Mixed Q_5S_6 , S_7O_8 , N_9D_{10} functional equations

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Abstract

In this paper, the authors introduce and establish the stability of single variable mixed quintic-sextic, septic-octic and nonic-decic functional equation of the form

$$\begin{aligned}\vartheta(2p) &= 48\vartheta(p) + 16\vartheta(-p); \\ \eta(2q) &= 192\eta(q) + 64\eta(-q); \\ \psi(2r) &= 768\psi(r) + 256\psi(-r);\end{aligned}$$

in Banach space with the help of Ulam - Hyers and Radus methods.

Keywords

Quintic, sextic, septic, octic, nonic, decic functional equations, generalized Hyers - Ulam - Rassias stability.

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1. Introduction

The source for the development of stability of functional equations is the question solicited by Ulam [28]. Hyers [16] presented an excellent answer to the question of Ulam. Later, the result of Hyers was generalized and refined further by many great mathematicians like Aoki [2], Th.M. Rassias [14], J.M. Rassias [27] and Gavruta [13] in various directions. The progress of the theory of stability of various types of functional equations such as quadratic, cubic, quartic, quintic, sextic, septic, octic, nonic, decic, undecic, duodecic, tredecic, quattuordecic have been dealt by many mathematicians and

there are lot of interesting and significant results available in the literature.

The stability problems of the following functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [3, 21, 29, 30]).

Recently, M.Arunkumar et.al., test the generalized Ulam - Hyers stability of the additive-quadratic and cubic-quartic functional equations

$$f(2x) = 3f(x) + f(-x) \quad (1.1)$$

$$g(2x) = 12g(x) + 4g(-x) \quad (1.2)$$

via Quasi-Beta Banach space and Intuitionistic fuzzy Banach space using direct and fixed point methods.

In this paper, the authors introduce and establish the stability of single variable mixed quintic-sextic, septic-octic and nonic-decic functional equation of the form

$$\vartheta(2p) = 48\vartheta(p) + 16\vartheta(-p) \quad (1.3)$$

$$\eta(2q) = 192\eta(q) + 64\eta(-q) \quad (1.4)$$

$$\psi(2r) = 768\psi(r) + 256\psi(-r) \quad (1.5)$$

in Banach space with the help of Ulam - Hyers and Radus methods.

Hereafter through out this paper, let us consider X and Y to be a normed linear space and a Banach space, respectively.

2. Stability Results: Direct Method : Odd Case

In this section, the generalized Ulam-Hyers stability of the functional equation (1.3), (1.4) and (1.5) odd case is investigated.

Theorem 2.1. Let $j \in \{-1, 1\}$ and $\xi, \zeta, \varsigma : X \rightarrow [0, \infty)$ be a function such that

$$\lim_{n \rightarrow \infty} \frac{\xi(2^{nj}p)}{2^{5nj}} = 0 \quad (2.1)$$

$$\lim_{n \rightarrow \infty} \frac{\zeta(2^{nj}q)}{2^{7nj}} = 0 \quad (2.2)$$

$$\lim_{n \rightarrow \infty} \frac{\varsigma(2^{nj}r)}{2^{9nj}} = 0 \quad (2.3)$$

for all $p, q, r \in X$. Let $\vartheta, \eta, \psi : X \rightarrow Y$ be an odd function satisfying the inequality

$$\|\vartheta(2p) - 48\vartheta(p) - 16\vartheta(-p)\| \leq \xi(p) \quad (2.4)$$

$$\|\eta(2q) - 192\eta(q) - 64\eta(-q)\| \leq \zeta(q) \quad (2.5)$$

$$\|\psi(2r) - 768\psi(r) - 256\psi(-r)\| \leq \varsigma(r) \quad (2.6)$$

for all $p, q, r \in X$. Then there exists a unique quintic, septic, nonic mapping $Q_5, S_7, N_9 : X \rightarrow Y$ and satisfying the functional equation (1.3), (1.4) and (1.5) such that

$$\|\vartheta(p) - Q_5(p)\| \leq \frac{1}{32} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\xi(2^{kj})}{2^{5kj}} \quad (2.7)$$

$$\|\eta(q) - S_7(q)\| \leq \frac{1}{128} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\zeta(2^{kj})}{2^{7kj}} \quad (2.8)$$

$$\|\psi(r) - N_9(r)\| \leq \frac{1}{512} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\varsigma(2^{kj})}{2^{9kj}} \quad (2.9)$$

for all $p, q, r \in X$. The mapping $Q_5(p)$, $S_7(q)$ and $N_9(r)$ is defined by

$$Q_5(p) = \lim_{n \rightarrow \infty} \frac{\vartheta(2^{nj}p)}{2^{5nj}} \quad (2.10)$$

$$S_7(q) = \lim_{n \rightarrow \infty} \frac{\zeta(2^{nj}q)}{2^{7nj}} \quad (2.11)$$

$$N_9(r) = \lim_{n \rightarrow \infty} \frac{\varsigma(2^{nj}r)}{2^{9nj}} \quad (2.12)$$

for all $p, q, r \in X$.

Proof. Assume $j = 1$. Using Oddness of ϑ, η and ψ in (2.4), (2.5) and (2.6) we get

$$\|\vartheta(2p) - 32\vartheta(p)\| \leq \xi(p) \quad (2.13)$$

$$\|\eta(2q) - 128\eta(q)\| \leq \zeta(q) \quad (2.14)$$

$$\|\psi(2r) - 512\psi(r)\| \leq \varsigma(r) \quad (2.15)$$

for all $p, q, r \in X$. It follows from above inequalities

$$\left\| \vartheta(p) - \frac{\vartheta(2p)}{2^5} \right\| \leq \frac{\xi(p)}{32} \quad (2.16)$$

$$\left\| \eta(q) - \frac{\eta(2q)}{2^7} \right\| \leq \frac{\zeta(q)}{128} \quad (2.17)$$

$$\left\| \psi(r) - \frac{\psi(2r)}{2^9} \right\| \leq \frac{\varsigma(r)}{512} \quad (2.18)$$

for all $p, q, r \in X$. Now replacing p by $2p$ and dividing by 32 in (2.16), q by $2q$ and dividing by 128 in (2.17), r by $2r$ and dividing by 512 in (2.18) we get

$$\left\| \frac{\vartheta(2p)}{2^5} - \frac{\vartheta(2^2p)}{2^{10}} \right\| \leq \frac{\xi(2p)}{2^{10}} \quad (2.19)$$

$$\left\| \frac{\eta(2q)}{2^7} - \frac{\eta(2^2q)}{2^{14}} \right\| \leq \frac{\zeta(2q)}{2^{14}} \quad (2.20)$$

$$\left\| \frac{\psi(2r)}{2^9} - \frac{\psi(2^2r)}{2^{18}} \right\| \leq \frac{\varsigma(2r)}{2^{18}} \quad (2.21)$$

for all $p, q, r \in X$. With the help of triangle inequality from (2.16) and (2.19); (2.17) and (2.20); (2.18) and (2.21); we achieve the subsequent inequalities

$$\begin{aligned} & \left\| \vartheta(p) - \frac{\vartheta(2^2 p)}{2^{10}} \right\| \\ & \leq \left\| \vartheta(p) - \frac{\vartheta(2p)}{2^5} \right\| + \left\| \frac{\vartheta(2p)}{2^5} - \frac{\vartheta(2^2 p)}{2^{10}} \right\| \\ & \leq \frac{1}{2^5} \left[\xi(p) + \frac{\xi(2p)}{2^5} \right] \end{aligned} \quad (2.22)$$

$$\begin{aligned} & \left\| \eta(q) - \frac{\eta(2^2 q)}{2^{14}} \right\| \\ & \leq \left\| \eta(q) - \frac{\eta(2q)}{2^7} \right\| + \left\| \frac{\eta(2q)}{2^7} - \frac{\eta(2^2 q)}{2^{14}} \right\| \\ & \leq \frac{1}{2^7} \left[\zeta(q) + \frac{\zeta(2q)}{2^{14}} \right] \end{aligned} \quad (2.23)$$

$$\begin{aligned} & \left\| \psi(r) - \frac{\psi(2^2 r)}{2^{18}} \right\| \\ & \leq \left\| \psi(r) - \frac{\psi(2r)}{2^9} \right\| + \left\| \frac{\psi(2r)}{2^9} - \frac{\psi(2^2 r)}{2^{18}} \right\| \\ & \leq \frac{1}{2^9} \left[\varsigma(r) + \frac{\varsigma(2r)}{2^{18}} \right] \end{aligned} \quad (2.24)$$

for all $p, q, r \in X$. In general for any positive integer n , we get

$$\left\| \vartheta(p) - \frac{\vartheta(2^n p)}{2^{5n}} \right\| \leq \frac{1}{32} \sum_{k=0}^{n-1} \frac{\xi(2^k p)}{2^{5p}} \quad (2.25)$$

$$\left\| \eta(q) - \frac{\eta(2^n q)}{2^{7n}} \right\| \leq \frac{1}{128} \sum_{k=0}^{n-1} \frac{\zeta(2^k q)}{2^{7q}} \quad (2.26)$$

$$\left\| \psi(r) - \frac{\psi(2^n r)}{2^{9n}} \right\| \leq \frac{1}{512} \sum_{k=0}^{n-1} \frac{\varsigma(2^k r)}{2^{9r}} \quad (2.27)$$

for all $p, q, r \in X$. In order to prove the convergence of the sequence

$$\left\{ \frac{\vartheta(2^n p)}{2^{5n}} \right\}, \left\{ \frac{\eta(2^n q)}{2^{7n}} \right\}, \left\{ \frac{\psi(2^n r)}{2^{9n}} \right\},$$

