20x) - 2024f(19x)\\ &\quad + 10626f(18x) - 42504f(17x)\\ &\quad + 134596f(16x) - 346104f(15x)\\ &\quad + 735471f(14x) - 1307504f(13x)\\ &\quad + 1961256f(12x) - 2496144f(11x)\\ &\quad + 2704156f(10x) - 2496144f(9x)\\ &\quad + 1961256f(8x) - 1307504f(7x)\\ &\quad + 735471f(6x) - 346104f(5x)\\ &\quad + 134596f(4x) - 42504f(3x) + 10627f(2x)\\ &\quad - 620448401733239439362048f(x)\Big\| \leq \mathscr{V}(10x,x)\end{aligned} \quad (4.9)$$

$$\begin{aligned}&\Big\| f(21x) - 24f(20x) + 276f(19x) - 2024f(18x)\\ &\quad + 10626f(17x) - 42504f(16x)\\ &\quad + 134596f(15x) - 346104f(14x)\\ &\quad + 735471f(13x) - 1307504f(12x)\\ &\quad + 1961256f(11x) - 2496144f(10x)\\ &\quad + 2704156f(9x) - 2496144f(8x)\\ &\quad + 1961256f(7x) - 1307504f(6x)\\ &\quad + 735471f(5x) - 346104f(4x)\\ &\quad + 134597f(3x) - 42528f(2x)\\ &\quad - 620448401733239439349098f(x)\Big\| \leq \mathscr{V}(9x,x)\end{aligned} \quad (4.10)$$

$$\begin{aligned}\Big\| f(20x) &- 24f(19x) + 276f(18x) - 2024f(17x) \\ &+ 10626f(16x) - 42504f(15x) \\ &+ 134596f(14x) - 346104f(13x) \\ &+ 735471f(12x) - 1307504f(11x) \\ &+ 1961256f(10x) - 2496144f(9x) \\ &+ 2704156f(8x) - 2496144f(7x) \\ &+ 1961256f(6x) - 1307504f(5x) \\ &+ 735472f(4x) - 346128f(3x) + 134872f(2x) \\ &- 620448401733239439404528f(x)\Big\| \leq \mathscr{V}(8x,x)\end{aligned} \tag{4.11}$$

$$\begin{aligned}\Big\| f(19x) &- 24f(18x) + 276f(17x) - 2024f(16x) \\ &+ 10626f(15x) - 42504f(14x) \\ &+ 134596f(13x) - 346104f(12x) \\ &+ 735471f(11x) - 1307504f(10x) \\ &+ 1961256f(9x) - 2496144f(8x) \\ &+ 2704156f(7x) - 2496144f(6x) \\ &+ 1961257f(5x) - 1307528f(4x) \\ &+ 735747f(3x) - 348128f(2x) \\ &- 620448401733239439214778f(x)\Big\| \leq \mathscr{V}(7x,x)\end{aligned} \tag{4.12}$$

$$\begin{aligned}\Big\| f(18x) &- 24f(17x) + 276f(16x) - 2024f(15x) \\ &+ 10626f(14x) - 42504f(13x) \\ &+ 134596f(12x) - 346104f(11x) \\ &+ 735471f(10x) - 1307504f(9x) \\ &+ 1961256f(8x) - 2496144f(7x) \\ &+ 2704157f(6x) - 2496168f(5x) \\ &+ 1961532f(4x) - 1309528f(3x) + 746097f(2x) \\ &- 620448401733239439748608f(x)\Big\| \leq \mathscr{V}(6x,x)\end{aligned} \tag{4.13}$$

$$\begin{aligned}\Big\| f(17x) &- 24f(16x) + 276f(15x) - 2024f(14x) \\ &+ 10626f(13x) - 42504f(12x) \\ &+ 134596f(11x) - 346104f(10x) \\ &+ 735471f(9x) - 1307504f(8x) \\ &+ 1961257f(7x) - 2496168f(6x) \\ &+ 2704432f(5x) - 2498168f(4x) \\ &+ 1971882f(3x) - 1350008f(2x) \\ &- 620448401733239438489933f(x)\Big\| \leq \mathscr{V}(5x,x)\end{aligned} \tag{4.14}$$

