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Ulam stability of a alternate additive-quadratic functional equation in IFB space

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Abstract

In this paper, we establish the various Ulam stability of a alternate additive - quadratic functional equation in Intuitionistic Fuzzy Banach spaces via two different substitutions.

Keywords

Additive functional equation, quadratic functional equation, additive - quadratic functional equation, Ulam stability, IFB space.

AMS Subject Classification

39B52, 39B82.

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Contents

1. Introduction

S.M. Ulam [43] is the pioneer for the stability problem in functional equations. In 1940, while he was delivering a talk before the Mathematics Club of University of Wisconsin, he discussed a number of unanswered problems. Among those was the following question concerning the stability of homomorphisms:

Let G be a group and G' be a metric group with metric $\rho(.\,,\,.)$. Given $\varepsilon > 0$ does there exist a $\delta > 0$ such that if a function $f : G \to G'$ satisfies the inequality $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then there exists a homomorphism $h : G \to G'$ with $\rho(f(x), h(x)) < \varepsilon$ for all $x \in G$?

In 1941, D.H. Hyers [22] gave a partial answer to the question of Ulam. In 1950, T. Aoki [2] and in 1978, Th.M. Rassias [36] explored the further generalization of Hyers theorem for linear mappings by considering an unbounded Cauchy difference for sum of powers of norms The result of Rassias has influenced the development of what is now called the Ulam - Hyers - Rassias stability theory for functional equations.

In 1982, a similar stability theorem was proved by J.M.Rassias [34] in which he replaced the term sum of powers of norms by product of norms. Later this stability result is called the Ulam - Gavruta - Rassias stability of functional equations.

All the above stability results are further generalized by P. Gavruta [19] in 1994 considering the control function as function of variables and proved the following theorem.

Theorem 1.1. *[19] Let $(G, +)$ be an Abelian group, $(X, ||\cdot||)$ be a Banach space and $\phi : G \times G \to [0, \infty)$ be a mapping such that*

$$\Phi(x, y) = \sum_{k=0}^{\infty} 2^{-k} \phi(2^k x, 2^k y) < \infty. \tag{1.1}$$

If a function $f : G \to E$ satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \phi(x, y) \tag{1.2}$$

for any $x, y \in G$, then there exists a unique additive function $T : G \to E$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{2}\Phi(x, x) \tag{1.3}$$

for all x in G. If moreover $f(tx)$ is continuous in t for each fixed $x \in G$, then T is linear.

This stability result is called Generalized Ulam - Hyers - Rassias stability of functional equations.

Very recently J.M. Rassias [38] introduced a new concept on stability by introducing the mixed type product-sum of powers of norms. This stability result is called JMRassias stability of functional equations.

The solution and stability of following additive quadratic functional equations

$$f(2x+y)+f(2x-y) = 2f(x+y)+2f(x-y)+2f(2x)-4f(x) \quad (1.4)$$

$$f(2x+y)+f(2x-y) = f(x+y)+f(x-y)+2f(2x)-2f(x) \quad (1.5)$$

$$g(x+y)+g(x-y)=2g(x)+g(y)+g(-y) \quad (1.6)$$

$$f(x-t)+f(y-t)+f(z-t) = 3f\left(\frac{x+y+z}{3}-t\right)+f\left(\frac{2x-y-z}{3}\right) + f\left(\frac{-x+2y-z}{3}\right)+f\left(\frac{-x-y+2z}{3}\right) \quad (1.7)$$

$$f(x+2y+3z)+f(x-2y+3z) + f(x+2y-3z)+f(x-2y-3z) = 4f(x)+8[f(y)+f(-y)]+18[f(z)+f(-z)] \quad (1.8)$$

were investigated by A. Najati, M.B.Moghimi [31], M.E. Gordji et. al., [18], M. J. Rassias et. al., [33], M.Arunkumar, J.M. Rassias [4] and M. Arunkumar, P. Agilan [5, 6].

In this paper, we establish the Ulam stability of a alternate additive - quadratic functional equation of the form

$$h(z_1+2z_2+3z_3)-h(z_1-2z_2+3z_3) + h(z_1+2z_2-3z_3)-h(z_1-2z_2-3z_3) = 8h(z_1)-8h(z_1-z_2)+4[h(z_2)+h(-z_2)] \quad (1.9)$$

in Intuitionistic Fuzzy Banach spaces via two different substitutions.

2. Basic Definitions of IFB Space

Now, we recall the basic definitions and notations in the setting of intuitionistic fuzzy normed space which was introduced by Saadati and Park [39].

Definition 2.1. *[39] A binary operation $*:[0,1]\times[0,1]\longrightarrow[0,1]$ is said to be continuous t-norm if $*$ satisfies the following conditions:*

(1) $*$ is commutative and associative;

(2) $*$ is continuous;

(3) $a*1=a$ for all $a\in[0,1]$;

(4) $a*b\le c*d$ whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.

Definition 2.2. *[39] A binary operation $\diamond:[0,1]\times[0,1]\longrightarrow[0,1]$ is said to be continuous t-conorm if $\diamond$ satisfies the following conditions:*

(1') $\diamond$ is commutative and associative;

(2') $\diamond$ is continuous;

(3') $a\diamond 0=a$ for all $a\in[0,1]$;

(4') $a\diamond b\le c\diamond d$ whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.

Definition 2.3. *[39] The five-tuple $(X,\mu,\nu,*,\diamond)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if $\mathscr{A}$ is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t−conorm, and μ,ν are fuzzy sets on $X\times(0,\infty)$ satisfying the following conditions. For every $x,y\in\mathscr{A}$ and $s,t>0$*

(IFN1) $\mu(x,t)+\nu(x,t)\le 1$,

(IFN2) $\mu(x,t)>0$,

(IFN3) $\mu(x,t)=1$, *if and only if* $z=0$.

(IFN4) $\mu(\alpha x,t)=\mu\left(x,\frac{t}{\alpha}\right)$ *for each* $\alpha\ne 0$,

(IFN5) $\mu(x,t)*\mu(y,s)\le\mu(x+y,t+s)$,

(IFN6) $\mu(x,\cdot):(0,\infty)\to[0,1]$ *is continuous,*

(IFN7) $\lim_{t\to\infty}\mu(x,t)=1$ *and* $\lim_{t\to 0}\mu(x,t)=0$,

(IFN8) $\nu(x,t)<1$,

(IFN9) $\nu(x,t)=0$, *if and only if* $z=0$.

(IFN10) $\nu(\alpha x,t)=\nu\left(x,\frac{t}{\alpha}\right)$ *for each* $\alpha\ne 0$,

(IFN11) $\nu(x,t)\diamond\nu(y,s)\ge\nu(x+y,t+s)$,

(IFN12) $\nu(x,\cdot):(0,\infty)\to[0,1]$ *is continuous,*

(IFN13) $\lim_{t\to\infty}\nu(x,t)=0$ *and* $\lim_{t\to 0}\nu(x,t)=1$.

In this case, (μ,ν) is called an intuitionistic fuzzy norm.

Example 2.4. *[39] Let $(X,\|\cdot\|)$ be a normed space. Let $a*b=ab$ and $a\diamond b=\min\{a+b,1\}$ for all $a,b\in[0,1]$. For all $x\in\mathscr{A}$ and every $t>0$, consider*

$$\mu(x,t)=\begin{cases}\frac{t}{t+\|x\|} & if\quad t>0;\\ 0 & if\quad t\le 0;\end{cases}$$

and

$$\nu(x,t)=\begin{cases}\frac{\|x\|}{t+\|x\|} & if\quad t>0;\\ 0 & if\quad t\le 0.\end{cases}$$

Then $(X,\mu,\nu,*,\diamond)$ is an IFN-space.

Definition 2.5. *[39] Let $(X,\mu,\nu,*,\diamond)$ be an IFNS. Then, a sequence $x=\{x_k\}$ is said to be intuitionistic fuzzy convergent to a point $L\in\mathscr{A}$ if*

$$\lim \quad \mu(x_k-L,t)=1 \quad and \quad \lim \quad \nu(x_k-L,t)=0$$

for all $t>0$. In this case, we write

$$x_k \xrightarrow{IF} L \qquad as \qquad k\to\infty$$

Definition 2.6. *[39] Let $(X,\mu,\nu,*,\diamond)$ be an IFN-space. Then, $x=\{x_k\}$ is said to be intuitionistic fuzzy Cauchy sequence if*

$$\mu\left(x_{k+p}-x_k,t\right)=1 \qquad and \qquad \nu\left(x_{k+p}-x_k,t\right)=0$$

for all $t>0$, and $p=1,2\cdots$.

Definition 2.7. *[39] Let $(X,\mu,\nu,*,\diamond)$ be an IFN-space. Then $(X,\mu,\nu,*,\diamond)$ is said to be complete if every intuitionistic fuzzy Cauchy sequence in $(X,\mu,\nu,*,\diamond)$ is intuitionistic fuzzy convergent $(X,\mu,\nu,*,\diamond)$.*

Here and subsequently, assume that

- $\mathscr{A}$ - linear space;
- $(\mathscr{C},\mu',\nu')$ - intuitionistic fuzzy normed space;
- $(\mathscr{B},\mu,\nu)$ - intuitionistic fuzzy Banach space.

Now, we use the following notation for a given mapping $h:\mathscr{A}\longrightarrow\mathscr{B}$ such that

$$\begin{aligned}H(z_1,z_2,z_3)=&h(z_1+2z_2+3z_3)-h(z_1-2z_2+3z_3)\\&+h(z_1+2z_2-3z_3)-h(z_1-2z_2-3z_3)\\&-8h(z_1)+8h(z_1-z_2)\\&-4[h(z_2)+h(-z_2)]\end{aligned}$$

for all $z_1,z_2,z_3\in\mathscr{A}$.

3. IFBS: Stability Results: Substitution - 1

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.9) in IFB space.