replace p by $2^m p$ and dividing by 2^{5m} in (2.25), q by $2^m q$ and dividing by 2^{7m} in (2.26), and r by $2^m r$ and dividing by 2^{9m} in (2.27), for any $m, n > 0$, we deduce

$$\begin{aligned} & \left\| \frac{\vartheta(2^m p)}{2^{5m}} - \frac{\vartheta(2^{n+m} p)}{2^{5(n+m)}} \right\| = \frac{1}{2^{5m}} \left\| \vartheta(2^m p) - \frac{\vartheta(2^n \cdot 2^m p)}{2^{5n}} \right\| \\ & \leq \frac{1}{32} \sum_{k=0}^{n-1} \frac{\xi(2^{k+m} p)}{2^{5(k+m)}} \\ & \leq \frac{1}{32} \sum_{k=0}^{\infty} \frac{\xi(2^{k+m} p)}{2^{5(k+m)}} \\ & \rightarrow 0 \text{ as } m \rightarrow \infty \end{aligned}$$

$$\begin{aligned} & \left\| \frac{\eta(2^m q)}{2^{7m}} - \frac{\eta(2^{n+m} q)}{2^{7(n+m)}} \right\| = \frac{1}{2^{7m}} \left\| \eta(2^m q) - \frac{\eta(2^n \cdot 2^m q)}{2^{7n}} \right\| \\ & \leq \frac{1}{128} \sum_{k=0}^{n-1} \frac{\zeta(2^{k+m} q)}{2^{7(k+m)}} \\ & \leq \frac{1}{128} \sum_{k=0}^{\infty} \frac{\zeta(2^{k+m} q)}{2^{7(k+m)}} \\ & \rightarrow 0 \text{ as } m \rightarrow \infty \end{aligned}$$

$$\begin{aligned} & \left\| \frac{\psi(2^m r)}{2^{9m}} - \frac{\psi(2^{n+m} r)}{2^{9(n+m)}} \right\| = \frac{1}{2^{9m}} \left\| \psi(2^m r) - \frac{\psi(2^n \cdot 2^m r)}{2^{9n}} \right\| \\ & \leq \frac{1}{512} \sum_{k=0}^{n-1} \frac{\varsigma(2^{k+m} r)}{2^{9(k+m)}} \\ & \leq \frac{1}{512} \sum_{k=0}^{\infty} \frac{\varsigma(2^{k+m} r)}{2^{9(k+m)}} \\ & \rightarrow 0 \text{ as } m \rightarrow \infty \end{aligned}$$

for all $p, q, r \in X$. Hence the sequence

$$\left\{ \frac{\vartheta(2^n p)}{2^{5n}} \right\}, \left\{ \frac{\eta(2^n q)}{2^{7n}} \right\}, \left\{ \frac{\psi(2^n r)}{2^{9n}} \right\}$$

is Cauchy sequence. Since Y is complete, there exists a mapping $Q_5, S_7, N_9 : X \rightarrow Y$ such that

$$Q_5(p) = \lim_{n \rightarrow \infty} \frac{\vartheta(2^n p)}{2^{5n}} \quad \forall p \in X.$$

$$S_7(q) = \lim_{n \rightarrow \infty} \frac{\eta(2^n q)}{2^{7n}} \quad \forall q \in X.$$

$$N_9(r) = \lim_{n \rightarrow \infty} \frac{\psi(2^n r)}{2^{9n}} \quad \forall r \in X.$$

Letting $n \rightarrow \infty$ in (2.25), (2.26) and (2.27), we see that (2.7), (2.8) and (2.9) holds for all $p, q, r \in X$. To prove that Q_5, S_7, N_9 satisfies (1.3), (1.4), and (1.5), replace p by $2^n p$ and dividing by 2^{5n} in (2.4), q by $2^n q$ and dividing by 2^{7n} in (2.5), and r by $2^n r$ and dividing by 2^{9n} in (2.6), we obtain

$$\frac{1}{2^{5n}} \left\| \vartheta(2.2^n p) - 48\vartheta(2^n p) - 16\vartheta(-2^n p) \right\| \leq \frac{1}{2^{5n}} \xi(2^n p)$$

$$\frac{1}{2^{7n}} \left\| \rho(2.2^n q) - 192\rho(2^n q) - 64\rho(-2^n q) \right\| \leq \frac{1}{2^{7n}} \zeta(2^n q)$$

$$\frac{1}{2^{9n}} \left\| \psi(2.2^n r) - 768\psi(2^n r) - 256\psi(-2^n r) \right\| \leq \frac{1}{2^{9n}} \varsigma(2^n r)$$

for all $p, q, r \in X$. Letting $n \rightarrow \infty$ in the above inequality and using the definition of $Q_5(p)$, $S_7(q)$ and $N_9(r)$, we see that

$$Q_5(2p) = 48Q_5(p) + 16Q_5(-p)$$

$$S_7(2q) = 192S_7(q) + 64S_7(-q)$$

$$N_9(2r) = 768N_9(r) + 256N_9(-r)$$

Hence $Q_5(p)$, $S_7(q)$ and $N_9(r)$ satisfies (1.3), (1.4), and (1.5) for all $p, q, r \in X$. To prove $Q_5(p)$, $S_7(q)$ and $N_9(r)$ is unique, we let $\check{Q}_5(p)$, $\check{S}_7(q)$ and $\check{N}_9(r)$ be another mapping satisfying (1.3), (1.4), and (1.5) and (2.7), (2.8) and (2.9) then

$$\begin{aligned} & \|Q_5(p) - \check{Q}_5(p)\| \\ &= \frac{1}{2^{5n}} \|Q_5(2^n p) - \check{Q}_5(2^n p)\| \\ &\leq \frac{1}{2^{5n}} \{ \|Q_5(2^n p) - \vartheta(2^n p)\| + \|\vartheta(2^n p) - \check{Q}_5(2^n p)\| \} \\ &\leq \sum_{k=0}^{\infty} \frac{2\xi(2^{k+n}p)}{2^{5(k+n)}} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

$$\begin{aligned} & \|S_7(q) - \check{S}_7(q)\| \\ &= \frac{1}{2^{7n}} \|S_7(2^n q) - \check{S}_7(2^n q)\| \\ &\leq \frac{1}{2^{7n}} \{ \|S_7(2^n q) - \eta(2^n q)\| + \|\eta(2^n q) - \check{S}_7(2^n q)\| \} \\ &\leq \sum_{k=0}^{\infty} \frac{2\xi(2^{k+n}q)}{2^{7(k+n)}} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

$$\begin{aligned} & \|N_9(r) - \check{N}_9(r)\| \\ &= \frac{1}{2^{9n}} \|N_9(2^n r) - \check{N}_9(2^n r)\| \\ &\leq \frac{1}{2^{9n}} \{ \|N_9(2^n r) - \psi(2^n r)\| + \|\psi(2^n r) - \check{N}_9(2^n r)\| \} \\ &\leq \sum_{k=0}^{\infty} \frac{2\varsigma(2^{k+n}r)}{2^{9(k+n)}} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

for all $p, q, r \in X$. Hence Q_5 , S_7 and N_9 is unique.

For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem. $\square$

The following Corollary is an immediate consequence of Theorem 2.1 concerning the Ulam-Hyers [16], Ulam-Hyers-Rassias [27] and Ulam-JRassias [26] stabilities of (1.3), (1.4), and (1.5).

Corollary 2.2. Let λ and s be nonnegative real numbers. Let an odd function $\vartheta, \eta, \psi : X \rightarrow Y$ satisfies the inequality

$$\|\vartheta(2p) - 48\vartheta(p) - 16\vartheta(-p)\| \leq \begin{cases} \lambda, \\ \lambda\|p\|^s, & s \neq 5; \end{cases} \quad (2.28)$$

$$\|\eta(2q) - 192\eta(q) - 64\eta(-q)\| \leq \begin{cases} \lambda, \\ \lambda\|q\|^s, & s \neq 7; \end{cases} \quad (2.29)$$

$$\|\psi(2r) - 768\psi(r) - 256\psi(-r)\| \leq \begin{cases} \lambda, \\ \lambda\|r\|^s, & s \neq 9; \end{cases} \quad (2.30)$$

for all $p, q, r \in X$. Then there exists a unique quintic, septic, nonic mapping $Q_5, S_7, N_9 : X \rightarrow Y$ such that

$$\|\vartheta(p) - Q_5(p)\| \leq \begin{cases} \frac{\lambda}{31}, \\ \frac{\lambda\|p\|^s}{|2^5 - 2^s|}, \end{cases} \quad (2.31)$$

$$\|\eta(p) - S_7(q)\| \leq \begin{cases} \frac{\lambda}{127}, \\ \frac{\lambda\|p\|^s}{|2^7 - 2^s|}, \end{cases} \quad (2.32)$$

$$\|\psi(r) - N_9(r)\| \leq \begin{cases} \frac{\lambda}{511}, \\ \frac{\lambda\|p\|^s}{|2^9 - 2^s|}, \end{cases} \quad (2.33)$$

for all $p, q, r \in X$.

3. Stability Results : Even Case

In this section, the generalized Ulam-Hyers stability of the functional equation (1.3), (1.4) and (1.5) even case is investigated.