$$\begin{aligned}\Big\| f(16x) &- 24f(15x) + 276f(14x) - 2024f(13x) \\ &+ 10626f(12x) - 42504f(11x) \\ &+ 134596f(10x) - 346104f(9x) \\ &+ 735472f(8x) - 1307528f(7x) \\ &+ 1961532f(6x) - 2498168f(5x) \\ &+ 2714782f(4x) - 25388648f(3x) + 2095852f(2x) \\ &- 620448401733239441013608f(x)\Big\| \leq \mathscr{V}(4x,x)\end{aligned} \tag{4.15}$$

$$\begin{aligned}\Big\| f(15x) &- 24f(14x) + 276f(13x) - 2024f(12x) \\ &+ 10626f(11x) - 42504f(10x) \\ &+ 134597f(9x) - 346128f(8x) \\ &+ 735747f(7x) - 1309528f(6x) \\ &+ 1971882f(5x) - 2538648f(4x) \\ &+ 2838752f(3x) - 2842248f(2x) \\ &- 620448401733239436663273f(x)\Big\| \leq \mathscr{V}(3x,x)\end{aligned} \tag{4.16}$$

$$\begin{aligned}\Big\| f(14x) &- 24f(13x) + 276f(12x) - 2024f(11x) \\ &+ 10627f(10x) - 42528f(9x) \\ &+ 134872f(8x) - 348128f(7x) \\ &+ 746097f(6x) - 1350008f(5x) \\ &+ 2095852f(4x) - 2842248f(3x) + 3439627f(2x) \\ &- 620448401733239443163648f(x)\Big\| \leq \mathscr{V}(2x,x)\end{aligned} \tag{4.17}$$

$$\begin{aligned}\Big\| f(12x) &- 24f(11x) + 276f(10x) - 2024f(9x) \\ &+ 10626f(8x) - 42504f(7x) \\ &+ 134596f(6x) - 346104f(5x) \\ &+ 735471f(4x) - 1307504f(3x) + 1961256f(2x) \\ &- 310224200866619722176144f(x)\Big\| \leq \mathscr{V}(0,x)\end{aligned} \tag{4.18}$$

$$\begin{aligned}\Big\| f(24x) &- 24f(22x) + 276f(20x) - 2024f(18x) \\ &+ 10626f(16x) - 42504f(14x) \\ &+ 134596f(12x) - 346104f(10x) \\ &+ 735471f(8x) - 1307504f(6x) \\ &+ 1961256f(4x) \\ &- 310224200866619722176144f(2x)\Big\| \leq \mathscr{V}(0,2x)\end{aligned} \tag{4.19}$$

for all $x \in \Gamma_1$. Combining (4.19) and (4.7), we get

$$\begin{aligned}
&\Big\| 24f(23x) - 300f(22x) + 2040f(21x) - 10350f(20x) \\
&\quad + 42504f(19x) - 136620f(18x) + 346104f(17x) \\
&\quad - 724845f(16x) + 1307504f(15x) - 2003760f(14x) \\
&\quad + 2496144f(13x) - 2569560f(12x) + 2496144f(11x) \\
&\quad - 2307630f(10x) + 1307504f(9x) + 346104f(7x) \\
&\quad - 1442100f(6x) + 42504f(5x) + 1950630f(4x) \\
&\quad + 2024f(3x) - 31022420086661972217642 0f(2x) \\
&\quad - 620448401733239439360024f(x) \Big\| \\
&\le K\left(\mathscr{V}(0,2x) + \mathscr{V}(12x,x)\right) \qquad (4.20)
\end{aligned}$$

for all $x \in \Gamma_1$. Multiplying (4.8) by 24 and subtracting from (4.20)

$$\begin{aligned}
&\Big\| 276f(22x) - 4600f(21x) + 38226f(20x) - 212520f(19x) \\
&\quad + 883476f(18x) - 2884200f(17x) \\
&\quad + 7581651f(16x) - 16343800f(15x) \\
&\quad + 29376336f(14x) - 44574000f(13x) \\
&\quad + 57337896f(12x) - 62403600f(11x) \\
&\quad + 57600096f(10x) - 45762640f(9x) \\
&\quad + 31380096f(8x) - 17305200f(7x) \\
&\quad + 6864396f(6x) - 3187800f(5x) \\
&\quad + 2970726f(4x) - 253000f(3x) \\
&\quad - 310224200866619722127844f(2x) \\
&\quad + 15511210043330985983993376f(x) \Big\| \\
&\le K^2\mathscr{V}(0,2x) + K^2\mathscr{V}(12x,x) + K\ 24\mathscr{V}(11x,x) \qquad (4.21)
\end{aligned}$$