Theorem 3.1. *Let $\Delta\in\{1,-1\}$. Let $H:\mathscr{A}\to\mathscr{B}$ be an odd function satisfying the inequality*

$$\left.\begin{aligned}\mu\left(H(z_1,z_2,z_3),t\right)\ge\mu'\left(\Omega_\mu\left(z_1,z_2,z_3\right),t\right)\\ \nu\left(H(z_1,z_2,z_3),t\right)\le\nu'\left(\Omega_\nu\left(z_1,z_2,z_3\right),t\right)\end{aligned}\right\}\tag{3.1}$$

for all $z_1,z_2,z_3\in\mathscr{A}$ and all $t>0$. Then there exists a unique additive mapping $\mathscr{H}_1:\mathscr{A}\longrightarrow\mathscr{B}$ satisfying (1.9) and

$$\left.\begin{aligned}\mu\left(\mathscr{H}_1(z)-h(z),t\right)\ge\mu'\left(\Omega_\mu(z,z,0),t|3-p|\right)\\ \nu\left(\mathscr{H}_1(z)-h(z),t\right)\le\nu'\left(\Omega_\nu(z,z,0),t|3-p|\right)\end{aligned}\right\}\tag{3.2}$$

for all $z\in\mathscr{A}$ and all $t>0$. Here $\Omega_\mu,\Omega_\nu:\mathscr{A}\times\mathscr{A}\times\mathscr{A}\longrightarrow\mathscr{C}$ are functions such that for some $0<\left(\frac{p}{3}\right)^\Delta<1$ with

$$\left.\begin{aligned}\mu'\left(\Omega_\mu\left(3^{n\Delta}z,3^{n\Delta}z,3^{n\Delta}z\right),t\right)\ge\mu'\left(p^{n\Delta}\Omega_\mu\left(z,z,z\right),t\right)\\ \nu'\left(\Omega_\nu\left(3^{n\Delta}z,3^{n\Delta}z,3^{n\Delta}z\right),t\right)\le\nu'\left(p^{n\Delta}\Omega_\nu\left(z,z,z\right),t\right)\end{aligned}\right\}\tag{3.3}$$

for all $z\in\mathscr{A}$ and all $t>0$ and

$$\left.\begin{aligned}\lim_{n\to\infty}\mu'\left(\Omega_\mu\left(3^{\Delta n}z_1,3^{\Delta n}z_2,3^{n\Delta}z_3\right),3^{\Delta n}t\right)=1\\ \lim_{n\to\infty}\nu'\left(\Omega_\nu\left(3^{\Delta n}z_1,3^{\Delta n}z_2,3^{n\Delta}z_3\right),3^{\Delta n}t\right)=0\end{aligned}\right\}\tag{3.4}$$

for all $z_1,z_2,z_3\in\mathscr{A}$ and all $t>0$.

Proof. **Case (1):**

Let $\Delta=1$. Using oddness of h and transform (z_1,z_2,z_3) by $(z,z,0)$ in (3.1), we have

$$\left.\begin{aligned}\mu\left(h(3z)-3h(z),t\right)\ge\mu'\left(\Omega_\mu\left(z,z,0\right),t\right)\\ \nu\left(h(3z)-3h(z),t\right)\le\nu'\left(\Omega_\nu\left(z,z,0\right),t\right)\end{aligned}\right\}\tag{3.5}$$

for all $z\in\mathscr{A}$ and all $t>0$. Using (IFN4) and (IFN10) in (3.5), we arrive

$$\left.\begin{aligned}\mu\left(\frac{h(3z)}{3}-h(z),\frac{t}{3}\right)\ge\mu'\left(\Omega_\mu(z,z,0),t\right)\\ \nu\left(\frac{h(3z)}{3}-h(z),\frac{t}{3}\right)\le\nu'\left(\Omega_\nu(z,z,0),t\right)\end{aligned}\right\}\tag{3.6}$$

for all $z\in\mathscr{A}$ and all $t>0$. Substituting z by 3^nz in (3.6), we have

$$\left.\begin{aligned}\mu\left(\frac{h(3^{n+1}z)}{3}-h(3^nz),\frac{t}{3}\right)\ge\mu'\left(\Omega_\mu(3^nz,3^nz,0),t\right)\\ \nu\left(\frac{h(3^{n+1}z)}{3}-h(3^nz),\frac{t}{3}\right)\le\nu'\left(\Omega_\nu(3^nz,3^nz,0)),t\right)\end{aligned}\right\}\tag{3.7}$$

for all $z\in\mathscr{A}$ and all $t>0$. It is easy to verify from (3.7) and using (3.3), (IFN4), (IFN10) that

$$\left.\begin{aligned}\mu\left(\frac{h(3^{n+1}z)}{3^{(n+1)}}-\frac{h(3^nz)}{3^n},\frac{t}{3\cdot3^n}\right)\ge\mu'\left(\Omega_\mu(z,z,0),\frac{t}{p^n}\right)\\ \nu\left(\frac{h(3^{n+1}z)}{3^{(n+1)}}-\frac{h(3^nz)}{3^n},\frac{t}{3\cdot3^n}\right)\le\nu'\left(\Omega_\nu(z,z,0),\frac{t}{p^n}\right)\end{aligned}\right\}\tag{3.8}$$

for all $z\in\mathscr{A}$ and all $t>0$. Interchanging t into p^nt in (3.8), we have

$$\left.\begin{aligned}\mu\left(\frac{h(3^{n+1}z)}{3^{(n+1)}}-\frac{h(3^nz)}{3^n},\frac{t\cdot p^n}{3\cdot3^n}\right)\ge\mu'\left(\Omega_\mu(z,z,0),t\right)\\ \nu\left(\frac{h(3^{n+1}z)}{3^{(n+1)}}-\frac{h(3^nz)}{3^n},\frac{t\cdot p^n}{3\cdot3^n}\right)\le\nu'\left(\Omega_\nu(z,z,0),t\right)\end{aligned}\right\}\tag{3.9}$$

for all $z\in\mathscr{A}$ and all $t>0$. It is easy to see that

$$\frac{h(3^nz)}{3^n}-h(z)=\sum_{i=0}^{n-1}\frac{h(3^{i+1}z)}{3^{(i+1)}}-\frac{h(3^iz)}{3^i}\tag{3.10}$$

for all $z \in \mathscr{A}$. From equations (3.8) and (3.10), we arrive

$$\left.\begin{array}{l}\mu\left(\frac{h(3^n z)}{3^n}-h(z),\sum_{i=0}^{n-1}\frac{t\ p^i}{3\cdot 3^i}\right)\\ \qquad \geq \prod_{i=0}^{n-1}\mu\left(\frac{h(3^{i+1}z)}{3^{(i+1)}}-\frac{h(3^i z)}{3^i},\frac{t\ p^i}{3\cdot 3^i}\right)\\ \nu\left(\frac{h(3^n z)}{3^n}-h(z),\sum_{i=0}^{n-1}\frac{t\ p^i}{3\cdot 3^i}\right)\\ \qquad \leq \prod_{i=0}^{n-1}\nu\left(\frac{h(3^{i+1}z)}{3^{(i+1)}}-\frac{h(3^i z)}{3^i},\frac{t\ p^i}{3\cdot 3^i}\right)\end{array}\right\} \tag{3.11}$$

with

$$\prod_{i=0}^{n-1} c_j = c_1 * c_2 * \cdots * c_n \quad and \quad \prod_{i=0}^{n-1} d_j = d_1 \diamond d_2 \diamond \cdots \diamond d_n$$

for all $z \in \mathscr{A}$ and all $t>0$. Thus from (3.9) and (3.11), we get

$$\left.\begin{array}{l}\mu\left(\frac{h(3^n z)}{3^n}-h(z),\sum_{i=0}^{n-1}\frac{t\ p^i}{3\cdot 3^i}\right)\\ \geq \prod_{i=0}^{n-1}\mu'\left(\Omega_\mu(z,z,0),t\right)=\mu'\left(\Omega_\mu(z,z,0),t\right)\\ \nu\left(\frac{h(3^n z)}{3^n}-h(z),\sum_{i=0}^{n-1}\frac{t\ p^i}{3\cdot 3^i}\right)\\ \leq \prod_{i=0}^{n-1}\nu'\left(\Omega_\nu(z,z,0),t\right)=\nu'\left(\Omega_\nu(z,z,0),t\right)\end{array}\right\} \tag{3.12}$$

for all $z \in \mathscr{A}$ and all $t>0$. Replacing z by $3^m z$ in (3.12) and using (3.3), (IFN4), (IFN10), we obtain

$$\left.\begin{array}{l}\mu\left(\frac{h(3^{n+m}z)}{3^{(n+m)}}-\frac{h(3^m z)}{3^m},\sum_{i=0}^{n-1}\frac{t\ p^i}{3\cdot 3^{(i+m)}}\right)\\ \geq \mu'\left(\Omega_\mu(3^m z,3^m z,0),t\right)\\ =\mu'\left(\Omega_\mu(z,z,0),\frac{t}{p^m}\right)\\ \nu\left(\frac{h(3^{n+m}z)}{3^{(n+m)}}-\frac{h(3^m z)}{3^m},\sum_{i=0}^{n-1}\frac{t\ p^i}{3\cdot 3^{(i+m)}}\right)\\ \leq \nu'\left(\Omega_\nu(3^m z,3^m z,0),t\right)\\ =\nu'\left(\Omega_\nu(z,z,0),\frac{t}{p^m}\right)\end{array}\right\} \tag{3.13}$$

for all $z \in \mathscr{A}$ and all $t>0$ and all $m,n \geq 0$. Interchanging t by $p^m t$ in (3.13), we arrive

$$\left.\begin{array}{l}\mu\left(\frac{h(3^{n+m}z)}{3^{(n+m)}}-\frac{h(3^m z)}{3^m},\sum_{i=0}^{n-1}\frac{t\ p^{i+m}}{3\cdot 3^{(i+m)}}\right)\\ \geq \mu'\left(\Omega_\mu(z,z,0),t\right)\\ \nu\left(\frac{h(3^{n+m}z)}{3^{(n+m)}}-\frac{h(3^m z)}{3^m},\sum_{i=0}^{n-1}\frac{t\ p^{i+m}}{3\cdot 3^{(i+m)}}\right)\\ \leq \nu'\left(\Omega_\nu(z,z,0),t\right)\end{array}\right\} \tag{3.14}$$