Theorem 3.1. Let $j \in \{-1, 1\}$ and $\xi, \zeta, \varsigma : X \rightarrow [0, \infty)$ be a function such that

$$\lim_{n \rightarrow \infty} \frac{\xi(2^{nj}p)}{2^{6nj}} = 0 \quad (3.1)$$

$$\lim_{n \rightarrow \infty} \frac{\zeta(2^{nj}q)}{2^{8nj}} = 0 \quad (3.2)$$

$$\lim_{n \rightarrow \infty} \frac{\varsigma(2^{nj}r)}{2^{10nj}} = 0 \quad (3.3)$$

for all $p, q, r \in X$. Let $\vartheta, \eta, \psi : X \rightarrow Y$ be an even function satisfying the inequality

$$\|\vartheta(2p) - 48\vartheta(p) - 16\vartheta(-p)\| \leq \xi(p) \quad (3.4)$$

$$\|\eta(2q) - 192\eta(q) - 64\eta(-q)\| \leq \zeta(q) \quad (3.5)$$

$$\|\psi(2r) - 768\psi(r) - 256\psi(-r)\| \leq \varsigma(r) \quad (3.6)$$

for all $p, q, r \in X$. Then there exists a unique sextic, octic, decic mapping $S_6, O_8, D_{10} : X \rightarrow Y$ and satisfying the functional equation (1.3), (1.4) and (1.5) such that

$$\|\vartheta(p) - S_6(p)\| \leq \frac{1}{64} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\xi(2^{kj})}{2^{6kj}} \quad (3.7)$$

$$\|\eta(q) - O_8(q)\| \leq \frac{1}{256} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\xi(2^{kj})}{2^{8kj}} \quad (3.8)$$

$$\|\psi(r) - D_{10}(r)\| \leq \frac{1}{1024} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\xi(2^{kj})}{2^{10kj}} \quad (3.9)$$

for all $p, q, r \in X$. The mapping $S_6(p)$, $O_8(q)$ and $D_{10}(r)$ is defined by

$$S_6(p) = \lim_{n \rightarrow \infty} \frac{\vartheta(2^{nj}x)}{2^{6nj}} \quad (3.10)$$

$$O_8(q) = \lim_{n \rightarrow \infty} \frac{\zeta(2^{nj}x)}{2^{8nj}} \quad (3.11)$$

$$D_{10}(r) = \lim_{n \rightarrow \infty} \frac{\varsigma(2^{nj}x)}{2^{10nj}} \quad (3.12)$$

for all $p, q, r \in X$.

Proof. Assume $j = 1$. Using evenness of ϑ , η and ψ in (3.4), (3.5) and (3.6) we get

$$\|\vartheta(2p) - 64\vartheta(p)\| \leq \xi(p) \quad (3.13)$$

$$\|\eta(2q) - 256\eta(q)\| \leq \zeta(q) \quad (3.14)$$

$$\|\psi(2r) - 1024\psi(r)\| \leq \varsigma(r) \quad (3.15)$$

□

for all $p, q, r \in X$. The Rest of the Proof is similar to that of Theorem 2.1.

Corollary 3.2. Let λ and s be nonnegative real numbers. Let an even function $\vartheta, \eta, \psi : X \rightarrow Y$ satisfies the inequality

$$\|\vartheta(2p) - 48\vartheta(p) - 16\vartheta(-p)\| \leq \begin{cases} \lambda, \\ \lambda\|p\|^s, \end{cases} \quad s \neq 6; \quad (3.16)$$

$$\|\eta(2q) - 192\eta(q) - 64\eta(-q)\| \leq \begin{cases} \lambda, \\ \lambda\|q\|^s, \end{cases} \quad s \neq 8; \quad (3.17)$$

$$\|\psi(2r) - 768\psi(r) - 256\psi(-r)\| \leq \begin{cases} \lambda, \\ \lambda\|r\|^s, \end{cases} \quad s \neq 10; \quad (3.18)$$

for all $p, q, r \in X$. Then there exists a unique sextic, octic, decic mapping $S_6, O_8, D_{10} : X \rightarrow Y$ such that

$$\|\vartheta(p) - S_6(p)\| \leq \begin{cases} \frac{\lambda}{63}, \\ \frac{\lambda\|p\|^s}{|2^6 - 2^s|}, \end{cases} \quad (3.19)$$

$$\|\eta(p) - O_8(q)\| \leq \begin{cases} \frac{\lambda}{253}, \\ \frac{\lambda\|p\|^s}{|2^8 - 2^s|}, \end{cases} \quad (3.20)$$

$$\|\psi(r) - D_{10}(r)\| \leq \begin{cases} \frac{\lambda}{1023}, \\ \frac{\lambda\|p\|^s}{|2^{10} - 2^s|}, \end{cases} \quad (3.21)$$

for all $p, q, r \in X$.

4. Stability Results : Mixed Case

In this section, the generalized Ulam-Hyers stability of the functional equation (1.3), (1.4) and (1.5) mixed case is proved.

Theorem 4.1. Let $j \in \{-1, 1\}$ and $\xi, \zeta, \varsigma : X \rightarrow [0, \infty)$ be a function satisfying (2.1), (2.2) and (2.3) also (3.1), (3.2) and (3.3) for all $p, q, r \in X$. Let $\vartheta, \eta, \psi : X \rightarrow Y$ be a function satisfying the inequality

$$\|\vartheta(2p) - 48\vartheta(p) - 16\vartheta(-p)\| \leq \xi(p) \quad (4.1)$$

$$\|\eta(2q) - 192\eta(q) - 64\eta(-q)\| \leq \zeta(q) \quad (4.2)$$

$$\|\psi(2r) - 768\psi(r) - 256\psi(-r)\| \leq \varsigma(r) \quad (4.3)$$

for all $p, q, r \in X$. Then there exists a unique quintic, septic, nonic mapping $Q_5, S_7, N_9 : X \rightarrow Y$ and a unique sextic, octic, decic mapping $S_6, O_8, D_{10} : X \rightarrow Y$ such that

$$\begin{aligned} & \|\vartheta(p) - Q_5(p) - S_6(p)\| \\ & \leq \frac{1}{2} \left[\frac{1}{32} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\xi(2^{kj}p)}{2^{5kj}} + \frac{\xi(-2^{kj}p)}{2^{5kj}} \right) \right. \\ & \quad \left. + \frac{1}{64} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\xi(2^{kj}p)}{2^{6kj}} + \frac{\xi(-2^{kj}p)}{2^{6kj}} \right) \right] \quad (4.4) \end{aligned}$$

$$\begin{aligned} & \|\eta(q) - S_7(q) - O_8(q)\| \\ & \leq \frac{1}{2} \left[\frac{1}{128} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\xi(2^{kj}q)}{2^{7kj}} + \frac{\xi(-2^{kj}q)}{2^{7kj}} \right) \right. \\ & \quad \left. + \frac{1}{256} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\xi(2^{kj}q)}{2^{8kj}} + \frac{\xi(-2^{kj}q)}{2^{8kj}} \right) \right] \quad (4.5) \end{aligned}$$

$$\begin{aligned} & \|\psi(r) - N_9(r) - D_{10}(r)\| \\ & \leq \frac{1}{2} \left[\frac{1}{512} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\xi(2^{kj}r)}{2^{9kj}} + \frac{\xi(-2^{kj}r)}{2^{9kj}} \right) \right. \\ & \quad \left. + \frac{1}{1024} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\xi(2^{kj}r)}{2^{10kj}} + \frac{\xi(-2^{kj}r)}{2^{10kj}} \right) \right] \quad (4.6) \end{aligned}$$

for all $p, q, r \in X$. The mapping $Q_5(p)$, $S_7(q)$ and $N_9(r)$ also $S_6(p)$, $O_8(q)$ and $D_{10}(r)$ are defined in (2.10), (2.11) and (2.12); (3.10), (3.11) and (3.11) respectively for all $p, q, r \in X$.

Proof. Let

$$\vartheta_o(p) = \frac{\vartheta(p) - \vartheta(-p)}{2}$$

$$\eta_o(q) = \frac{\eta(q) - \eta(-q)}{2}$$

$$\psi_o(r) = \frac{\psi(r) - \psi(-r)}{2}$$

for all $p, q, r \in X$. Then $\vartheta_o(0) = 0$, $\eta_o(0) = 0$ and $\psi_o(0) = 0$ and $\vartheta_o(-p) = -\vartheta_o(p)$, $\eta_o(-q) = -\eta_o(q)$, $\psi_o(-r) = -\psi_o(r)$ for all $p, q, r \in X$. Hence

$$\|\vartheta_o(2p) - 48\vartheta_o(p) - 16\vartheta_o(-p)\| \leq \frac{\xi(p)}{2} + \frac{\xi(-p)}{2} \quad (4.7)$$

$$\|\eta_o(2q) - 192\eta_o(q) - 64\eta_o(-q)\| \leq \frac{\xi(q)}{2} + \frac{\xi(-q)}{2} \quad (4.8)$$

$$\|\psi_o(2r) - 768\psi_o(r) - 256\psi_o(-r)\| \leq \frac{\xi(r)}{2} + \frac{\xi(-r)}{2} \quad (4.9)$$

for all $p, q, r \in X$. By Theorem 2.1, we have

$$\|\vartheta(p) - Q_5(p)\| \leq \frac{1}{32} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\xi(2^{kj})}{2^{5kj}} + \frac{\xi(-2^{kj})}{2^{5kj}} \right) \quad (4.10)$$

$$\|\eta(q) - S_7(q)\| \leq \frac{1}{128} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\xi(2^{kj})}{2^{7kj}} + \frac{\xi(-2^{kj})}{2^{7kj}} \right) \quad (4.11)$$