for all $x \in \Gamma_1$. Multiplying (4.9) by 276 and subtracting from (4.21)

$$\begin{aligned}
&\Big\| 2024f(21x) - 37950f(20x) + 346104f(19x) \\
&\quad - 2049300f(18x) + 8846904f(17x) \\
&\quad - 29566845f(16x) + 79180904f(15x) \\
&\quad - 173613660f(14x) + 316297104f(13x) \\
&\quad - 483968760f(12x) + 626532144f(11x) \\
&\quad - 688746960f(10x) + 643173104f(9x) \\
&\quad - 509926560f(8x) + 343565904f(7x) \\
&\quad - 196125600f(6x) + 92336904f(5x) \\
&\quad - 34177770f(4x) + 11478104f(3x) \\
&\quad - 310224200866619725060896f(2x) \\
&\quad - 186754968921705071247918624f(x) \Big\| \\
&\le K^3\mathscr{V}(0,2x) + K^3\mathscr{V}(12x,x) \\
&\quad + K^2 24\mathscr{V}(11x,x) + K\ 276\mathscr{V}(10x,x) \qquad (4.22)
\end{aligned}$$

for all $x \in \Gamma_1$. Multiplying (4.10) by 20247 and subtracting from (4.22)

$$\begin{aligned}
&\Big\| 10626f(20x) - 212520f(19x) + 2047276f(18x) \\
&\quad - 12660120f(17x) + 56461251f(16x) \\
&\quad - 193241400f(15x) + 526900836f(14x) \\
&\quad - 1172296200f(13x) + 2162419336f(12x) \\
&\quad - 3343050000f(11x) + 4363448496f(10x) \\
&\quad - 4830038640f(9x) + 4542268896f(8x) \\
&\quad - 3626016240f(7x) + 2450262496f(6x) \\
&\quad - 1396256400f(5x) + 666336726f(4x) \\
&\quad - 260946224f(3x) \\
&\quad + 310224200866619638984224f(2x) \\
&\quad - 1442542534029977816964904 92976f(x) \Big\| \\
&\le K^4\mathscr{V}(0,2x) + K^4\mathscr{V}(12x,x) + K^3 24\mathscr{V}(11x,x) \\
&\quad + K^2 276\mathscr{V}(10x,x) + K\ \ 20247\mathscr{V}(9x,x) \qquad (4.23)
\end{aligned}$$

for all $x \in \Gamma_1$. Multiplying (4.11) by 10626 and subtracting from (4.23)

$$\begin{aligned}
&\Big\| 42504f(19x) - 88550f(18x) + 8846904f(17x) \\
&\quad - 56450625f(16x) + 258406104f(15x) \\
&\quad - 903316260f(14x) + 2505404904f(13x) \\
&\quad - 5652695510f(12x) + 10550487504f(11x) \\
&\quad - 16476857760f(10x) + 21693987504f(9x) \\
&\quad - 24192092760f(8x) + 22898009904f(7x) \\
&\quad - 18390043760f(6x) + 12497281104f(5x) \\
&\quad - 7148788746f(4x) + 3417009904f(3x) \\
&\quad - 310224200866621072134096f(2x) \\
&\quad - 8035427250844718397960300 7504f(x) \Big\| \\
&\le K^5\mathscr{V}(0,2x) + K^5\mathscr{V}(12x,x) + K^4 24\mathscr{V}(11x,x) \\
&\quad + K^3 276\mathscr{V}(10x,x) + K^2 20247\mathscr{V}(9x,x) \\
&\quad + K\ \ 10626\mathscr{V}(8x,x) \qquad (4.24)
\end{aligned}$$

for all $x \in \Gamma_1$. Multiplying (4.12) by 42504 and subtracting from (4.24)