for all $z \in \mathscr{A}$ and all $t>0$ and all $m,n \geq 0$. Using (IFN4), (IFN10) in (3.14) implies that

$$\left.\begin{array}{l}\mu\left(\frac{h(3^{n+m}z)}{3^{(n+m)}}-\frac{h(3^m z)}{3^m},t\right)\geq \mu'\left(\Omega_\mu(z,z,0),\frac{t}{\sum_{i=m}^{n-1}\frac{p^i}{3\cdot 3^i}}\right)\\ \nu\left(\frac{h(3^{n+m}z)}{3^{(n+m)}}-\frac{h(3^m z)}{3^m},t\right)\leq \nu'\left(\Omega_\nu(z,z,0),\frac{t}{\sum_{i=m}^{n-1}\frac{p^i}{3\cdot 3^i}}\right)\end{array}\right\} \tag{3.15}$$

holds for all $z \in \mathscr{A}$ and all $t>0$ and all $m,n \geq 0$. Since $0<p<1$ and $\sum_{i=0}^{n}\left(\frac{p}{1}\right)^i<\infty$. The Cauchy criterion for convergence in IFNS shows that the sequence $\left\{\frac{h(3^n z)}{3^n}\right\}$ is Cauchy in $(\mathscr{B},\mu,\nu)$. Since $(\mathscr{B},\mu,\nu)$ is a complete IFN-space this sequence converges to some point $\mathscr{H}_1(z) \in \mathscr{B}$. So, one can define the mapping $\mathscr{H}_1:\mathscr{A}\longrightarrow\mathscr{B}$ by $\lim_{n\to\infty}\mu\left(\frac{h(3^n z)}{3^n}-\mathscr{H}_1(z),t\right)=1$, and $\lim_{n\to\infty}\nu\left(\frac{h(3^n z)}{3^n}-\mathscr{H}_1(z),t\right)=0$ for all $z\in\mathscr{A}$ and all $t>0$. Hence

$$\frac{h(3^n z)}{3^n}\xrightarrow{IF}\mathscr{H}_1(z), \quad as \quad n\to\infty.$$

Letting $m=0$ and n tend to infinity in (3.15), we arrive

$$\left.\begin{array}{l}\mu\left(\mathscr{H}_1(z)-h(z),t\right)\geq\mu'\left(\Omega_\mu(z,z,0),t(3-p)\right)\\ \nu\left(\mathscr{H}_1(z)-h(z),t\right)\leq\nu'\left(\Omega_\nu(z,z,0),t(3-p)\right)\end{array}\right\} \tag{3.16}$$

for all $z \in \mathscr{A}$ and all $t>0$. To prove $\mathscr{A}$ satisfies (1.9), replacing (z_1,z_2,z_3) by $(3^n z_1,3^n z_2,3^n z_3)$ in (3.1) respectively, we obtain

$$\left.\begin{array}{l}\mu\left(\frac{1}{3^n}H(3^n z_1,3^n z_2,3^n z_3),t\right)\\ \qquad\geq\mu'\left(\Omega_\mu(3^n z_1,3^n z_2,3^n z_3),3^n t\right)\\ \nu\left(\frac{1}{3^n}H(3^n z_1,3^n z_2,3^n z_3),t\right)\\ \qquad\leq\nu'\left(\Omega_\nu(3^n z_1,3^n z_2,3^n z_3),t\right)\end{array}\right\} \tag{3.17}$$

for all $z \in \mathscr{A}$ and all $t>0$. Now,

$$\begin{aligned}&\mu\Big(\mathscr{H}_1(z_1+2z_2+3z_3)-\mathscr{H}_1(z_1-2z_2+3z_3)\\&\qquad+\mathscr{H}_1(z_1+2z_2-3z_3)-\mathscr{H}_1(z_1-2z_2-3z_3)\\&\qquad-8\mathscr{H}_1(z_1)+8\mathscr{H}_1(z_1-z_2)-4\left[h(z_2)+h(-z_2)\right],t\Big)\geq\\&\mu\Big(\mathscr{H}_1(z_1+2z_2+3z_3)-\frac{1}{3^n}h_o\left(3^n(z_1+2z_2+3z_3)\right),\frac{t}{8}\Big)*\\&\mu\Big(-\mathscr{H}_1(z_1-2z_2+3z_3)+\frac{1}{3^n}h_o\left(3^n(z_1-2z_2+3z_3)\right),\frac{t}{8}\Big)*\\&\mu\Big(\mathscr{H}_1(z_1+2z_2-3z_3)-\frac{1}{3^n}h_o\left(3^n(z_1+2z_2-3z_3)\right),\frac{t}{8}\Big)*\\&\mu\Big(-\mathscr{H}_1(z_1-2z_2-3z_3)-\frac{1}{3^n}+h_o\left(3^n(z_1-2z_2-3z_3)\right),\frac{t}{8}\Big)*\\&\mu\Big(-8\mathscr{H}_1(z_1)+\frac{8}{3^n}h_o\left(3^n z_1\right),\frac{t}{8}\Big)*\\&\mu\Big(8\mathscr{H}_1(z_1-z_2)-\frac{8}{3^n}h_o\left(3^n(z_1-z_2)\right),\frac{t}{8}\Big)*\\&\mu\Big(-4\left[\mathscr{H}_1(z_2)+\mathscr{H}_1(-z_2)\right]\\&\qquad+\frac{4}{3^n}\left[h_o\left(3^n z_2\right)+h_o\left(-3^n z_2\right)\right],\frac{t}{8}\Big)*\\&\mu\Big(\frac{1}{3^n}h_o\left(3^n(z_1+2z_2+3z_3)\right)-\frac{1}{3^n}h_o\left(3^n(z_1-2z_2+3z_3)\right)\\&\qquad+\frac{1}{3^n}h_o\left(3^n(z_1+2z_2-3z_3)\right)-\frac{1}{3^n}h_o\left(3^n(z_1-2z_2-3z_3)\right)\\&\qquad-\frac{8}{3^n}h_o\left(3^n z_1\right)+\frac{8}{3^n}h_o\left(3^n(z_1-z_2)\right)\\&\qquad-\frac{4}{3^n}\left[h_o\left(3^n z_2\right)+h_o\left(-3^n z_2\right)\right],\frac{t}{8}\Big)\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
&\nu\Big(\mathscr{H}_1(z_1+2z_2+3z_3)-\mathscr{H}_1(z_1-2z_2+3z_3)\\
&+\mathscr{H}_1(z_1+2z_2-3z_3)-\mathscr{H}_1(z_1-2z_2-3z_3)\\
&-8\mathscr{H}_1(z_1)+8\mathscr{H}_1(z_1-z_2)-4[h(z_2)+h(-z_2)],t\Big)\\
&\geq \nu\Big(\mathscr{H}_1(z_1+2z_2+3z_3)\\
&\qquad -\frac{1}{3^n}h_o(3^n(z_1+2z_2+3z_3)),\frac{t}{8}\Big)\diamond\\
&\quad \nu\Big(-\mathscr{H}_1(z_1-2z_2+3z_3)\\
&\qquad +\frac{1}{3^n}h_o(3^n(z_1-2z_2+3z_3)),\frac{t}{8}\Big)\diamond\\
&\quad \nu\Big(\mathscr{H}_1(z_1+2z_2-3z_3)\\
&\qquad -\frac{1}{3^n}h_o(3^n(z_1+2z_2-3z_3)),\frac{t}{8}\Big)\diamond\\
&\quad \nu\Big(-\mathscr{H}_1(z_1-2z_2-3z_3)\\
&\qquad -\frac{1}{3^n}+h_o(3^n(z_1-2z_2-3z_3)),\frac{t}{8}\Big)\diamond\\
&\quad \nu\Big(-8\mathscr{H}_1(z_1)+\frac{8}{3^n}h_o(3^nz_1),\frac{t}{8}\Big)\diamond\\
&\qquad \nu\Big(8\mathscr{H}_1(z_1-z_2)-\frac{8}{3^n}h_o(3^n(z_1-z_2)),\frac{t}{8}\Big)\diamond\\
&\quad \nu\Big(-4[\mathscr{H}_1(z_2)+\mathscr{H}_1(-z_2)]\\
&\qquad +\frac{4}{3^n}[h_o(3^nz_2)+h_o(-3^nz_2)],\frac{t}{8}\Big)\diamond\\
&\quad \nu\Big(\frac{1}{3^n}h_o(3^n(z_1+2z_2+3z_3))\\
&\qquad -\frac{1}{3^n}h_o(3^n(z_1-2z_2+3z_3))\\
&\qquad +\frac{1}{3^n}h_o(3^n(z_1+2z_2-3z_3))\\
&\qquad -\frac{1}{3^n}h_o(3^n(z_1-2z_2-3z_3))\\
&\qquad -\frac{8}{3^n}h_o(3^nz_1)+\frac{8}{3^n}h_o(3^n(z_1-z_2))\\
&\quad -\frac{4}{3^n}[h_o(3^nz_2)+h_o(-3^nz_2)],\frac{t}{8}\Big) \qquad (3.19)
\end{aligned}$$

or all $z_1,z_2,z_3\in\mathscr{A}$ and all $t>0$. Letting $n\to\infty$ in (3.17), we arrive

$$\left.\begin{aligned}
\lim_{n\to\infty}\mu\Big(\tfrac{1}{3^n}H(3^nz_1,3^nz_2,3^nz_3),\tfrac{t}{6}\Big)=1\\
\lim_{n\to\infty}\nu\Big(\tfrac{1}{3^n}H(3^nz_1,3^nz_2,3^nz_3),\tfrac{t}{6}\Big)=0
\end{aligned}\right\} \qquad (3.20)$$