$$\|\psi(r) - N_9(r)\| \leq \frac{1}{512} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\xi(2^{kj})}{2^{9kj}} + \frac{\xi(-2^{kj})}{2^{9kj}} \right) \quad (4.12)$$

for all $p, q, r \in X$. Also, Let

$$\vartheta_e(p) = \frac{\vartheta(p) + \vartheta(-p)}{2}$$

$$\eta_e(q) = \frac{\eta(q) + \eta(-q)}{2}$$

$$\psi_e(r) = \frac{\psi(r) + \psi(-r)}{2}$$

for all $p, q, r \in X$. Then $\vartheta_e(0) = 0$, $\eta_e(0) = 0$ and $\psi_e(0) = 0$ and $\vartheta_e(-p) = \vartheta_e(p)$, $\eta_e(-q) = \eta_e(q)$, $\psi_e(-r) = \psi_e(r)$ for all $p, q, r \in X$. Hence

$$\|\vartheta_e(2p) - 48\vartheta_e(p) - 16\vartheta_e(-p)\| \leq \frac{\xi(p)}{2} + \frac{\xi(-p)}{2} \quad (4.13)$$

$$\|\eta_e(2q) - 192\eta_e(q) - 64\eta_e(-q)\| \leq \frac{\xi(q)}{2} + \frac{\xi(-q)}{2} \quad (4.14)$$

$$\|\psi_e(2r) - 768\psi_e(r) - 256\psi_e(-r)\| \leq \frac{\xi(r)}{2} + \frac{\xi(-r)}{2} \quad (4.15)$$

for all $p, q, r \in X$. By Theorem 3.1, we have

$$\|\vartheta(p) - S_6(p)\| \leq \frac{1}{64} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\xi(2^{kj})}{2^{6kj}} + \frac{\xi(-2^{kj})}{2^{6kj}} \right) \quad (4.16)$$

$$\|\eta(q) - O_8(q)\| \leq \frac{1}{256} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\xi(2^{kj})}{2^{8kj}} + \frac{\xi(-2^{kj})}{2^{8kj}} \right) \quad (4.17)$$

$$\|\psi(r) - D_{10}(r)\| \leq \frac{1}{1024} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\xi(2^{kj})}{2^{10kj}} + \frac{\xi(-2^{kj})}{2^{10kj}} \right) \quad (4.18)$$

for all $p, q, r \in X$. Define

$$\vartheta(p) = \vartheta_e(p) + \vartheta_o(p) \quad (4.19)$$

$$\eta(q) = \eta_e(q) + \eta_o(q) \quad (4.20)$$

$$\psi(r) = \psi_e(r) + \psi_o(r) \quad (4.21)$$

for all $p, q, r \in X$. From (4.10), (4.11) and (4.12); (4.16), (4.17) and (4.18); (4.19) (4.20) and (4.21) we arrive

$$\begin{aligned} & \|\vartheta(p) - Q_5(p) - S_6(p)\| \\ &= \|\vartheta_e(p) + \vartheta_o(p) - Q_5(p) - S_6(p)\| \\ &\leq \|\vartheta_o(p) - Q_5(p)\| + \|\vartheta_e(p) - S_6(p)\| \\ &\leq \frac{1}{64} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\xi(2^{kj}p)}{2^{5kj}} + \frac{\xi(-2^{kj}p)}{2^{5kj}} \right) \\ &\quad + \frac{1}{128} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\xi(2^{kj}p)}{2^{6kj}} + \frac{\xi(-2^{kj}p)}{2^{6kj}} \right) \end{aligned}$$

$$\begin{aligned} & \|\eta(q) - S_7(q) - O_8(q)\| \\ &= \|\eta_e(q) + \eta_o(q) - S_7(q) - O_8(q)\| \\ &\leq \|\eta_o(q) - S_7(q)\| + \|\eta_e(q) - O_8(q)\| \\ &\leq \frac{1}{256} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\zeta(2^{kj}q)}{2^{7kj}} + \frac{\zeta(-2^{kj}q)}{2^{7kj}} \right) \\ &\quad + \frac{1}{1024} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\zeta(2^{kj}q)}{2^{8kj}} + \frac{\zeta(-2^{kj}q)}{2^{8kj}} \right) \end{aligned}$$

$$\begin{aligned} & \|\psi(r) - N_9(r) - D_{10}(r)\| \\ &= \|\psi_e(r) + \psi_o(r) - N_9(r) - D_{10}(r)\| \\ &\leq \|\psi_o(r) - N_9(r)\| + \|\psi_e(r) - D_{10}(r)\| \\ &\leq \frac{1}{1024} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\varsigma(2^{kj}r)}{2^{9kj}} + \frac{\varsigma(-2^{kj}r)}{2^{9kj}} \right) \\ &\quad + \frac{1}{2048} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\varsigma(2^{kj}r)}{2^{10kj}} + \frac{\varsigma(-2^{kj}r)}{2^{10kj}} \right) \end{aligned}$$

for all $p, q, r \in X$. Hence the theorem is proved. $\square$

Using Corollaries 2.2 and 3.2, we have the following Corollary concerning the stability of (1.3), (1.4), and (1.5).

Corollary 4.2. Let λ and s be nonnegative real numbers. Let an even function $\vartheta, \eta, \psi : X \rightarrow Y$ satisfies the inequality

$$\|\vartheta(2p) - 48\vartheta(p) - 16\vartheta(-p)\| \leq \begin{cases} \lambda, \\ \lambda\|p\|^s, \end{cases} \quad s \neq 5, 6; \quad (4.22)$$

$$\|\eta(2q) - 192\eta(q) - 64\eta(-q)\| \leq \begin{cases} \lambda, \\ \lambda\|q\|^s, \end{cases} \quad s \neq 7, 8; \quad (4.23)$$

$$\|\psi(2r) - 768\psi(r) - 256\psi(-r)\| \leq \begin{cases} \lambda, \\ \lambda\|r\|^s, \end{cases} \quad s \neq 9, 10; \quad (4.24)$$

for all $p, q, r \in X$. Then there exists a unique quintic, septic, nonic mapping $Q_5, S_7, N_9 : X \rightarrow Y$ and a unique sextic, octic, decic mapping $S_6, O_8, D_{10} : X \rightarrow Y$ such that

$$\|\vartheta(p) - Q_5(p) - S_6(p)\| \leq \begin{cases} \left(\frac{\lambda}{31} + \frac{\lambda}{63} \right), \\ \left(\frac{\lambda\|p\|^s}{|2^5 - 2^s|} + \frac{\lambda\|p\|^s}{|2^6 - 2^s|} \right), \end{cases} \quad (4.25)$$

$$\|\eta(q) - S_7(q) - O_8(q)\| \leq \begin{cases} \left(\frac{\lambda}{127} + \frac{\lambda}{253} \right), \\ \left(\frac{\lambda\|q\|^s}{|2^7 - 2^s|} + \frac{\lambda\|q\|^s}{|2^8 - 2^s|} \right), \end{cases} \quad (4.26)$$

$$\|\psi(r) - N_9(r) - D_{10}(r)\| \leq \begin{cases} \left(\frac{\lambda}{511} + \frac{\lambda}{1023} \right), \\ \left(\frac{\lambda\|r\|^s}{|2^9 - 2^s|} + \frac{\lambda\|r\|^s}{|2^{10} - 2^s|} \right), \end{cases} \quad (4.27)$$

for all $p, q, r \in X$.

Theorem 4.3. [20] (The alternative of fixed point) Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \rightarrow X$ with Lipschitz constant L . Then, for each given element $x \in X$, either

(B₁) $d(T^n x, T^{n+1} x) = \infty \quad \forall n \geq 0$,
or

(B₂) there exists a natural number n_0 such that:

(i) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;

(ii) The sequence $(T^n x)$ is convergent to a fixed point y^* of T

(iii) y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0} x, y) < \infty\}$;

(iv) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in Y$.

5. Stability Results : Fixed Point Method: Odd Case

In this section, the generalized Ulam-Hyers stability of the functional equation (1.3), (1.4) and (1.5) odd case is proved by using fixed point method.

Theorem 5.1. Let $\vartheta, \eta, \psi : X \rightarrow Y$ be a mapping for which there exist a function $\xi, \zeta, \varsigma : X \rightarrow [0, \infty)$ with the condition

$$\lim_{k \rightarrow \infty} \frac{\xi(\mu_i^k p)}{\mu_i^{5k}} = 0 \quad (5.1)$$

$$\lim_{k \rightarrow \infty} \frac{\zeta(\mu_i^k q)}{\mu_i^{7k}} = 0 \quad (5.2)$$

$$\lim_{k \rightarrow \infty} \frac{\varsigma(\mu_i^k r)}{\mu_i^{9k}} = 0 \quad (5.3)$$

where $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$ such that the functional inequality with

$$\|\vartheta(2p) - 48\vartheta(p) - 16\vartheta(-p)\| \leq \xi(p) \quad (5.4)$$

$$\|\eta(2q) - 192\eta(q) - 64\eta(-q)\| \leq \zeta(q) \quad (5.5)$$

$$\|\psi(2r) - 768\psi(r) - 256\psi(-r)\| \leq \varsigma(r) \quad (5.6)$$

for all $p, q, r \in X$. If there exists $L = L(i) < 1$ such that the function

$$p \rightarrow \beta(p) = \xi\left(\frac{p}{2}\right),$$

$$q \rightarrow \beta(q) = \zeta\left(\frac{q}{2}\right),$$

$$r \rightarrow \beta(r) = \varsigma\left(\frac{r}{2}\right),$$

has the property

$$L\beta(p) \leq \frac{1}{\mu_i^5} \beta(p\mu_i) \quad (5.7)$$

$$L\beta(q) \leq \frac{1}{\mu_i^7} \beta(q\mu_i) \quad (5.8)$$

$$L\beta(r) \leq \frac{1}{\mu_i^9} \beta(r\mu_i) \quad (5.9)$$

for all $p, q, r \in X$. Then there exists a unique quintic, septic, nonic mapping $Q_5, S_7, N_9 : X \rightarrow Y$ satisfying the functional equation (1.3), (1.4) and (1.5) such that

$$\|\vartheta(p) - Q_5(p)\| \leq \frac{L^{1-i}}{1-L} \beta(p) \quad (5.10)$$

$$\|\eta(q) - S_7(q)\| \leq \frac{L^{1-i}}{1-L} \beta(q) \quad (5.11)$$

$$\|\psi(r) - N_9(r)\| \leq \frac{L^{1-i}}{1-L} \beta(r) \quad (5.12)$$

holds for all $p, q, r \in X$.