$$\begin{aligned}&\Big\| 134596f(18x)-2884200f(17x)+29577471f(16x)\\&\quad-193241400f(15x)+903273756f(14x)\\&\quad-3215463480f(13x)+9058108906f(12x)\\&\quad-20709971880f(11x)+390997292256f(10x)\\&\quad-61667237520f(9x)+81904011816f(8x)\\&\quad-92039436720f(7x)+87706060816f(6x)\\&\quad-70863986424f(5x)+48426381366f(4x)\\&\quad-27855180584f(3x)\\&\quad+310224200866606275301584f(2x)\\&\quad-3440696611811679310398793161 6f(x)\Big\|\\&\le K^6\mathscr{V}(0,2x)+K^6\mathscr{V}(12x,x)+K^5 24\mathscr{V}(11x,x)\\&\quad+K^4 276\mathscr{V}(10x,x)+K^3 20247\mathscr{V}(9x,x)\\&\quad+K^2 10626\mathscr{V}(8x,x)+K\ 42504\mathscr{V}(7x,x)\end{aligned}\tag{4.25}$$

for all $x\in\Gamma_1$. Multiplying (4.13) by 134596 and subtracting from (4.25)

$$\begin{aligned}&\Big\| 346104f(17x)-7571025f(16x)+79180904f(15x)\\&\quad-526943340f(14x)+2505404904f(13x)\\&\quad-9057974310f(12x)+25874242104f(11x)\\&\quad-59894162460f(10x)+114317570864f(9x)\\&\quad-182073200760f(8x)+243931561104f(7x)\\&\quad-276262654756f(6x)+265110241704f(5x)\\&\quad-215587979706f(4x)+148402050104f(3x)\\&\quad-310224200866706696973396f(2x)\\&\quad-11791683919780388873639157398 4f(x)\Big\|\\&\le K^7\mathscr{V}(0,2x)+K^7\mathscr{V}(12x,x)+K^6 24\mathscr{V}(11x,x)\\&\quad+K^5 276\mathscr{V}(10x,x)+K^4 20247\mathscr{V}(9x,x)\\&\quad+K^3 10626\mathscr{V}(8x,x)+K^2 42504\mathscr{V}(7x,x)\\&\quad+K\ 134596\mathscr{V}(6x,x)\end{aligned}\tag{4.26}$$

for all $x\in\Gamma_1$. Multiplying (4.14) by 346104 and subtracting from (4.26)

$$\begin{aligned}&\Big\| 735471f(16x)-16343800f(15x)+173571156f(14x)\\&\quad-1172296200f(13x)+5652830106f(12x)\\&\quad-20709971880f(11x)+59893816356f(10x)\\&\quad-140231884120f(9x)+270459163656f(8x)\\&\quad-434867331624f(7x)+587671074716f(6x)\\&\quad-670904491224f(5x)+649037957766f(4x)\\&\quad-534074197624f(3x)\\&\quad+310224200866239453804564f(2x)\\&\quad-33265651283128499135551134501 6f(x)\Big\|\\&\le K^8\mathscr{V}(0,2x)+K^8\mathscr{V}(12x,x)+K^7 24\mathscr{V}(11x,x)\\&\quad+K^6 276\mathscr{V}(10x,x)+K^5 20247\mathscr{V}(9x,x)\\&\quad+K^4 10626\mathscr{V}(8x,x)+K^3 42504\mathscr{V}(7x,x)\\&\quad+K^2 134596\mathscr{V}(6x,x)+K\ 346104\mathscr{V}(5x,x)\end{aligned}\tag{4.27}$$

for all $x\in\Gamma_1$. Multiplying (4.15) by 735471 and subtracting from (4.27)