for all $z\in\mathscr{A}$ and all $t>0$. Letting $n\to\infty$ in (3.18), (3.19) and using (3.20), we observe that $\mathscr{H}_1$ fulfills (1.9). Therefore, $\mathscr{H}_1$ is a additive mapping. In order to prove $\mathscr{H}_1(z)$ is unique, let $\mathscr{H}_1'(z)$ be another additive functional equation satisfying (1.9) and (3.2). Hence,

$$\begin{aligned}
&\mu(\mathscr{H}_1(z)-\mathscr{H}_1'(z),t)\\
&\geq \mu\left(\mathscr{H}_1(3^nz)-h(3^nz),\frac{t.3^n}{2}\right)*\\
&\mu\left(h(3^nz)-\mathscr{H}_1'(3^nz),\frac{t.3^n}{2}\right)\\
&\geq \mu'\left(\Omega_\mu(z,z,0)(3^nz),\frac{t\,3^n(3-p)}{2}\right)\\
&\geq \mu'\left(\Omega_\mu(z,z,0),\frac{t\,3^n(3-p)}{2\cdot p^n}\right)\\
&\nu(\mathscr{H}_1(z)-\mathscr{H}_1'(z),t)\\
&\leq \nu\left(\mathscr{H}_1(3^nz)-h(3^nz),\frac{t.3^n}{2}\right)\diamond\\
&\nu\left(h(3^nz)-\mathscr{H}_1'(3^nz),\frac{t.3^n}{2}\right)\\
&\leq \nu'\left(\Omega_\nu(z,z,0)(3^nz),\frac{t\,3^n(3-p)}{2}\right)\\
&\leq \nu'\left(\Omega_\nu(z,z,0),\frac{t\,3^n(3-p)}{2\cdot p^n}\right)
\end{aligned}$$

for all $z\in\mathscr{A}$ and all $t>0$.

Since $\lim_{n\to\infty}\dfrac{t\,3^n(3-p)}{2\,p^n}=\infty$, we obtain

$$\left.\begin{aligned}
\lim_{n\to\infty}\mu'\left(\Omega_\mu(z),\tfrac{t\,3^n(3-p)}{2\,p^n}\right)=1\\
\lim_{n\to\infty}\nu'\left(\Omega_\nu(z),\tfrac{t\,3^n(3-p)}{2\,p^n}\right)=0
\end{aligned}\right\}$$

for all $z\in\mathscr{A}$ and all $t>0$. Thus

$$\left.\begin{aligned}
\mu(\mathscr{H}_1(z)-\mathscr{H}_1'(z),t)=1\\
\nu(\mathscr{H}_1(z)-\mathscr{H}_1'(z),t)=0
\end{aligned}\right\}$$

for all $z\in\mathscr{A}$ and all $t>0$. Hence, $\mathscr{H}_1(z)=\mathscr{H}_1'(z)$. Therefore, $\mathscr{H}_1(z)$ is unique.

Case 2:

For $\Delta=-1$. Putting z by $\frac{z}{3}$ in (3.5), we get

$$\left.\begin{aligned}
\mu\left(h(z)-3h\left(\tfrac{z}{3}\right),t\right)\geq\mu'\left(\Omega_\mu\left(\tfrac{z}{3},\tfrac{z}{3},0\right),t\right)\\
\nu\left(h(z)-3h\left(\tfrac{z}{3}\right),t\right)\leq\nu'\left(\Omega_\nu\left(\tfrac{z}{3},\tfrac{z}{3},0\right),t\right)
\end{aligned}\right\} \qquad (3.21)$$

for all $z\in\mathscr{A}$ and all $t>0$. The rest of the proof is similar to that of Case 1. This completes the proof. □

The following corollary is an immediate consequence of Theorem 3.1, regarding some stabilities of (1.9).

Corollary 3.2. *Assume an odd function $H:\mathscr{A}\longrightarrow\mathscr{B}$ satisfies the double inequality*

$$\left.\begin{array}{l}\mu\left(H(z_1,z_2,z_3),t\right)\\ \geq\left\{\begin{array}{l}\mu'\left(\lambda,t\right),\\ \mu'\left(\lambda\sum_{i=1}^{3}||z_i||^{a},t\right),\\ \mu'\left(\lambda\sum_{i=1}^{3}||z_i||^{a_i},t\right),\end{array}\right.\\ \nu\left(H(z_1,z_2,z_3),t\right)\\ \leq\left\{\begin{array}{l}\nu'\left(\lambda,t\right),\\ \nu'\left(\lambda\sum_{i=1}^{3}||z_i||^{a},t\right),\\ \nu'\left(\lambda\sum_{i=1}^{3}||z_i||^{a_i},t\right),\end{array}\right.\end{array}\right\}\tag{3.22}$$

for all $z_1,z_2,z_3\in\mathscr{A}$ and all $t>0$, where $\lambda,a,a_i's$ are constants with $\lambda>0$. Then there exists a unique additive mapping $\mathscr{H}_1:\mathscr{A}\longrightarrow\mathscr{B}$ such that

$$\left.\begin{array}{l}\mu\left(h(z)-\mathscr{H}_1(z),t\right)\\ \geq\left\{\begin{array}{l}\mu'\left(\lambda,|3-p|t\right),\\ \mu'\left(2\lambda||z||^{a},t|3-p|\right),\\ \mu'\left(\lambda\left[||z||^{a_1}+||z||^{a_2}\right],t|3-p|\right),\end{array}\right.\\ \nu\left(h(z)-\mathscr{H}_1(z),t\right)\\ \leq\left\{\begin{array}{l}\nu'\left(\lambda,|3-p|t\right),\\ \nu'\left(2\lambda||z||^{a},t|3-p|\right),\\ \nu'\left(\lambda\left[||z||^{a_1}+||z||^{a_2}\right],t|3-p|\right),\end{array}\right.\end{array}\right\}\tag{3.23}$$

for all $z\in\mathscr{A}$ and all $t>0$.

Theorem 3.3. *Let $\Delta\in\{1,-1\}$. Let $H:\mathscr{A}\to\mathscr{B}$ be an even function satisfying the inequality*

$$\left.\begin{array}{l}\mu\left(H(z_1,z_2,z_3),t\right)\geq\mu'\left(\Omega_\mu\left(z_1,z_2,z_3\right),t\right)\\ \nu\left(H(z_1,z_2,z_3),t\right)\leq\nu'\left(\Omega_\nu\left(z_1,z_2,z_3\right),t\right)\end{array}\right\}\tag{3.24}$$

for all $z_1,z_2,z_3\in\mathscr{A}$ and all $t>0$. Then there exists a unique quadratic mapping $\mathscr{H}_2:\mathscr{A}\longrightarrow\mathscr{B}$ satisfying (1.9) and

$$\left.\begin{array}{l}\mu\left(\mathscr{H}_2(z)-h(z),t\right)\geq\mu'\left(\Omega_\mu(z,z,0),t|9-p|\right)\\ \nu\left(\mathscr{H}_2(z)-h(z),t\right)\leq\nu'\left(\Omega_\nu(z,z,0),t|9-p|\right)\end{array}\right\}\tag{3.25}$$

for all $z\in\mathscr{A}$ and all $t>0$. Here $\Omega_\mu,\Omega_\nu:\mathscr{A}\times\mathscr{A}\times\mathscr{A}\longrightarrow\mathscr{C}$ are functions such that for some $0<\left(\frac{p}{3}\right)^{\Delta}<1$ with

$$\left.\begin{array}{l}\mu'\left(\Omega_\mu\left(3^{n\Delta}z,3^{n\Delta}z,3^{n\Delta}z\right),t\right)\geq\mu'\left(p^{n\Delta}\Omega_\mu\left(z,z,z\right),t\right)\\ \nu'\left(\Omega_\nu\left(3^{n\Delta}z,3^{n\Delta}z,3^{n\Delta}z\right),t\right)\leq\nu'\left(p^{n\Delta}\Omega_\nu\left(z,z,z\right),t\right)\end{array}\right\}\tag{3.26}$$

for all $z\in\mathscr{A}$ and all $t>0$ and

$$\left.\begin{array}{l}\lim_{n\to\infty}\mu'\left(\Omega_\mu\left(3^{\Delta n}z_1,3^{\Delta n}z_2,3^{n\Delta}z_3\right),9^{\Delta n}t\right)=1\\ \lim_{n\to\infty}\nu'\left(\Omega_\nu\left(3^{\Delta n}z_1,3^{\Delta n}z_2,3^{n\Delta}z_3\right),9^{\Delta n}t\right)=0\end{array}\right\}\tag{3.27}$$

for all $z_1,z_2,z_3\in\mathscr{A}$ and all $t>0$.

Proof. **Case (1):** Let $\Delta=1$.

Using evenness of h and transform (z_1,z_2,z_3) by $(z,z,0)$ in (3.24), we have

$$\left.\begin{array}{l}\mu\left(h_e(3x)-9h_e(z),t\right)\geq\mu'\left(\Omega_\mu\left(z,z,0\right),t\right)\\ \nu\left(h_e(3x)-9h_e(z),t\right)\leq\nu'\left(\Omega_\nu\left(z,z,0\right),t\right)\end{array}\right\}\tag{3.28}$$

for all $z\in\mathscr{A}$ and all $t>0$. Using (IFN4) and (IFN10) in (3.28), we arrive

$$\left.\begin{array}{l}\mu\left(\frac{h(3z)}{9}-h(z),\frac{t}{9}\right)\geq\mu'\left(\Omega_\mu\left(z,z,0\right),t\right)\\ \nu\left(\frac{h(3z)}{9}-h(z),\frac{t}{9}\right)\leq\nu'\left(\Omega_\nu\left(z,z,0\right),t\right)\end{array}\right\}\tag{3.29}$$

for all $z\in\mathscr{A}$ and all $t>0$. The rest of the proof is similar to that of Theorem 3.1. □

The following corollary is an immediate consequence of Theorem 3.3, regarding some stabilities of (1.9).