Proof. Consider the set $\Psi = \{f/g : X \rightarrow Y, f(0) = 0\}$ and introduce the generalized metric on Ψ ,

$$d(f, g) = \inf\{K \in (0, \infty) : \|f(p) - g(p)\| \leq K\beta(p), p \in X\}.$$

$$d(f, g) = \inf\{K \in (0, \infty) : \|f(q) - g(q)\| \leq K\beta(q), q \in X\}.$$

$$d(f, g) = \inf\{K \in (0, \infty) : \|f(r) - g(r)\| \leq K\beta(r), r \in X\}.$$

It is easy to see that (Ψ, d) is complete. Define $T : \Psi \rightarrow \Psi$ by

$$Tf(p) = \frac{1}{\mu_i^5} f(\mu_i p)$$

$$Tf(q) = \frac{1}{\mu_i^7} f(\mu_i q)$$

$$Tf(r) = \frac{1}{\mu_i^9} f(\mu_i r)$$

for all $p, q, r \in \Psi$. Now $f, g \in \Psi$,

$$d(f, g) \leq K$$

$$\Rightarrow \|f(p) - g(p)\| \leq K\beta(p), p \in X.$$

$$\Rightarrow \left\| \frac{1}{\mu_i^5} f(\mu_i p) - \frac{1}{\mu_i^5} g(\mu_i p) \right\| \leq \frac{1}{\mu_i^5} K\beta(\mu_i p), p \in X,$$

$$\Rightarrow \left\| \frac{1}{\mu_i^5} f(\mu_i p) - \frac{1}{\mu_i^5} g(\mu_i p) \right\| \leq LK\beta(p), p \in X,$$

$$\Rightarrow \|Tf(p) - Tg(p)\| \leq LK\beta(p), p \in X,$$

$$\Rightarrow d(Tf, Tg) \leq LK.$$

$$d(f, g) \leq K$$

$$\Rightarrow \|f(q) - g(q)\| \leq K\beta(q), q \in X.$$

$$\Rightarrow \left\| \frac{1}{\mu_i^7} f(\mu_i q) - \frac{1}{\mu_i^7} g(\mu_i q) \right\| \leq \frac{1}{\mu_i^7} K\beta(\mu_i q), q \in X,$$

$$\Rightarrow \left\| \frac{1}{\mu_i^7} f(\mu_i q) - \frac{1}{\mu_i^7} g(\mu_i q) \right\| \leq LK\beta(q), q \in X,$$

$$\Rightarrow \|Tf(q) - Tg(q)\| \leq LK\beta(q), q \in X,$$

$$\Rightarrow d(Tf, Tg) \leq LK.$$

$$d(f, g) \leq K$$

$$\Rightarrow \|f(r) - g(r)\| \leq K\beta(r), r \in X.$$

$$\Rightarrow \left\| \frac{1}{\mu_i^9} f(\mu_i r) - \frac{1}{\mu_i^9} g(\mu_i r) \right\| \leq \frac{1}{\mu_i^9} K\beta(\mu_i r), r \in X,$$

$$\Rightarrow \left\| \frac{1}{\mu_i^9} f(\mu_i r) - \frac{1}{\mu_i^9} g(\mu_i r) \right\| \leq LK\beta(r), r \in X,$$

$$\Rightarrow \|Tf(r) - Tg(r)\| \leq LK\beta(r), r \in X,$$

$$\Rightarrow d(Tf, Tg) \leq LK.$$

This implies $d(Tf, Tg) \leq Ld(f, g)$, for all $f, g \in \Psi$. i.e., T is a strictly contractive mapping on Ψ with Lipschitz constant L . Using oddness of ϑ , η and ψ in (5.4), (5.5) and (5.6) we get

$$\left\| \vartheta(p) - \frac{\vartheta(2p)}{2^5} \right\| \leq \frac{\xi(p)}{32} \quad (5.13)$$

$$\left\| \eta(q) - \frac{\eta(2q)}{2^7} \right\| \leq \frac{\zeta(q)}{128} \quad (5.14)$$

$$\left\| \psi(r) - \frac{\psi(2r)}{2^9} \right\| \leq \frac{\varsigma(r)}{512} \quad (5.15)$$

for all $p, q, r \in X$. for all $p, q, r \in X$. Using (5.7), (5.8) and (5.9) for the case $i = 0$ it reduces to

$$\left\| \vartheta(p) - \frac{\vartheta(2p)}{2^5} \right\| \leq \frac{1}{2} \beta(p)$$

$$\left\| \eta(q) - \frac{\eta(2q)}{2^7} \right\| \leq \frac{1}{2} \beta(q)$$

$$\left\| \psi(r) - \frac{\psi(2r)}{2^9} \right\| \leq \frac{1}{2} \beta(r)$$

for all $p, q, r \in X$.

$$\text{i.e., } d(\vartheta, T\vartheta) \leq \frac{1}{2} \Rightarrow d(\vartheta, T\vartheta) \leq \frac{1}{2} = L = L^1 < \infty.$$

$$\text{i.e., } d(\eta, T\eta) \leq \frac{1}{2} \Rightarrow d(\eta, T\eta) \leq \frac{1}{2} = L = L^1 < \infty.$$

$$\text{i.e., } d(\psi, T\psi) \leq \frac{1}{2} \Rightarrow d(\psi, T\psi) \leq \frac{1}{2} = L = L^1 < \infty.$$

Again replacing $p = \frac{p}{2}$ in (5.13), $q = \frac{q}{2}$ in (5.14) and $r = \frac{r}{2}$ in (5.15) we get

$$\left\| \vartheta(p) - 2^5 \vartheta\left(\frac{p}{2}\right) \right\| \leq \xi\left(\frac{p}{2}\right).$$

$$\left\| \eta(q) - 2^7 \eta\left(\frac{q}{2}\right) \right\| \leq \zeta\left(\frac{q}{2}\right).$$

$$\left\| \psi(r) - 2^9 \psi\left(\frac{r}{2}\right) \right\| \leq \varsigma\left(\frac{r}{2}\right).$$

for all $p, q, r \in X$. Using (5.7), (5.8) and (5.9) for the case $i = 1$ it reduces to

$$\left\| \vartheta(p) - 2^5 \vartheta\left(\frac{p}{2}\right) \right\| \leq \beta(p)$$

$$\left\| \eta(q) - 2^7 \eta\left(\frac{q}{2}\right) \right\| \leq \beta(q)$$

$$\left\| \psi(r) - 2^9 \psi\left(\frac{r}{2}\right) \right\| \leq \beta(r)$$

for all $p, q, r \in X$.

$$\text{i.e., } d(\vartheta, T\vartheta) \leq 1 \Rightarrow d(\vartheta, T\vartheta) \leq 1 = L^0 < \infty.$$

$$\text{i.e., } d(\eta, T\eta) \leq 1 \Rightarrow d(\eta, T\eta) \leq 1 = L^0 < \infty.$$

$$\text{i.e., } d(\psi, T\psi) \leq 1 \Rightarrow d(\psi, T\psi) \leq 1 = L^0 < \infty.$$

In above cases, we arrive

$$d(\vartheta, T\vartheta) \leq L^{1-i}$$

$$d(\eta, T\eta) \leq L^{1-i}$$

$$d(\psi, T\psi) \leq L^{1-i}$$

Therefore $(B_2(i))$ holds.

By $(B_2(ii))$, it follows that there exists a fixed point Q_5 , S_7 and N_9 of T in X such that

$$Q_5(p) = \lim_{k \rightarrow \infty} \frac{\vartheta(\mu_i^k p)}{\mu_i^k} \quad \forall p \in X. \quad (5.16)$$

$$S_7(q) = \lim_{k \rightarrow \infty} \frac{\eta(q)(\mu_i^k q)}{\mu_i^k} \quad \forall q \in X. \quad (5.17)$$

$$N_9(r) = \lim_{k \rightarrow \infty} \frac{\psi(\mu_i^k r)}{\mu_i^k} \quad \forall r \in X. \quad (5.18)$$

In order to prove $Q_5, S_7, N_9 : V \rightarrow B$ is quintic, septic, nonic. Replacing p by $(\mu_i^k p)$, q by $(\mu_i^k q)$ and r by $(\mu_i^k r)$ in (5.4), (5.5) and (5.6) and dividing by μ_i^k , it follows from (5.1), (5.2) and (5.3); and (5.16), (5.17) and (5.18) Q_5 , S_7 and N_9 satisfies (1.3), (1.4) and (1.5) for all $p, q, r \in X$. i.e., Q_5 , S_7 and N_9 satisfies the functional equation (1.3), (1.4) and (1.5).