$$\begin{aligned}&\Big\| 1307504f(15x)-29418840f(14x)+316297104f(13x)\\&\quad-2162284740f(12x)+10550487504f(11x)\\&\quad-39097638360f(10x)+114317570864f(9x)\\&\quad-270459163656f(8x)+526781594064f(7x)\\&\quad-854978826856f(6x)+1166425625904f(5x)\\&\quad-1347605474556f(4x)+1333027785584f(3x)\\&\quad-310224200867780892170856f(2x)\\&\quad+78897831930243233627723063438 4f(x)\Big\|\\&\le K^9\mathscr{V}(0,2x)+K^9\mathscr{V}(12x,x)+K^8 24\mathscr{V}(11x,x)\\&\quad+K^7 276\mathscr{V}(10x,x)+K^6 20247\mathscr{V}(9x,x)\\&\quad+K^5 10626\mathscr{V}(8x,x)+K^4 42504\mathscr{V}(7x,x)\\&\quad+K^3 134596\mathscr{V}(6x,x)+K^2 346104\mathscr{V}(5x,x)\\&\quad+K\ 735471\mathscr{V}(4x,x)\end{aligned}\tag{4.28}$$

for all $x\in\Gamma_1$. Multiplying (4.16) by 1307504 and subtracting from (4.28)

$$\begin{aligned}
&\Big\| 1961256f(14x)-44574000f(13x)+484103356f(12x)\\
&\quad-3343050000f(11x)+16476511656f(10x)\\
&\quad-61668545024f(9x)+182104580856f(8x)\\
&\quad-435210551424f(7x)+857234271256f(6x)\\
&\quad-1411817976624f(5x)+1971686940036f(4x)\\
&\quad-2378651809424f(3x)\\
&\quad+310224200864064641541864f(2x)\\
&\quad+1600217086362249832672206734976f(x)\Big\|\\
&\quad\le K^{10}\mathscr{V}(0,2x)+K^{10}\mathscr{V}(12x,x)+K^{9}24\mathscr{V}(11x,x)\\
&\quad+K^{8}276\mathscr{V}(10x,x)+K^{7}20247\mathscr{V}(9x,x)\\
&\quad+K^{6}10626\mathscr{V}(8x,x)+K^{5}42504\mathscr{V}(7x,x)\\
&\quad+K^{4}134596\mathscr{V}(6x,x)+K^{3}346104\mathscr{V}(5x,x)\\
&\quad+K^{2}735471\mathscr{V}(4x,x)+K\ 1307504\mathscr{V}(3x,x) \qquad (4.29)
\end{aligned}$$

for all $x\in\Gamma_1$. Multiplying (4.17) by 1961256 , using evenness of f and subtracting from (4.29)

$$\begin{aligned}
&\Big\| 2496144f(13x)-57203300f(12x)\\
&\quad+626532144f(11x)-4365755856f(10x)\\
&\quad+21739750144f(9x)-82413938376f(8x)\\
&\quad+247557577344f(7x)-606052946576f(6x)\\
&\quad+1235893313424f(5x)-2138815370076f(4x)\\
&\quad+3195724134064f(3x)\\
&\quad-310224200870810630633376f(2x)\\
&\quad+28170752366951976090013570356864f(x)\Big\|\\
&\quad\le K^{11}\mathscr{V}(0,2x)+K^{11}\mathscr{V}(12x,x)+K^{10}24\mathscr{V}(11x,x)\\
&\quad+K^{9}276\mathscr{V}(10x,x)+K^{8}20247\mathscr{V}(9x,x)\\
&\quad+K^{7}10626\mathscr{V}(8x,x)+K^{6}42504\mathscr{V}(7x,x)\\
&\quad+K^{5}134596\mathscr{V}(6x,x)+K^{4}346104\mathscr{V}(5x,x)\\
&\quad+K^{3}735471\mathscr{V}(4x,x)+K^{2}1307504\mathscr{V}(3x,x)\\
&\quad+K\ 1961256\mathscr{V}(2x,x) \qquad (4.30)
\end{aligned}$$

for all $x\in\Gamma_1$. Multiplying (4.6) by 2496144 and subtracting from (4.30)