Corollary 3.4. *Assume an even function $H:\mathscr{A}\longrightarrow\mathscr{B}$ satisfies the double inequality*

$$\left.\begin{array}{l}\mu\left(H(z_1,z_2,z_3),t\right)\geq\left\{\begin{array}{l}\mu'\left(\lambda,t\right),\\ \mu'\left(\lambda\sum_{i=1}^{3}||z_i||^{a},t\right),\\ \mu'\left(\lambda\sum_{i=1}^{3}||z_i||^{a_i},t\right),\end{array}\right.\\ \nu\left(H(z_1,z_2,z_3),t\right)\leq\left\{\begin{array}{l}\nu'\left(\lambda,t\right),\\ \nu'\left(\lambda\sum_{i=1}^{3}||z_i||^{a},t\right),\\ \nu'\left(\lambda\sum_{i=1}^{3}||z_i||^{a_i},t\right),\end{array}\right.\end{array}\right\}\tag{3.30}$$

for all $z_1,z_2,z_3\in\mathscr{A}$ and all $t>0$, where $\lambda,a,a_i's$ are constants with $\lambda>0$. Then there exists a unique quadratic mapping $\mathscr{H}_2:\mathscr{A}\longrightarrow\mathscr{B}$ such that

$$\left.\begin{array}{l}\mu\left(h(z)-\mathscr{H}_2(z),t\right)\\ \geq\left\{\begin{array}{l}\mu'\left(\lambda,|9-p|t\right),\\ \mu'\left(2\lambda||z||^{a},t|9-p|\right),\\ \mu'\left(\lambda\left[||z||^{a_1}+||z||^{a_2}\right],t|9-p|\right),\end{array}\right.\\ \nu\left(h(z)-\mathscr{H}_2(z),t\right)\\ \leq\left\{\begin{array}{l}\nu'\left(\lambda,|9-p|t\right),\\ \nu'\left(\lambda||z||^{a},t|9-p|\right),\\ \nu'\left(\lambda\left[||z||^{a_1}+||z||^{a_2}\right],t|9-p|\right),\end{array}\right.\end{array}\right\}\tag{3.31}$$

for all $z\in\mathscr{A}$ and all $t>0$.

Theorem 3.5. *Let $\Delta\in\{1,-1\}$. Let $H:\mathscr{A}\to\mathscr{B}$ be a function satisfying the inequality*

$$\left.\begin{array}{l}\mu\left(H(z_1,z_2,z_3),t\right)\geq\mu'\left(\Omega_\mu\left(z_1,z_2,z_3\right),t\right)\\ \nu\left(H(z_1,z_2,z_3),t\right)\leq\nu'\left(\Omega_\nu\left(z_1,z_2,z_3\right),t\right)\end{array}\right\}\tag{3.32}$$

for all $z_1,z_2,z_3\in\mathscr{A}$ and all $t>0$. Then there exists a unique additive mapping $\mathscr{H}_1:\mathscr{A}\longrightarrow\mathscr{B}$ and a unique quadratic *mapping $\mathscr{H}_2 : \mathscr{A} \longrightarrow \mathscr{B}$ satisfying (1.9) and*

$$\left.\begin{array}{l}
\mu\Big(h(z)-\mathscr{H}_1(z)-\mathscr{H}_2(z),2t\Big)\\
\geq \mu'\left(\Omega_\mu(z,z,0),t|3-p|\right)*\\
\quad \mu'\left(\Omega_\mu(z,z,0),t|3-p|\right)*\\
\quad \mu'\left(\Omega_\mu(-z,-z,0),t|3-p|\right)*\\
\quad \mu'\left(\Omega_\mu(-z,-z,0),t|3-p|\right)*\\
\quad \mu'\left(\Omega_\mu(z,z,0),t|9-p|\right)*\\
\quad \mu'\left(\Omega_\mu(z,z,0),t|9-p|\right)*\\
\quad \mu'\left(\Omega_\mu(-z,-z,0),t|9-p|\right)*\\
\quad \mu'\left(\Omega_\mu(-z,-z,0),t|9-p|\right)\\
\nu\Big(h(z)-\mathscr{H}_1(z)-\mathscr{H}_2(z),2t\Big)\\
\leq \nu'\left(\Omega_\nu(z,z,0),t|3-p|\right)\diamond\\
\quad \nu'\left(\Omega_\nu(z,z,0),t|3-p|\right)\diamond\\
\quad \nu'\left(\Omega_\nu(-z,-z,0),t|3-p|\right)\diamond\\
\quad \nu'\left(\Omega_\nu(-z,-z,0),t|3-p|\right)\diamond\\
\quad \nu'\left(\Omega_\nu(z,z,0),t|9-p|\right)\diamond\\
\quad \nu'\left(\Omega_\nu(z,z,0),t|9-p|\right)\diamond\\
\quad \nu'\left(\Omega_\nu(-z,-z,0),t|9-p|\right)\diamond\\
\quad \nu'\left(\Omega_\nu(-z,-z,0),t|9-p|\right)
\end{array}\right\} \tag{3.33}$$

for all $z \in \mathscr{A}$ and all $t > 0$. Here $\Omega_\mu, \Omega_\nu : \mathscr{A}\times\mathscr{A}\times\mathscr{A} \longrightarrow \mathscr{C}$ are functions such that for some $0 < \left(\frac{p}{3}\right)^\Delta < 1$; $0 < \left(\frac{p}{4}\right)^\Delta < 1$ with conditions (3.3), (3.26) and (3.4), (3.27) for all $z \in \mathscr{A}$ and all $t > 0$ and for all $z_1, z_2, z_3 \in \mathscr{A}$ and all $t > 0$.

Proof. Let $h_O(z) = \frac{h(z)-h(-z)}{2}$ for all $z \in \mathscr{A}$. It is easy to verify that $h_O(0) = 0$ and $h_O(-z) = -h_O(z)$ for all $z \in \mathscr{A}$. By definition of $h_O(z)$, we have

$$\left.\begin{array}{l}
\mu\left(h_O(z_1,z_2,z_3),t\right)\\
=\mu\left(h(z_1,z_2,z_3)-h(-z_1,-z_2,-z_3),2t\right)\\
\geq \mu\left(h(z_1,z_2,z_3),t\right)*\\
\qquad \mu\left(h(-z_1,-z_2,-z_3),t\right)\\
\geq \mu'\left(\Omega_\mu\left(z_1,z_2,z_3\right),t\right)*\\
\qquad \mu'\left(\Omega_\mu\left(-z_1,-z_2,-z_3\right),t\right)\\
\nu\left(h_O(z_1,z_2,z_3),t\right)\\
=\nu\left(h(z_1,z_2,z_3)-h(-z_1,-z_2,-z_3),2t\right)\\
\leq \nu\left(h(z_1,z_2,z_3),t\right)\diamond\\
\qquad \nu\left(h(-z_1,-z_2,-z_3),t\right)\\
\leq \nu'\left(\Omega_\nu\left(z_1,z_2,z_3\right),t\right)\diamond\\
\qquad \nu'\left(\Omega_\nu\left(-z_1,-z_2,-z_3\right),t\right)
\end{array}\right\} \tag{3.34}$$

for all $z_1, z_2, z_3 \in \mathscr{A}$ and all $t > 0$. By Theorem 3.1, there exists a unique additive mapping $\mathscr{H}_1 : \mathscr{A} \longrightarrow \mathscr{B}$ satisfying (1.9) and

$$\left.\begin{array}{l}
\mu\Big(h_O(z)-\mathscr{H}_1(z),t\Big)\\
\geq \mu'\left(\Omega_\mu(z,z,0),t|3-p|\right)*\\
\quad \mu'\left(\Omega_\mu(z,z,0),t|3-p|\right)*\\
\quad \mu'\left(\Omega_\mu(-z,-z,0),t|3-p|\right)*\\
\quad \mu'\left(\Omega_\mu(-z,-z,0),t|3-p|\right)\\
\nu\Big(h_O(z)-\mathscr{H}_1(z),t\Big)\\
\leq \nu'\left(\Omega_\nu(z,z,0),t|3-p|\right)\diamond\\
\quad \nu'\left(\Omega_\nu(z,z,0),t|3-p|\right)\diamond\\
\quad \nu'\left(\Omega_\nu(-z,-z,0),t|3-p|\right)\diamond\\
\quad \nu'\left(\Omega_\nu(-z,-z,0),t|3-p|\right)
\end{array}\right\} \tag{3.35}$$

for all $z \in \mathscr{A}$ and all $t > 0$.

Also, let $h_E(z) = \frac{h(z)+h(-z)}{2}$ for all $z \in \mathscr{A}$. It is easy to verify that $h_E(0) = 0$ and $h_E(-z) = h_E(z)$ for all $z \in \mathscr{A}$. By definition of $h_E(z)$, we have

$$\left.\begin{array}{l}
\mu\left(h_E(z_1,z_2,z_3),t\right)\\
=\mu\left(h(z_1,z_2,z_3)+h(-z_1,-z_2,-z_3),2t\right)\\
\geq \mu\left(h(z_1,z_2,z_3),t\right)*\mu\left(h(-z_1,-z_2,-z_3),t\right)\\
\geq \mu'\left(\Omega_\mu\left(z_1,z_2,z_3\right),t\right)*\mu'\left(\Omega_\mu\left(-z_1,-z_2,-z_3\right),t\right)\\
\nu\left(h_E(z_1,z_2,z_3),t\right)\\
=\nu\left(h(z_1,z_2,z_3)+h(-z_1,-z_2,-z_3),2t\right)\\
\leq \nu\left(h(z_1,z_2,z_3),t\right)\diamond\nu\left(h(-z_1,-z_2,-z_3),t\right)\\
\leq \nu'\left(\Omega_\nu\left(z_1,z_2,z_3\right),t\right)\diamond\nu'\left(\Omega_\nu\left(-z_1,-z_2,-z_3\right),t\right)
\end{array}\right\} \tag{3.36}$$

for all $z_1, z_2, z_3 \in \mathscr{A}$ and all $t > 0$. Also, by Theorem 3.3, there exists a unique quadratic mapping $\mathscr{H}_2 : \mathscr{A} \longrightarrow \mathscr{B}$ satisfying (1.9) and

$$\left.\begin{array}{l}
\mu\Big(h_E(z)-\mathscr{H}_2(z),t\Big)\\
\geq \mu'\left(\Omega_\mu(z,z,0),t|9-p|\right)*\mu'\left(\Omega_\mu(z,z,0),t|9-p|\right)*\\
\mu'\left(\Omega_\mu(-z,-z,0),t|9-p|\right)*\mu'\left(\Omega_\mu(-z,-z,0),t|9-p|\right)\\
\nu\Big(h_E(z)-\mathscr{H}_2(z),t\Big)\\
\leq \nu'\left(\Omega_\nu(z,z,0),t|9-p|\right)\diamond\nu'\left(\Omega_\nu(z,z,0),t|9-p|\right)\diamond\\
\nu'\left(\Omega_\nu(-z,-z,0),t|9-p|\right)\diamond\nu'\left(\Omega_\nu(-z,-z,0),t|9-p|\right)
\end{array}\right\} \tag{3.37}$$

for all $z \in \mathscr{A}$ and all $t > 0$.