By $(B_2(iii))$, Q_5 , S_7 and N_9 is the unique fixed point of T in the set

$$Y = \{Q_5(p) \in X : d(T\vartheta(p), Q_5(p)) < \infty\},$$

$$Y = \{S_7(q) \in X : d(T\eta(q), S_7(q)) < \infty\},$$

$$Y = \{N_9(r) \in X : d(T\psi(r), N_9(r)) < \infty\},$$

using the fixed point alternative result Q_5 , S_7 and N_9 is the unique function such that

$$\|\vartheta(p) - Q_5(p)\| \leq K\beta(p)$$

$$\|\eta(q) - S_7(q)\| \leq K\beta(q)$$

$$\|\psi(r) - N_9(r)\| \leq K\beta(r)$$

for all $p, q, r \in X$ and $K > 0$. Finally by $(B_2(iv))$, we obtain

$$d(\vartheta, Q_5) \leq \frac{1}{1-L} d(\vartheta, T\vartheta)$$

$$d(\eta, S_7) \leq \frac{1}{1-L} d(\eta, T\eta)$$

$$d(\psi, N_9) \leq \frac{1}{1-L} d(\psi, T\psi)$$

this implies

$$d(\vartheta, Q_5) \leq \frac{L^{1-i}}{1-L}.$$

$$d(\eta, S_7) \leq \frac{L^{1-i}}{1-L}$$

$$d(\psi, N_9) \leq \frac{L^{1-i}}{1-L}$$

Hence we conclude that

$$\|\vartheta(p) - Q_5(p)\| \leq \frac{L^{1-i}}{1-L} \beta(p)$$

$$\|\eta(q) - S_7(q)\| \leq \frac{L^{1-i}}{1-L} \beta(q),$$

$$\|\psi(r) - N_9(r)\| \leq \frac{L^{1-i}}{1-L} \beta(r).$$

for all $p, q, r \in X$. This completes the proof of the theorem. $\square$

Corollary 5.2. Let λ and s be nonnegative real numbers. Let an odd function $\vartheta, \eta, \psi : X \rightarrow Y$ satisfies the inequality

$$\|\vartheta(2p) - 48\vartheta(p) - 16\vartheta(-p)\| \leq \begin{cases} \lambda, \\ \lambda \|p\|^s, & s \neq 5; \end{cases} \quad (5.19)$$

$$\|\eta(2q) - 192\eta(q) - 64\eta(-q)\| \leq \begin{cases} \lambda, \\ \lambda \|q\|^s, & s \neq 7; \end{cases} \quad (5.20)$$

$$\|\psi(2r) - 768\psi(r) - 256\psi(-r)\| \leq \begin{cases} \lambda, \\ \lambda \|r\|^s, & s \neq 9; \end{cases} \quad (5.21)$$

for all $p, q, r \in X$. Then there exists a unique quintic, septic, nonic mapping $Q_5, S_7, N_9 : X \rightarrow Y$ such that

$$\|\vartheta(p) - Q_5(p)\| \leq \begin{cases} \frac{\lambda}{31}, \\ \frac{\lambda \|p\|^s}{|2^5 - 2^s|}, \end{cases} \quad (5.22)$$

$$\|\eta(p) - S_7(p)\| \leq \begin{cases} \frac{\lambda}{127}, \\ \frac{\lambda \|p\|^s}{|2^7 - 2^s|}, \end{cases} \quad (5.23)$$

$$\|\psi(r) - N_9(r)\| \leq \begin{cases} \frac{\lambda}{511}, \\ \frac{\lambda \|r\|^s}{|2^9 - 2^s|}, \end{cases} \quad (5.24)$$

for all $p, q, r \in X$.

Proof. Setting

$$\xi(x) = \begin{cases} \lambda \\ \lambda \{|x|^s\}, \end{cases}$$

$$\zeta(x) = \begin{cases} \lambda \\ \lambda \{|x|^s\}, \end{cases}$$

$$\varsigma(x) = \begin{cases} \lambda \\ \lambda \{|x|^s\} \end{cases}$$

for all $p, q, r \in X$. Now

$$\begin{aligned} \frac{\xi(\mu_i^k p)}{\mu_i^{5k}} &= \begin{cases} \frac{\lambda}{\mu_i^{5k}} \\ \frac{\lambda}{\mu_i^{5k}} \{| \mu_i^k p |^s \} \end{cases} \\ &= \begin{cases} \rightarrow 0 \text{ as } k \rightarrow \infty, \\ \rightarrow 0 \text{ as } k \rightarrow \infty \end{cases} \end{aligned}$$

$$\begin{aligned} \frac{\zeta(\mu_i^k q)}{\mu_i^{7k}} &= \begin{cases} \frac{\lambda}{\mu_i^{7k}} \\ \frac{\lambda}{\mu_i^{7k}} \{| \mu_i^k q |^s \} \end{cases} \\ &= \begin{cases} \rightarrow 0 \text{ as } k \rightarrow \infty, \\ \rightarrow 0 \text{ as } k \rightarrow \infty \end{cases} \end{aligned}$$

$$\begin{aligned} \frac{\varsigma(\mu_i^k r)}{\mu_i^{9k}} &= \begin{cases} \frac{\lambda}{\mu_i^{9k}} \\ \frac{\lambda}{\mu_i^{9k}} \{| \mu_i^k r |^s \} \end{cases} \\ &= \begin{cases} \rightarrow 0 \text{ as } k \rightarrow \infty, \\ \rightarrow 0 \text{ as } k \rightarrow \infty \end{cases} \end{aligned}$$

i.e., (5.1), (5.2) and (5.3) is holds. But we have $\beta(p) = \xi\left(\frac{p}{2}\right)$, $\beta(q) = \zeta\left(\frac{q}{2}\right)$ and $\beta(r) = \varsigma\left(\frac{r}{2}\right)$. has the property

$$L\beta(p) \leq \frac{1}{\mu_i^5} \beta(p\mu_i)$$

$$L\beta(q) \leq \frac{1}{\mu_i^7} \beta(q\mu_i)$$

$$L\beta(r) \leq \frac{1}{\mu_i^9} \beta(r\mu_i)$$

for all $p, q, r \in V$. Hence

$$\beta(p) = \xi\left(\frac{p}{2}\right) = \begin{cases} \frac{\lambda}{2^s} \|p\|^s, \end{cases}$$

$$\beta(q) = \zeta\left(\frac{q}{2}\right) = \begin{cases} \frac{\lambda}{2^s} \|q\|^s, \end{cases}$$

$$\beta(r) = \varsigma\left(\frac{r}{2}\right) = \begin{cases} \frac{\lambda}{2^s} \|r\|^s, \end{cases}$$

Also,

$$\begin{aligned}\frac{1}{\mu_i^5}\beta(\mu_i p) &= \begin{cases} \frac{\lambda}{\mu_i^5} \\ \frac{\lambda}{\mu_i^5} \|\mu_i p\|^s \end{cases} \\ &= \begin{cases} \mu_i^{-5}\beta(p) \\ \mu_i^{s-5}\beta(p) \end{cases}\end{aligned}$$

$$\begin{aligned}\frac{1}{\mu_i^7}\beta(\mu_i p) &= \begin{cases} \frac{\lambda}{\mu_i^7} \\ \frac{\lambda}{\mu_i^7} \|\mu_i q\|^s \end{cases} \\ &= \begin{cases} \mu_i^{-7}\beta(q) \\ \mu_i^{s-7}\beta(q) \end{cases}\end{aligned}$$

$$\begin{aligned}\frac{1}{\mu_i^9}\beta(\mu_i p) &= \begin{cases} \frac{\lambda}{\mu_i^9} \\ \frac{\lambda}{\mu_i^9} \|\mu_i r\|^s \end{cases} \\ &= \begin{cases} \mu_i^{-9}\beta(r) \\ \mu_i^{s-9}\beta(r) \end{cases}\end{aligned}$$

Now from (5.10), (5.11) and (5.12), we prove the following cases

Case:1 $L = 2^{-5}$, if $i = 0$

$$\|\vartheta(p) - Q_5(p)\| \leq \frac{L^{1-i}}{1-L}\beta(p) = \frac{(2^{-5})^{1-0}}{1-(2)^{-5}}\lambda = \frac{\lambda}{31}.$$

Case:2 $L = \frac{1}{2^{-5}}$, if $i = 1$

$$\|\vartheta(p) - Q_5(p)\| \leq \frac{L^{1-i}}{1-L}\beta(p) = \frac{\left(\frac{1}{2^{-5}}\right)^{1-1}}{1-\frac{1}{2^{-5}}}\lambda = \frac{\lambda}{-31}.$$

Case:1 $L = 2^{-7}$, if $i = 0$

$$\|\eta(q) - S_7(q)\| \leq \frac{L^{1-i}}{1-L}\beta(q) = \frac{(2^{-7})^{1-0}}{1-(2)^{-7}}\lambda = \frac{\lambda}{127}.$$

Case:2 $L = \frac{1}{2^{-7}}$, if $i = 1$

$$\|\eta(q) - S_7(q)\| \leq \frac{L^{1-i}}{1-L}\beta(q) = \frac{\left(\frac{1}{2^{-7}}\right)^{1-1}}{1-\frac{1}{2^{-7}}}\lambda = \frac{\lambda}{-127}.$$

Case:1 $L = 2^{-9}$, if $i = 0$

$$\|\psi(p) - N_9(r)\| \leq \frac{L^{1-i}}{1-L}\beta(r) = \frac{(2^{-9})^{1-0}}{1-(2)^{-9}}\lambda = \frac{\lambda}{511}.$$