$$\begin{aligned}
&\Big\| 2704156f(12x)-64899744f(11x)\\
&\quad+746347056f(10x)-5473211744f(9x)\\
&\quad+28734361656f(8x)-114937446624f(7x)\\
&\quad+363968580976f(6x)-935919208224f(5x)\\
&\quad+1988828317476f(4x)-3535694786624f(3x)\\
&\quad-310224200861316177500064f(2x)\\
&\quad+4365803792247991305489858025536f(x)\Big\|\\
&\quad\le K^{12}\mathscr{V}(0,2x)+K^{12}\mathscr{V}(12x,x)+K^{11}24\mathscr{V}(11x,x)\\
&\quad+K^{10}276\mathscr{V}(10x,x)+K^{9}20247\mathscr{V}(9x,x)\\
&\quad+K^{8}10626\mathscr{V}(8x,x)+K^{7}42504\mathscr{V}(7x,x)\\
&\quad+K^{6}134596\mathscr{V}(6x,x)+K^{5}346104\mathscr{V}(5x,x)\\
&\quad+K^{4}735471\mathscr{V}(4x,x)+K^{3}1307504\mathscr{V}(3x,x)\\
&\quad+K^{2}1961256\mathscr{V}(2x,x)+K\ 2496144\mathscr{V}(x,x) \qquad (4.31)
\end{aligned}$$

for all $x\in\Gamma_1$. Multiplying (4.18) by 2704156 and subtracting from (4.31)

$$\begin{aligned}
&\Big\| -310224200866619719680000f(2x)\\
&\quad+520469842636666226930810880000f(x)\Big\|\\
&\quad\le K^{13}\mathscr{V}(0,2x)+K^{13}\mathscr{V}(12x,x)+K^{12}24\mathscr{V}(11x,x)\\
&\quad+K^{11}276\mathscr{V}(10x,x)+K^{10}20247\mathscr{V}(9x,x)\\
&\quad+K^{9}10626\mathscr{V}(8x,x)+K^{8}42504\mathscr{V}(7x,x)\\
&\quad+K^{7}134596\mathscr{V}(6x,x)+K^{6}346104\mathscr{V}(5x,x)\\
&\quad+K^{5}735471\mathscr{V}(4x,x)+K^{4}1307504\mathscr{V}(3x,x)\\
&\quad+K^{3}1961256\mathscr{V}(2x,x)+K^{2}2496144\mathscr{V}(x,x)\\
&\quad+K\ 2704156\mathscr{V}(0,x) \qquad (4.32)
\end{aligned}$$

for all $x\in\Gamma_1$. We achieve from (4.31) that and one can achieve from Theorem 2.1, that

$$\begin{aligned}
&\left\| f(2x)-\left(\frac{520469842636666226930810880000}{310224200866619719680000}\right)f(x)\right\|\\
&\quad=\left\| f(2x)-16777216\ f(x)\right\|\\
&\quad=\left\| f(2x)-2^{24}f(x)\right\|\\
&\quad\le K^{13}\mathscr{V}(0,2x)+K^{13}\mathscr{V}(12x,x)+K^{12}24\mathscr{V}(11x,x)\\
&\quad+K^{11}276\mathscr{V}(10x,x)+K^{10}20247\mathscr{V}(9x,x)\\
&\quad+K^{9}10626\mathscr{V}(8x,x)+K^{8}42504\mathscr{V}(7x,x)\\
&\quad+K^{7}134596\mathscr{V}(6x,x)+K^{6}346104\mathscr{V}(5x,x)\\
&\quad+K^{5}735471\mathscr{V}(4x,x)+K^{4}1307504\mathscr{V}(3x,x)\\
&\quad+K^{3}1961256\mathscr{V}(2x,x)+K^{2}2496144\mathscr{V}(x,x)\\
&\quad+K\ 2704156\mathscr{V}(0,x) \qquad (4.33)
\end{aligned}$$

for all $x \in \Gamma_1$. Define

$$\begin{aligned}\mathscr{V}_{24}^{D}(x,x) = & K^{13}\mathscr{V}(0,2x) + K^{13}\mathscr{V}(12x,x)\\ & + K^{12}24\mathscr{V}(11x,x)\\ & + K^{11}276\mathscr{V}(10x,x) + K^{10}20247\mathscr{V}(9x,x)\\ & + K^{9}10626\mathscr{V}(8x,x) + K^{8}42504\mathscr{V}(7x,x)\\ & + K^{7}134596\mathscr{V}(6x,x) + K^{6}346104\mathscr{V}(5x,x)\\ & + K^{5}735471\mathscr{V}(4x,x) + K^{4}1307504\mathscr{V}(3x,x)\\ & + K^{3}1961256\mathscr{V}(2x,x) + K^{2}2496144\mathscr{V}(x,x)\\ & + K\ \ 2704156\mathscr{V}(0,x)\end{aligned} \tag{4.34}$$