Suppose if we define a function $h(z)$ by

$$h(z) = h_O(z) + h_E(z) \tag{3.38}$$

for all $z \in \mathscr{A}$. It follows from (3.35), (3.37), (3.38), we arrive

$$
\left.\begin{array}{l}
\mu\Big(h(z)-\mathscr{H}_1(z)-\mathscr{H}_2(z),2t\Big)\\
\geq \mu\Big(h_O(z)-\mathscr{H}_1(z),t\Big)*\mu\Big(h_E(z)-\mathscr{H}_2(z),t\Big)\\
\geq \mu'\left(\Omega_\mu(z,z,0),t|3-p|\right)*\\
\qquad \mu'\left(\Omega_\mu(z,z,0),t|3-p|\right)*\\
\qquad \mu'\left(\Omega_\mu(-z,-z,0),t|3-p|\right)*\\
\qquad \mu'\left(\Omega_\mu(-z,-z,0),t|3-p|\right)*\\
\qquad \mu'\left(\Omega_\mu(z,z,0),t|9-p|\right)*\\
\qquad \mu'\left(\Omega_\mu(z,z,0),t|9-p|\right)*\\
\qquad \mu'\left(\Omega_\mu(-z,-z,0),t|9-p|\right)*\\
\qquad \mu'\left(\Omega_\mu(-z,-z,0),t|9-p|\right)\\
\nu\Big(h(z)-\mathscr{H}_1(z)-\mathscr{H}_2(z),2t\Big)\\
\leq \nu\Big(h_O(z)-\mathscr{H}_1(z),t\Big)\diamond\nu\Big(h_E(z)-\mathscr{H}_2(z),t\Big)\\
\leq \nu'\left(\Omega_\nu(z,z,0),t|3-p|\right)\diamond\\
\qquad \nu'\left(\Omega_\nu(z,z,0),t|3-p|\right)\diamond\\
\qquad \nu'\left(\Omega_\nu(-z,-z,0),t|3-p|\right)\diamond\\
\qquad \nu'\left(\Omega_\nu(-z,-z,0),t|3-p|\right)\diamond\\
\qquad \nu'\left(\Omega_\nu(z,z,0),t|9-p|\right)\diamond\\
\qquad \nu'\left(\Omega_\nu(z,z,0),t|9-p|\right)\diamond\\
\qquad \nu'\left(\Omega_\nu(-z,-z,0),t|9-p|\right)\diamond\\
\qquad \nu'\left(\Omega_\nu(-z,-z,0),t|9-p|\right)
\end{array}\right\}
$$

for all $z \in \mathscr{A}$ and all $t>0$. □

The following corollary is an immediate consequence of Theorem 3.5, regarding some stabilities of (1.9).

Corollary 3.6. *Assume a function $H:\mathscr{A}\longrightarrow\mathscr{B}$ satisfies the double inequality*

$$
\left.\begin{array}{l}
\mu\left(H(z_1,z_2,z_3),t\right)\geq\left\{\begin{array}{l}\mu'\left(\lambda,t\right),\\ \mu'\left(\lambda\sum_{i=1}^{3}||z_i||^{a},t\right),\\ \mu'\left(\lambda\sum_{i=1}^{3}||z_i||^{a_i},t\right),\end{array}\right.\\
\nu\left(H(z_1,z_2,z_3),t\right)\leq\left\{\begin{array}{l}\nu'\left(\lambda,t\right),\\ \nu'\left(\lambda\sum_{i=1}^{3}||z_i||^{a},t\right),\\ \nu'\left(\lambda\sum_{i=1}^{3}||z_i||^{a_i},t\right),\end{array}\right.
\end{array}\right\}
\tag{3.39}
$$

for all $z_1,z_2,z_3\in\mathscr{A}$ and all $t>0$, where λ, a, a_i's are constants with $\lambda>0$. Then there exists a unique additive mapping $\mathscr{H}_1:\mathscr{A}\longrightarrow\mathscr{B}$ and unique quadratic mapping $\mathscr{H}_2:\mathscr{A}\longrightarrow\mathscr{B}$ such that

$$
\left.\begin{array}{l}
\mu\left(h(z)-\mathscr{H}_1(z)-\mathscr{H}_2(z),2t\right)\\
\geq\left\{\begin{array}{l}\mu'\left(\lambda,|3-p|t+|9-p|t\right),\\ \mu'\left(2\lambda||z||^{a},t|3-p|+t|9-p|\right),\\ \mu'\left(\lambda\left[||z||^{a_1}+||z||^{a_2}\right],t|3-p|+t|9-p|\right),\end{array}\right.\\
\nu\left(h(z)-\mathscr{H}_1(z)-\mathscr{H}_2(z),2t\right)\\
\leq\left\{\begin{array}{l}\mu'\left(\lambda,|3-p|t+|9-p|t\right),\\ \mu'\left(2\lambda||z||^{a},t|3-p|+t|9-p|\right),\\ \mu'\left(\lambda\left[||z||^{a_1}+||z||^{a_2}\right],t|3-p|+t|9-p|\right),\end{array}\right.
\end{array}\right\}
\tag{3.40}
$$

all $z\in\mathscr{A}$ and all $t>0$.

4. IFBS: Stability Results: Substitution - 2

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.9) in IFB space via another substituton .

Theorem 4.1. *Let $\Delta\in\{1,-1\}$. Let $H:\mathscr{A}\to\mathscr{B}$ be an odd function satisfying the inequality*

$$
\left.\begin{array}{l}
\mu\left(H(z_1,z_2,z_3),t\right)\geq\mu'\left(\Omega_\mu\left(z_1,z_2,z_3\right),t\right)\\
\nu\left(H(z_1,z_2,z_3),t\right)\leq\nu'\left(\Omega_\nu\left(z_1,z_2,z_3\right),t\right)
\end{array}\right\}
\tag{4.1}
$$

for all $z_1,z_2,z_3\in\mathscr{A}$ and all $t>0$. Then there exists a unique additive mapping $\mathscr{H}_1:\mathscr{A}\longrightarrow\mathscr{B}$ satisfying (1.9) and

$$
\left.\begin{array}{l}
\mu\Big(\mathscr{H}_1(z)-h(z),t\Big)\geq\mu'\left(\Omega^{O}_{\mu}(z,z,z),\tfrac{t}{2}|3-p|\right)\\
\nu\Big(\mathscr{H}_1(z)-h(z),t\Big)\leq\nu'\left(\Omega^{O}_{\nu}(z,z,z),\tfrac{t}{2}|3-p|\right)
\end{array}\right\}
\tag{4.2}
$$

where

$$
\left.\begin{array}{l}
\mu'\left(\Omega^{O}_{\mu}(z,z,z),t\right)\\
=\mu'\left(\Omega_\mu\left(\tfrac{z}{2},\tfrac{z}{2},\tfrac{z}{2}\right),t\right)*\mu'\left(\Omega_\mu\left(\tfrac{z}{2},\tfrac{z}{2},\tfrac{z}{6}\right),t\right)\\
\nu'\left(\Omega^{O}_{\nu}(z,z,z),t\right)\\
=\nu'\left(\Omega_\nu\left(\tfrac{z}{2},\tfrac{z}{2},\tfrac{z}{2}\right),t\right)\diamond\nu'\left(\Omega_\nu\left(\tfrac{z}{2},\tfrac{z}{2},\tfrac{z}{6}\right),t\right)
\end{array}\right\}
\tag{4.3}
$$

for all $z\in\mathscr{A}$ and all $t>0$. Here $\Omega_\mu,\Omega_\nu:\mathscr{A}\times\mathscr{A}\times\mathscr{A}\longrightarrow\mathscr{C}$ are functions such that for some $0<\left(\dfrac{p}{3}\right)^{\Delta}<1$ with

$$
\left.\begin{array}{l}
\mu'\left(\Omega_\mu\left(3^{n\Delta}z,3^{n\Delta}z,3^{n\Delta}z\right),t\right)\geq\mu'\left(p^{n\Delta}\Omega_\mu\left(z,z,z\right),t\right)\\
\nu'\left(\Omega_\nu\left(3^{n\Delta}z,3^{n\Delta}z,3^{n\Delta}z\right),t\right)\leq\nu'\left(p^{n\Delta}\Omega_\nu\left(z,z,z\right),t\right)
\end{array}\right\}
\tag{4.4}
$$

for all $z\in\mathscr{A}$ and all $t>0$ and

$$
\left.\begin{array}{l}
\lim\limits_{n\to\infty}\mu'\left(\Omega_\mu\left(3^{\Delta n}z_1,3^{\Delta n}z_2,3^{n\Delta}z_3\right),3^{\Delta n}t\right)=1\\
\lim\limits_{n\to\infty}\nu'\left(\Omega_\nu\left(3^{\Delta n}z_1,3^{\Delta n}z_2,3^{n\Delta}z_3\right),3^{\Delta n}t\right)=0
\end{array}\right\}
\tag{4.5}
$$

for all $z_1,z_2,z_3\in\mathscr{A}$ and all $t>0$.

Proof. **Case (1):** Let $\Delta=1$.