Case:2 $L = \frac{1}{2^{-9}}$, if $i = 1$

$$\|\psi(p) - N_9(r)\| \leq \frac{L^{1-i}}{1-L}\beta(r) = \frac{\left(\frac{1}{2^{-9}}\right)^{1-1}}{1-\frac{1}{2^{-9}}}\lambda = \frac{\lambda}{-511}.$$

Case:1 $L = 2^{s-5}$, if $i = 0$

$$\begin{aligned}\|\vartheta(p) - Q_5(p)\| &\leq \frac{L^{1-i}}{1-L}\beta(p) \\ &= \frac{(2^{s-5})^{1-0}}{1-(2)^{s-5}}\frac{\lambda}{2^s}\|p\|^s \\ &= \frac{\lambda\|p\|^s}{2^5 - 2^s}.\end{aligned}$$

Case:2 $L = \frac{1}{2^{s-5}}$, if $i = 1$

$$\begin{aligned}\|\vartheta(p) - Q_5(p)\| &\leq \frac{L^{1-i}}{1-L}\beta(p) \\ &= \frac{\left(\frac{1}{2^{s-5}}\right)^{1-1}}{1-\frac{1}{2^{s-5}}}\frac{\lambda}{2^s}\|p\|^s \\ &= \frac{\lambda\|p\|^s}{2^s - 2^5}.\end{aligned}$$

Case:1 $L = 2^{s-7}$, if $i = 0$

$$\begin{aligned}\|\eta(q) - S_7(q)\| &\leq \frac{L^{1-i}}{1-L}\beta(q) \\ &= \frac{(2^{s-7})^{1-0}}{1-(2)^{s-7}}\frac{\lambda}{2^s}\|q\|^s \\ &= \frac{\lambda\|q\|^s}{2^7 - 2^s}.\end{aligned}$$

Case:2 $L = \frac{1}{2^{s-7}}$, if $i = 1$

$$\begin{aligned}\|\eta(q) - S_7(q)\| &\leq \frac{L^{1-i}}{1-L}\beta(q) \\ &= \frac{\left(\frac{1}{2^{s-7}}\right)^{1-1}}{1-\frac{1}{2^{s-7}}}\frac{\lambda}{2^s}\|q\|^s \\ &= \frac{\lambda\|q\|^s}{2^s - 2^7}.\end{aligned}$$

Case:1 $L = 2^{s-9}$, if $i = 0$

$$\begin{aligned}\|\psi(p) - N_9(r)\| &\leq \frac{L^{1-i}}{1-L}\beta(r) \\ &= \frac{(2^{s-9})^{1-0}}{1-(2)^{s-9}}\frac{\lambda}{2^s}\|r\|^s \\ &= \frac{\lambda\|r\|^s}{2^9 - 2^s}.\end{aligned}$$

Case:2 $L = \frac{1}{2^{s-9}}$, if $i = 1$

$$\begin{aligned}\|\psi(p) - N_9(r)\| &\leq \frac{L^{1-i}}{1-L}\beta(r) \\ &= \frac{\left(\frac{1}{2^{s-9}}\right)^{1-1}}{1-\frac{1}{2^{s-9}}}\frac{\lambda}{2^s}\|r\|^s \\ &= \frac{\lambda\|r\|^s}{2^s - 2^9}.\end{aligned}$$

Hence the proof of the corollary. $\square$

6. Stability Results: Even Case

In this section, the generalized Ulam-Hyers stability of the functional equation (1.3), (1.4) and (1.5) even case is proved by using fixed point method. The proofs of the following Theorem and Corollary is similar tracing to that of Theorem 5.1 and Corollary 5.2. Hence we omit the proofs.

Theorem 6.1. Let $\vartheta, \eta, \psi : X \rightarrow Y$ be a mapping for which there exist a function $\xi, \zeta, \varsigma : X \rightarrow [0, \infty)$ with the condition

$$\lim_{k \rightarrow \infty} \frac{\xi(\mu_i^k p)}{\mu_i^{6k}} = 0 \quad (6.1)$$

$$\lim_{k \rightarrow \infty} \frac{\zeta(\mu_i^k q)}{\mu_i^{8k}} = 0 \quad (6.2)$$

$$\lim_{k \rightarrow \infty} \frac{\varsigma(\mu_i^k r)}{\mu_i^{10k}} = 0 \quad (6.3)$$

where $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$ such that the functional inequality with

$$\|\vartheta(2p) - 48\vartheta(p) - 16\vartheta(-p)\| \leq \xi(p) \quad (6.4)$$

$$\|\eta(2q) - 192\eta(q) - 64\eta(-q)\| \leq \zeta(q) \quad (6.5)$$

$$\|\psi(2r) - 768\psi(r) - 256\psi(-r)\| \leq \varsigma(r) \quad (6.6)$$

for all $p, q, r \in X$. If there exists $L = L(i) < 1$ such that the function

$$\begin{aligned} p &\rightarrow \beta(p) = \xi\left(\frac{p}{2}\right), \\ q &\rightarrow \beta(q) = \zeta\left(\frac{q}{2}\right), \\ r &\rightarrow \beta(r) = \varsigma\left(\frac{r}{2}\right), \end{aligned}$$

has the property

$$L\beta(p) \leq \frac{1}{\mu_i^6} \beta(p\mu_i) \quad (6.7)$$

$$L\beta(q) \leq \frac{1}{\mu_i^8} \beta(q\mu_i) \quad (6.8)$$

$$L\beta(r) \leq \frac{1}{\mu_i^{10}} \beta(r\mu_i) \quad (6.9)$$

for all $p, q, r \in X$. Then there exists a unique sextic, octic, decic mapping $S_6, O_8, D_{10} : X \rightarrow Y$ satisfying the functional equation (1.3), (1.4) and (1.5) such that

$$\|\vartheta(p) - S_6(p)\| \leq \frac{L^{1-i}}{1-L} \beta(p) \quad (6.10)$$

$$\|\eta(q) - O_8(q)\| \leq \frac{L^{1-i}}{1-L} \beta(q) \quad (6.11)$$

$$\|\psi(r) - D_{10}(r)\| \leq \frac{L^{1-i}}{1-L} \beta(r) \quad (6.12)$$

holds for all $p, q, r \in X$.

Corollary 6.2. Let λ and s be nonnegative real numbers. Let an even function $\vartheta, \eta, \psi : X \rightarrow Y$ satisfies the inequality

$$\|\vartheta(2p) - 48\vartheta(p) - 16\vartheta(-p)\| \leq \begin{cases} \lambda, \\ \lambda \|p\|^s, \end{cases} \quad s \neq 6; \quad (6.13)$$

$$\|\eta(2q) - 192\eta(q) - 64\eta(-q)\| \leq \begin{cases} \lambda, \\ \lambda \|q\|^s, \end{cases} \quad s \neq 8; \quad (6.14)$$

$$\|\psi(2r) - 768\psi(r) - 256\psi(-r)\| \leq \begin{cases} \lambda, \\ \lambda \|r\|^s, \end{cases} \quad s \neq 10; \quad (6.15)$$

for all $p, q, r \in X$. Then there exists a unique quintic, septic, nonic mapping $S_7, O_8, D_{10} : X \rightarrow Y$ such that

$$\|\vartheta(p) - S_7(p)\| \leq \begin{cases} \frac{\lambda}{63}, \\ \frac{\lambda \|p\|^s}{|2^6 - 2^s|}, \end{cases} \quad (6.16)$$

$$\|\eta(p) - O_8(q)\| \leq \begin{cases} \frac{\lambda}{253}, \\ \frac{\lambda \|p\|^s}{|2^8 - 2^s|}, \end{cases} \quad (6.17)$$

$$\|\psi(r) - D_{10}(r)\| \leq \begin{cases} \frac{\lambda}{1023}, \\ \frac{\lambda \|r\|^s}{|2^{10} - 2^s|}, \end{cases} \quad (6.18)$$

for all $p, q, r \in X$.

7. Stability Results: Mixed Case

In this section, the generalized Ulam-Hyers stability of the functional equation (1.3), (1.4) and (1.5) mixed case is proved by using fixed point method.

Theorem 7.1. Let $\vartheta, \eta, \psi : X \rightarrow Y$ be a mapping for which there exist a function $\xi, \zeta, \varsigma : X \rightarrow [0, \infty)$ with the condition (5.1), (5.2) and (5.3); (6.1), (6.2) and (6.3) where $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$ such that the functional inequality with

$$\|\vartheta(2p) - 48\vartheta(p) - 16\vartheta(-p)\| \leq \xi(p) \quad (7.1)$$

$$\|\eta(2q) - 192\eta(q) - 64\eta(-q)\| \leq \zeta(q) \quad (7.2)$$

$$\|\psi(2r) - 768\psi(r) - 256\psi(-r)\| \leq \varsigma(r) \quad (7.3)$$

for all $x, y, z \in X$. If there exists $L = L(i) < 1$ such that the function

$$p \rightarrow \beta(p) = \xi\left(\frac{p}{2}\right),$$

$$q \rightarrow \beta(q) = \zeta\left(\frac{q}{2}\right),$$

$$r \rightarrow \beta(r) = \varsigma\left(\frac{r}{2}\right),$$

has the properties (5.7), (5.8) and (5.9); (6.7), (6.8) and (6.9); for all $p, q, r \in X$. Then there exists a unique quintic, septic, nonic mapping $Q_5, S_7, N_9 : X \rightarrow Y$ and a unique sextic, octic, decic mapping $S_6, O_8, D_{10} : X \rightarrow Y$ such that

$$\|\vartheta(p) - Q_5(p) - S_6(p)\| \leq \frac{L^{1-i}}{1-L} (\beta(p) + \beta(-p)) \quad (7.4)$$

$$\|\eta(q) - S_7(q) - O_8(q)\| \leq \frac{L^{1-i}}{1-L} (\beta(q) + \beta(-q)) \quad (7.5)$$

$$\|\psi(r) - N_9(r) - D_{10}(r)\| \leq \frac{L^{1-i}}{1-L} (\beta(r) + \beta(-r)) \quad (7.6)$$

holds for all $p, q, r \in X$.