for all $x \in \Gamma_1$. Using (4.34) in (4.33), we arrive

$$\left\|f(2x) - 2^{24} f(x)\right\| \le \mathscr{V}_{24}^{D}(x,x) \tag{4.35}$$

for all $x \in \Gamma_1$. It follows from (4.35) that

$$\left\|\frac{f(2x)}{2^{24}} - f(x)\right\| \le \frac{\mathscr{V}_{24}^{D}(x,x)}{2^{24\beta}} \tag{4.36}$$

for all $x \in \Gamma_1$. Now substitute x by $2x$ and dividing by 2^{24} in (4.36), we get

$$\left\|\frac{f(2^2x)}{2^{48}} - \frac{f(2x)}{2^{24}}\right\| \le \frac{\mathscr{V}_{24}^{D}(2x,2x)}{2^{24+24\beta}} \tag{4.37}$$

for all $x \in \Gamma_1$. From (4.36) and (4.37), we have

$$\left\|\frac{f(2^2x)}{2^{48}} - f(x)\right\| = \frac{K}{2^{24\beta}}\left[\mathscr{V}_{24}^{D}(x,x) + \frac{\mathscr{V}_{24}^{D}(2x,2x)}{2^{24}}\right] \tag{4.38}$$

for all $x \in \Gamma_1$. Generalizing, for a positive integer s, we obtain

$$\left\|\frac{f(2^sx)}{2^{24s}} - f(x)\right\| \le \frac{K^{s-1}}{2^{24\beta}}\sum_{r=0}^{s-1}\frac{\mathscr{V}_{24}^{D}(2^rx,2^rx)}{2^{24r}} \tag{4.39}$$

for all $x \in \Gamma_1$. The rest of the proof is similar to that of Theorem 3.1. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 4.3 concerning the stabilities of (1.1).

Corollary 4.4. *Let $f : \Gamma_1 \to \Gamma_2$ be a mapping. If there exist real numbers ϕ and ψ such that*

$$\|Df_{24}(x,y)\| \le \begin{cases}\phi, & \\ \phi\left\{||x||^{\psi} + ||y||^{\psi}\right\}, & \psi \neq 24;\\ \phi||x||^{\psi}||y||^{\psi}, & \psi \neq 12;\end{cases} \tag{4.40}$$

for all $x,y \in \Gamma_1$, then there exists a unique quattuorvigintic function $\mathscr{Q}_{24} : \Gamma_1 \to \Gamma_2$ such that

$$\|f(x) - \mathscr{Q}_{24}(x)\| \le \begin{cases}\dfrac{2^{24}K^{s-1}\phi_{\beta C}}{2^{24\beta}|2^{24}-1|},\\ \dfrac{2^{24}K^{s-1}\phi_{\beta S}||z||^{\psi}}{2^{24\beta}|2^{24}-2^{\psi\beta}|},\\ \dfrac{2^{24}K^{s-1}\phi_{\beta P}||z||^{2\psi}}{2^{24\beta}|2^{24}-2^{2\psi\beta}|},\end{cases} \tag{4.41}$$