Using oddness of h and transform (z_1,z_2,z_3) by $\left(\frac{z}{2},\frac{z}{2},\frac{z}{2}\right)$ and $\left(\frac{z}{2},\frac{z}{2},\frac{z}{6}\right)$ in (4.1), respectively, we have

$$
\left.\begin{array}{l}
\mu\left(h_o(3z)+h_o(2z)-h_o(z)-8h_o\left(\tfrac{z}{2}\right),t\right)\\
\geq\mu'\left(\Omega_\mu\left(\tfrac{z}{2},\tfrac{z}{2},\tfrac{z}{2}\right),t\right)\\
\nu\left(h_o(3z)+h_o(2z)-h_o(z)-8h_o\left(\tfrac{z}{2}\right),t\right)\\
\leq\nu'\left(\Omega_\nu\left(\tfrac{z}{2},\tfrac{z}{2},\tfrac{z}{2}\right),t\right)
\end{array}\right\}
\tag{4.6}
$$

and

$$
\left.\begin{array}{l}
\mu\left(h_o(2x)+h_o(z)+h_o(z)-8h_o\left(\tfrac{z}{2}\right),t\right)\\
\geq\mu'\left(\Omega_\mu\left(\tfrac{z}{2},\tfrac{z}{2},\tfrac{z}{6}\right),t\right)\\
\nu\left(h_o(2x)+h_o(z)+h_o(z)-8h_o\left(\tfrac{z}{2}\right),t\right)\\
\leq\nu'\left(\Omega_\nu\left(\tfrac{z}{2},\tfrac{z}{2},\tfrac{z}{6}\right),t\right)
\end{array}\right\}
\tag{4.7}
$$

for all $z \in \mathscr{A}$ and all $t > 0$. It follows from (4.6), (4.7) and [(IFN5)], [(IFN11)], we arrive at

$$\left.\begin{array}{l}\mu\left(h_o(3x)-3h_o(z),2t\right)\\ \geq \mu\left(h_o(3z)+h_o(2z)-h_o(z)-8h_o\left(\frac{z}{2}\right),t\right)*\\ \qquad \mu\left(h_o(2z)+h_o(z)+h_o(z)-8h_o\left(\frac{z}{2}\right),t\right)\\ \geq \mu'\left(\Omega_\mu\left(\frac{z}{2},\frac{z}{2},\frac{z}{2}\right),t\right)*\mu'\left(\Omega_\nu\left(\frac{z}{2},\frac{z}{2},\frac{z}{6}\right),t\right)\\ = \mu'\left(\Omega_\mu^O(z),t\right)\\ \nu\left(h_o(3x)-3h_o(z),2t\right)\\ \leq \nu\left(h_o(3z)+h_o(2z)-h_o(z)-8h_o\left(\frac{z}{2}\right),t\right)\diamond\\ \qquad \nu\left(h_o(2z)+h_o(z)+h_o(z)-8h_o\left(\frac{z}{2}\right),t\right)\\ \leq \nu'\left(\Omega_\mu\left(\frac{z}{2},\frac{z}{2},\frac{z}{2}\right),t\right)\diamond\nu'\left(\Omega_\nu\left(\frac{z}{2},\frac{z}{2},\frac{z}{6}\right),t\right)\\ = \nu'\left(\Omega_\nu^O(z),t\right)\end{array}\right\} \tag{4.8}$$

for all $z \in \mathscr{A}$ and all $t > 0$. Using (IFN4) and (IFN10) in (4.8), we arrive

$$\left.\begin{array}{l}\mu\left(\frac{h(3z)}{3}-h(z),\frac{2t}{3}\right)\geq \mu'\left(\Omega_\mu^O(z),t\right)\\ \nu\left(\frac{h(3z)}{3}-h(z),\frac{2t}{3}\right)\leq \nu'\left(\Omega_\nu^O(z),t\right)\end{array}\right\} \tag{4.9}$$

for all $z \in \mathscr{A}$ and all $t > 0$. Substituting z by $3^n z$ in (4.9), we have

$$\left.\begin{array}{l}\mu\left(\frac{h(3^{n+1}z)}{3}-h(3^n z),\frac{2t}{3}\right)\geq \mu'\left(\Omega_\mu^O(3^n z),t\right)\\ \nu\left(\frac{h(3^{n+1}z)}{3}-h(3^n z),\frac{2t}{3}\right)\leq \nu'\left(\Omega_\nu^O(3^n z),t\right)\end{array}\right\} \tag{4.10}$$

for all $z \in \mathscr{A}$ and all $t > 0$. It is easy to verify from (4.10) and using (4.4), (IFN4), (IFN10) that

$$\left.\begin{array}{l}\mu\left(\frac{h(3^{n+1}z)}{3^{(n+1)}}-\frac{h(3^n z)}{3^n},\frac{2t}{3\cdot 3^n}\right)\geq \mu'\left(\Omega_\mu^O(z),\frac{t}{p^n}\right)\\ \nu\left(\frac{h(3^{n+1}z)}{3^{(n+1)}}-\frac{h(3^n z)}{3^n},\frac{2t}{3\cdot 3^n}\right)\leq \nu'\left(\Omega_\nu^O(z),\frac{t}{p^n}\right)\end{array}\right\} \tag{4.11}$$

for all $z \in \mathscr{A}$ and all $t > 0$. Interchanging t into $p^n t$ in (4.11), we have

$$\left.\begin{array}{l}\mu\left(\frac{h(3^{n+1}z)}{3^{(n+1)}}-\frac{h(3^n z)}{3^n},\frac{2t\cdot p^n}{3\cdot 3^n}\right)\geq \mu'\left(\Omega_\mu^O(z),t\right)\\ \nu\left(\frac{h(3^{n+1}z)}{3^{(n+1)}}-\frac{h(3^n z)}{3^n},\frac{2t\cdot p^n}{3\cdot 3^n}\right)\leq \nu'\left(\Omega_\nu^O(z),t\right)\end{array}\right\} \tag{4.12}$$

for all $z \in \mathscr{A}$ and all $t > 0$. It is easy to see that

$$\frac{h(3^n z)}{3^n}-h(z)=\sum_{i=0}^{n-1}\frac{h(3^{i+1}z)}{3^{(i+1)}}-\frac{h(3^i z)}{3^i} \tag{4.13}$$

for all $z \in \mathscr{A}$. From equations (4.11) and (4.13), we get

$$\left.\begin{array}{l}\mu\left(\frac{h(3^n z)}{3^n}-h(z),\sum_{i=0}^{n-1}\frac{2t\ p^i}{3\cdot 3^i}\right)\\ \geq \prod_{i=0}^{n-1}\mu\left(\frac{h(3^{i+1}z)}{3^{(i+1)}}-\frac{h(3^i z)}{3^i},\frac{2t\ p^i}{3\cdot 3^i}\right)\\ \nu\left(\frac{h(3^n z)}{3^n}-h(z),\sum_{i=0}^{n-1}\frac{2t\ p^i}{3\cdot 3^i}\right)\\ \leq \prod_{i=0}^{n-1}\nu\left(\frac{h(3^{i+1}z)}{3^{(i+1)}}-\frac{h(3^i z)}{3^i},\frac{2t\ p^i}{3\cdot 3^i}\right)\end{array}\right\} \tag{4.14}$$

with

$$\prod_{i=0}^{n-1}c_j=c_1*c_2*\cdots*c_n \qquad and \qquad \prod_{i=0}^{n-1}d_j=d_1\diamond d_2\diamond\cdots\diamond d_n$$

for all $z \in \mathscr{A}$ and all $t > 0$. Thus

$$\left.\begin{array}{l}\mu\left(\frac{h(3^n z)}{3^n}-h(z),\sum_{i=0}^{n-1}\frac{2t\ p^i}{3\cdot 3^i}\right)\\ \geq \prod_{i=0}^{n-1}\mu'\left(\Omega_\mu^O(z),t\right)=\mu'\left(\Omega_\mu^O(z),t\right)\\ \nu\left(\frac{h(3^n z)}{3^n}-h(z),\sum_{i=0}^{n-1}\frac{2t\ p^i}{3\cdot 3^i}\right)\\ \leq \prod_{i=0}^{n-1}\nu'\left(\Omega_\nu^O(z),t\right)=\nu'\left(\Omega_\nu^O(z),t\right)\end{array}\right\} \tag{4.15}$$

for all $z \in \mathscr{A}$ and all $t > 0$. Replacing z by $3^m z$ in (4.15) and using (4.5), (IFN4), (IFN10), we obtain

$$\left.\begin{array}{l}\mu\left(\frac{h(3^{n+m}z)}{3^{(n+m)}}-\frac{h(3^m z)}{3^m},\sum_{i=0}^{n-1}\frac{2t\ p^i}{3\cdot 3^{(i+m)}}\right)\\ \geq \mu'\left(\Omega_\mu^O(z)(3^m z),t\right)=\mu'\left(\Omega_\mu^O(z),\frac{t}{p^m}\right)\\ \nu\left(\frac{h(3^{n+m}z)}{3^{(n+m)}}-\frac{h(3^m z)}{3^m},\sum_{i=0}^{n-1}\frac{2t\ p^i}{3\cdot 3^{(i+m)}}\right)\\ \leq \nu'\left(\Omega_\nu^O(z)(3^m z),t\right)=\nu'\left(\Omega_\nu^O(z),\frac{t}{p^m}\right)\end{array}\right\} \tag{4.16}$$

for all $z \in \mathscr{A}$ and all $t > 0$ and all $m,n \geq 0$. Interchanging t by $p^m t$ in (4.16), we get

$$\left.\begin{array}{l}\mu\left(\frac{h(3^{n+m}z)}{3^{(n+m)}}-\frac{h(3^m z)}{3^m},\sum_{i=0}^{n-1}\frac{2t\ p^{i+m}}{3\cdot 3^{(i+m)}}\right)\geq \mu'\left(\Omega_\mu^O(z),t\right)\\ \nu\left(\frac{h(3^{n+m}z)}{3^{(n+m)}}-\frac{h(3^m z)}{3^m},\sum_{i=0}^{n-1}\frac{2t\ p^{i+m}}{3\cdot 3^{(i+m)}}\right)\leq \nu'\left(\Omega_\nu^O(z),t\right)\end{array}\right\} \tag{4.17}$$