Proof. Let

$$\vartheta_o(p) = \frac{\vartheta(p) - \vartheta(-p)}{2}$$

$$\eta_o(q) = \frac{\eta(q) - \eta(-q)}{2}$$

$$\psi_o(r) = \frac{\psi(r) - \psi(-r)}{2}$$

for all $p, q, r \in X$. Then $\vartheta_o(0) = 0$, $\eta_o(0) = 0$ and $\psi_o(0) = 0$ and $\vartheta_o(-p) = -\vartheta_o(p)$, $\eta_o(-q) = -\eta_o(q)$, $\psi_o(-r) = -\psi_o(r)$ for all $p, q, r \in X$. Hence

$$\|\vartheta_o(2p) - 48\vartheta_o(p) - 16\vartheta_o(-p)\| \leq \frac{\xi(p)}{2} + \frac{\xi(-p)}{2} \quad (7.7)$$

$$\|\eta_o(2q) - 192\eta_o(q) - 64\eta_o(-q)\| \leq \frac{\zeta(q)}{2} + \frac{\zeta(-q)}{2} \quad (7.8)$$

$$\|\psi_o(2r) - 768\psi_o(r) - 256\psi_o(-r)\| \leq \frac{\varsigma(r)}{2} + \frac{\varsigma(-r)}{2} \quad (7.9)$$

for all $p, q, r \in X$. By Theorem 5.1, we have

$$\|\vartheta(p) - Q_5(p)\| \leq \frac{1}{2} \left[\frac{L^{1-i}}{1-L} (\beta(p) + \beta(-p)) \right] \quad (7.10)$$

$$\|\eta(q) - S_7(q)\| \leq \frac{1}{2} \left[\frac{L^{1-i}}{1-L} (\beta(q) + \beta(-q)) \right] \quad (7.11)$$

$$\|\psi(r) - N_9(r)\| \leq \frac{1}{2} \left[\frac{L^{1-i}}{1-L} (\beta(r) + \beta(-r)) \right] \quad (7.12)$$

for all $p, q, r \in X$. Also, Let

$$\vartheta_e(p) = \frac{\vartheta(p) + \vartheta(-p)}{2}$$

$$\eta_e(q) = \frac{\eta(q) + \eta(-q)}{2}$$

$$\psi_e(r) = \frac{\psi(r) + \psi(-r)}{2}$$

for all $p, q, r \in X$. Then $\vartheta_e(0) = 0$, $\eta_e(0) = 0$ and $\psi_e(0) = 0$ and $\vartheta_e(-p) = \vartheta_e(p)$, $\eta_e(-q) = \eta_e(q)$, $\psi_e(-r) = \psi_e(r)$ for all $p, q, r \in X$. Hence

$$\|\vartheta_e(2p) - 48\vartheta_e(p) - 16\vartheta_e(-p)\| \leq \frac{\xi(p)}{2} + \frac{\xi(-p)}{2} \quad (7.13)$$

$$\|\eta_e(2q) - 192\eta_e(q) - 64\eta_e(-q)\| \leq \frac{\zeta(q)}{2} + \frac{\zeta(-q)}{2} \quad (7.14)$$

$$\|\psi_e(2r) - 768\psi_e(r) - 256\psi_e(-r)\| \leq \frac{\varsigma(r)}{2} + \frac{\varsigma(-r)}{2} \quad (7.15)$$

for all $p, q, r \in X$. By Theorem 6.1, we have

$$\|\vartheta(p) - S_6(p)\| \leq \frac{1}{2} \left[\frac{L^{1-i}}{1-L} (\beta(p) + \beta(-p)) \right] \quad (7.16)$$

$$\|\eta(q) - O_8(q)\| \leq \frac{1}{2} \left[\frac{L^{1-i}}{1-L} (\beta(q) + \beta(-q)) \right] \quad (7.17)$$

$$\|\psi(r) - D_{10}(r)\| \leq \frac{1}{2} \left[\frac{L^{1-i}}{1-L} (\beta(r) + \beta(-r)) \right] \quad (7.18)$$

for all $p, q, r \in X$. Define

$$\vartheta(p) = \vartheta_e(p) + \vartheta_o(p) \quad (7.19)$$

$$\vartheta(q) = \vartheta_e(q) + \vartheta_o(q) \quad (7.20)$$

$$\vartheta(r) = \vartheta_e(r) + \vartheta_o(r) \quad (7.21)$$

for all $p, q, r \in X$. From (7.13), (7.14) and (7.15); (7.16), (7.17) and (7.18); (7.19) (7.20) and (7.21) we arrive

$$\begin{aligned} & \|\vartheta(p) - Q_5(p) - S_6(p)\| \\ &= \|\vartheta_e(p) + \vartheta_o(p) - Q_5(p) - S_6(p)\| \\ &\leq \|\vartheta_o(p) - Q_5(p)\| + \|\vartheta_e(p) - S_6(p)\| \\ &\leq \frac{1}{2} \left[\frac{L^{1-i}}{1-L} (\beta(p) + \beta(-p)) + \frac{1}{2} \frac{L^{1-i}}{1-L} (\beta(p) + \beta(-p)) \right] \\ &\leq \frac{L^{1-i}}{1-L} (\beta(p) + \beta(-p)) \end{aligned}$$

$$\begin{aligned} & \|\eta(q) - S_7(q) - O_8(q)\| \\ &= \|\eta_e(q) + \eta_o(q) - S_7(q) - O_8(q)\| \\ &\leq \|\eta_o(q) - S_7(q)\| + \|\eta_e(q) - O_8(q)\| \\ &\leq \frac{1}{2} \left[\frac{L^{1-i}}{1-L} (\beta(q) + \beta(-q)) + \frac{1}{2} \frac{L^{1-i}}{1-L} (\beta(q) + \beta(-q)) \right] \\ &\leq \frac{L^{1-i}}{1-L} (\beta(q) + \beta(-q)) \end{aligned}$$

$$\begin{aligned} & \|\psi(r) - N_9(r) - D_{10}(r)\| \\ &= \|\psi_e(r) + \psi_o(r) - N_9(r) - D_{10}(r)\| \\ &\leq \|\psi_o(r) - N_9(r)\| + \|\psi_e(r) - D_{10}(r)\| \\ &\leq \frac{1}{2} \left[\frac{L^{1-i}}{1-L} (\beta(r) + \beta(-r)) + \frac{1}{2} \frac{L^{1-i}}{1-L} (\beta(r) + \beta(-r)) \right] \\ &\leq \frac{L^{1-i}}{1-L} (\beta(r) + \beta(-r)) \end{aligned}$$

for all $p, q, r \in X$. Hence the theorem is proved. $\square$

Using Corollaries 5.2 and 6.2, we have the following Corollary concerning the stability of (1.3), (1.4), and (1.5).

Corollary 7.2. Let λ and s be nonnegative real numbers. Let an even function $\vartheta, \eta, \psi : X \rightarrow Y$ satisfies the inequality

$$\|\vartheta(2p) - 48\vartheta(p) - 16\vartheta(-p)\| \leq \begin{cases} \lambda, \\ \lambda \|p\|^s, & s \neq 5, 6; \end{cases} \quad (7.22)$$

$$\|\eta(2q) - 192\eta(q) - 64\eta(-q)\| \leq \begin{cases} \lambda, \\ \lambda \|q\|^s, & s \neq 7, 8; \end{cases} \quad (7.23)$$

$$\|\psi(2r) - 768\psi(r) - 256\psi(-r)\| \leq \begin{cases} \lambda, \\ \lambda \|r\|^s, & s \neq 9, 10; \end{cases} \quad (7.24)$$

for all $p, q, r \in X$. Then there exists a unique quintic, septic, nonic mapping $Q_5, S_7, N_9 : X \rightarrow Y$ and a unique sextic, octic, decic mapping $S_6, O_8, D_{10} : X \rightarrow Y$ such that

$$\|\vartheta(p) - Q_5(p) - S_6(p)\| \leq \begin{cases} \left(\frac{\lambda}{31} + \frac{\lambda}{63} \right), \\ \left(\frac{\lambda \|p\|^s}{|2^5 - 2^s|} + \frac{\lambda \|p\|^s}{|2^6 - 2^s|} \right), \end{cases} \quad (7.25)$$

$$\|\eta(q) - S_7(q) - O_8(q)\| \leq \begin{cases} \left(\frac{\lambda}{127} + \frac{\lambda}{253} \right), \\ \left(\frac{\lambda \|q\|^s}{|2^7 - 2^s|} + \frac{\lambda \|q\|^s}{|2^8 - 2^s|} \right), \end{cases} \quad (7.26)$$

$$\|\psi(r) - N_9(r) - D_{10}(r)\| \leq \begin{cases} \left(\frac{\lambda}{511} + \frac{\lambda}{1023} \right), \\ \left(\frac{\lambda \|r\|^s}{|2^9 - 2^s|} + \frac{\lambda \|r\|^s}{|2^{10} - 2^s|} \right), \end{cases} \quad (7.27)$$

for all $p, q, r \in X$.

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