where

$$\begin{aligned}\phi_{\beta C} = {} & \phi\Big(2K^{13} + K^{12}\ 24 + K^{11}\ 276 + K^{10}\ 20247\\ & + K^{9}\ 10626 + K^{8}\ 42504 + K^{7}\ 134596\\ & + K^{6}\ 346104 + K^{5}\ 735471 + K^{4}\ 1307504\\ & + K^{3}\ 1961256 + K^{2}\ 2496144 + K\ \ 2704156\Big),\\ \phi_{\beta S} = {} & \Big(K^{13}\Big(12^{\psi\beta} + 2^{\psi\beta} + 1\Big) + K^{12}24\Big(11^{\psi\beta} + 1\Big)\\ & + K^{11}276\Big(10^{\psi\beta} + 1\Big) + K^{10}20247\Big(9^{\psi\beta} + 1\Big)\\ & + K^{9}10626\Big(8^{\psi\beta} + 1\Big) + K^{8}42504\Big(7^{\psi\beta} + 1\Big)\\ & + K^{7}134596\Big(6^{\psi\beta} + 1\Big) + K^{6}346104\Big(5^{\psi\beta} + 1\Big)\\ & + K^{5}735471\Big(4^{\psi\beta} + 1\Big) + K^{4}1307504\Big(3^{\psi\beta} + 1\Big)\\ & + K^{3}1961256\Big(2^{\psi\beta} + 1\Big) + 2K^{2}\ 2496144 + K\ 2704156\Big),\\ \phi_{\beta P} = {} & \Big(K^{13}\ 12^{\psi\beta}\ + K^{12}24\ 11^{\psi\beta} + K^{11}276\ 10^{\psi\beta}\\ & + K^{10}20247\ 9^{\psi\beta} + K^{9}10626\ 8^{\psi\beta}\ + K^{8}42504\ 7^{\psi\beta}\\ & + K^{7}134596\ 6^{\psi\beta} + K^{6}346104\ 5^{\psi\beta}\\ & + K^{5}735471\ 4^{\psi\beta} + K^{4}1307504\ 3^{\psi\beta}\\ & + K^{3}1961256\ 2^{\psi\beta} + K^{2}\ 2496144\Big),\end{aligned} \tag{4.42}$$

for all $x \in \Gamma_1$.

4.3 Ulam - Radus Method

The proof of the following theorem and corollary is similar to that of results of Section 3.2.

Theorem 4.5. *Let $f : \Gamma_1 \to \Gamma_2$ be a function fulfilling the functional inequality*

$$\|Df_{24}(x,y)\| \le \mathscr{V}(x,y) \tag{4.43}$$

where $\mathscr{V} : \Gamma_1^2 \to [0,\infty)$ is a function satisfying the condition

$$\lim_{s\to\infty}\frac{1}{d_i^{24s}}\mathscr{V}(d_i^sx, d_i^sy) = 0 \tag{4.44}$$

with where

$$d_i = \begin{cases}2 & if \quad i = 0,\\ \frac{1}{2} & if \quad i = 1\end{cases} \tag{4.45}$$

for all $x,y \in \Gamma_1$. Assume that there exists $L = L(i)$ such that the function

$$\mathscr{V}_{24}^{F}(x,x) = \mathscr{V}_{24}^{D}\left(\frac{x}{2},\frac{x}{2}\right) \tag{4.46}$$

where $\mathscr{V}_{24}^{D}(x,x)$ is defined in (4.5) with the property

$$\frac{1}{d_i^{24}}\mathscr{V}_{24}^{F}(d_ix,d_ix) = L\ \ \mathscr{V}_{24}^{F}(x,x) \tag{4.47}$$

for all $x \in \Gamma_1$. Then there exists a unique quattuorvigintic function $\mathscr{Q}_{24} : \Gamma_1 \to \Gamma_2$ by

$$\mathscr{Q}_{24}(x) = \lim_{s\to\infty}\frac{f(d_i^sx)}{d_i^{24s}} \tag{4.48}$$

which satisfies (1.1) and

$$\|f(x)-\mathscr{Q}_{24}(x)\| \le \left(\frac{L^{1-i}}{1-L}\right)\mathscr{V}_{24}^{F}(x,x) \tag{4.49}$$

for all $x\in\Gamma_1$.

Corollary 4.6. *Let* $f:\Gamma_1\to\Gamma_2$ *be a mapping. If there exist real numbers* ϕ *and* ψ *satisfying (4.40) for all* $x,y\in\Gamma_1$, *then there exists a unique quattuorvigintic function* $\mathscr{Q}_{24}:\Gamma_1\to\Gamma_2$ *such that (4.41) holds*

$$\|f(x)-\mathscr{Q}_{24}(x)\| \le \begin{cases} \dfrac{\phi_{\beta C}}{|2^{24}-1|}, \\ \dfrac{\phi_{\beta S}||z||^{\psi}}{|2^{24}-2^{\psi}|}, \\ \dfrac{\phi_{\beta P}||z||^{2\psi}}{|2^{24}-2^{2\psi}|}, \end{cases} \tag{4.50}$$

where $\phi_{\beta C},\phi_{\beta S},\phi_{\beta P}$ *are defined in (4.42) for all* $x\in\Gamma_1$.

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