for all $z \in \mathscr{A}$ and all $t > 0$ and all $m,n \geq 0$. Using (IFN4), (IFN10) in (4.17) implies that

$$\left.\begin{array}{l}\mu\left(\frac{h(3^{n+m}z)}{3^{(n+m)}}-\frac{h(3^m z)}{3^m},t\right)\geq \mu'\left(\Omega_\mu^O(z),\frac{t}{\sum_{i=m}^{n-1}\frac{2\ p^i}{3\cdot 3^i}}\right)\\ \nu\left(\frac{h(3^{n+m}z)}{3^{(n+m)}}-\frac{h(3^m z)}{3^m},t\right)\leq \nu'\left(\Omega_\nu^O(z),\frac{t}{\sum_{i=m}^{n-1}\frac{2\ p^i}{3\cdot 3^i}}\right)\end{array}\right\} \tag{4.18}$$

holds for all $z \in \mathscr{A}$ and all $t > 0$ and all $m,n \geq 0$. Since $0 < p < 1$ and $\sum_{i=0}^{n}\left(\frac{p}{1}\right)^i < \infty$. The Cauchy criterion for convergence in IFNS shows that the sequence $\left\{\frac{h(3^n z)}{3^n}\right\}$ is Cauchy in $(\mathscr{B},\mu,\nu)$. Since $(\mathscr{B},\mu,\nu)$ is a complete IFN-space this sequence converges to some point $\mathscr{H}_1(z) \in \mathscr{B}$. So, one can define the mapping $\mathscr{H}_1 : \mathscr{A} \longrightarrow \mathscr{B}$ by

$$\lim_{n\to\infty}\mu\left(\frac{h(3^n z)}{3^n}-\mathscr{H}_1(z),t\right)=1,$$

$$\lim_{n\to\infty}\nu\left(\frac{h(3^n z)}{3^n}-\mathscr{H}_1(z),t\right)=0$$

for all $z \in \mathscr{A}$ and all $t > 0$. Hence

$$\frac{h(3^n z)}{3^n}\xrightarrow{IF}\mathscr{H}_1(z), \quad as \quad n\to\infty.$$

Letting $m=0$ in (4.18), we arrive

$$\left.\begin{array}{l}\mu\left(\frac{h(3^nz)}{3^n}-h(z),t\right)\geq\mu'\left(\Omega_\mu^O(z),\frac{t}{\sum_{i=0}^{n-1}\frac{2}{3}\frac{p^i}{3^i}}\right)\\ \nu\left(\frac{h(3^nz)}{3^n}-h(z),t\right)\leq\nu'\left(\Omega_\nu^O(z),\frac{t}{\sum_{i=0}^{n-1}\frac{2}{3}\frac{p^i}{3^i}}\right)\end{array}\right\}\tag{4.19}$$

for all $z\in\mathscr{A}$ and all $t>0$. Letting n tend to infinity in (4.19), we have

$$\left.\begin{array}{l}\mu\left(\mathscr{H}_1(z)-h(z),t\right)\geq\mu'\left(\Omega_\mu^O(z),\frac{t}{2}(3-p)\right)\\ \nu\left(\mathscr{H}_1(z)-h(z),t\right)\leq\nu'\left(\Omega_\nu^O(z),\frac{t}{2}(3-p)\right)\end{array}\right\}\tag{4.20}$$

for all $z\in\mathscr{A}$ and all $t>0$. To prove $\mathscr{H}_1$ satisfies (1.9) and it is unique the proof is similar to that of 3.1. □

The following corollary is an immediate consequence of Theorem regarding some the stabilities of (1.9)

Corollary 4.2. *Assume an odd function $H:\mathscr{A}\longrightarrow\mathscr{B}$ satisfies the double inequality*

$$\left.\begin{array}{l}\mu\left(H(z_1,z_2,z_3),t\right)\geq\begin{cases}\mu'(\lambda,t),\\ \mu'\left(\lambda\sum_{i=1}^3||z_i||^a,t\right),\\ \mu'\left(\lambda\sum_{i=1}^3||z_i||^{a_i},t\right),\\ \mu'\left(\lambda\prod_{i=1}^3||z_i||^a,t\right),\\ \mu'\left(\lambda\prod_{i=1}^3||z_i||^{a_i},t\right),\end{cases}\\ \nu\left(H(z_1,z_2,z_3),t\right)\leq\begin{cases}\nu'(\lambda,t),\\ \nu'\left(\lambda\sum_{i=1}^3||z_i||^a,t\right),\\ \nu'\left(\lambda\sum_{i=1}^3||z_i||^{a_i},t\right),\\ \nu'\left(\lambda\prod_{i=1}^3||z_i||^a,t\right),\\ \nu'\left(\lambda\prod_{i=1}^3||z_i||^{a_i},t\right),\end{cases}\end{array}\right\}\tag{4.21}$$

for all $z_1,z_2,z_3\in\mathscr{A}$ and all $t>0$, where $\lambda,a,a_i's$ are constants with $\lambda>0$. Then there exists a unique additive mapping $\mathscr{H}_1:\mathscr{A}\longrightarrow\mathscr{B}$ such that

$$\left.\begin{array}{l}\mu\left(h(z)-\mathscr{H}_1(z),t\right)\\ \geq\begin{cases}\mu'\left(2\lambda,|3-p|t\right),\\ \mu'\left(2\lambda||z||^a\left[\frac{5}{2^a}+\frac{1}{6^a}\right],t|3-p|\right),\\ \mu'\Big(2\lambda\left[\frac{2||z||^{a_1}}{2^{a_1}}+\frac{2||z||^{a_2}}{2^{a_2}}+\frac{||z||^{a_3}}{2^{a_3}}+\frac{||z||^{a_3}}{6^{a_3}}\right],\\ \qquad t|3-p|\Big),\\ \mu'\left(2\lambda||z||^{3a}\left[\frac{1}{2^{3a}}+\frac{1}{2^{2a}6^a}\right],t|3-p|\right),\\ \mu'\Big(2\lambda||z||^{a_1+a_2+a_3}\left[\frac{1}{2^{a_1+a_2+a_3}}+\frac{1}{2^{a_1+a_2}\cdot6^{a_3}}\right],\\ \qquad t|3-p|\Big),\end{cases}\\ \nu\left(h(z)-\mathscr{H}_1(z),t\right)\\ \leq\begin{cases}\nu'\left(2\lambda,|3-p|t\right),\\ \nu'\left(2\lambda||z||^a\left[\frac{5}{2^a}+\frac{1}{6^a}\right],t|3-p|\right),\\ \nu'\Big(2\lambda\left[\frac{2||z||^{a_1}}{2^{a_1}}+\frac{2||z||^{a_2}}{2^{a_2}}+\frac{||z||^{a_3}}{2^{a_3}}+\frac{||z||^{a_3}}{6^{a_3}}\right],\\ \qquad t|3-p|\Big),\\ \nu'\left(2\lambda||z||^{3a}\left[\frac{1}{2^{3a}}+\frac{1}{2^{2a}6^a}\right],t|3-p|\right),\\ \nu'\Big(2\lambda||z||^{a_1+a_2+a_3}\left[\frac{1}{2^{a_1+a_2+a_3}}+\frac{1}{2^{a_1+a_2}\cdot6^{a_3}}\right],\\ \qquad t|3-p|\Big),\end{cases}\end{array}\right\}\tag{4.22}$$

for all $z\in\mathscr{A}$ and all $t>0$.

Theorem 4.3. *Let $\Delta\in\{1,-1\}$. Let $H:\mathscr{A}\to\mathscr{B}$ be an even function satisfying the inequality*

$$\left.\begin{array}{l}\mu\left(H(z_1,z_2,z_3),t\right)\geq\mu'\left(\Omega_\mu\left(z_1,z_2,z_3\right),t\right)\\ \nu\left(H(z_1,z_2,z_3),t\right)\leq\nu'\left(\Omega_\nu\left(z_1,z_2,z_3\right),t\right)\end{array}\right\}\tag{4.23}$$

for all $z_1,z_2,z_3\in\mathscr{A}$ and all $t>0$. Then there exists a unique quadratic mapping $\mathscr{H}_2:\mathscr{A}\longrightarrow\mathscr{B}$ satisfying (1.9) and

$$\left.\begin{array}{l}\mu\left(\mathscr{H}_2(z)-h(z),t\right)\geq\mu'\left(\Omega_\mu^E(z),t|16-p|\right)\\ \nu\left(\mathscr{H}_2(z)-h(z),t\right)\leq\nu'\left(\Omega_\nu^E(z),t|16-p|\right)\end{array}\right\}\tag{4.24}$$

where

$$\left.\begin{array}{l}\mu'\left(\Omega_\mu^E(z),t\right)=\mu'\left(\Omega_\mu\left(\frac{z}{2},\frac{z}{2},\frac{z}{3}\right),t\right)\\ \nu'\left(\Omega_\mu^E(z),t\right)=\nu'\left(\Omega_\nu\left(\frac{z}{2},\frac{z}{2},\frac{z}{3}\right),t\right)\end{array}\right\}\tag{4.25}$$

for all $z\in\mathscr{A}$ and all $t>0$. Here $\Omega_\mu,\Omega_\nu:\mathscr{A}\times\mathscr{A}\times\mathscr{A}\longrightarrow\mathscr{C}$ are functions such that for some $0<\left(\frac{p}{4}\right)^\Delta<1$ with

$$\left.\begin{array}{l}\mu'\left(\Omega_\mu\left(4^{n\Delta}z,4^{n\Delta}z,4^{n\Delta}z\right),t\right)\geq\mu'\left(p^{n\Delta}\Omega_\mu\left(z,z,z\right),t\right)\\ \nu'\left(\Omega_\nu\left(4^{n\Delta}z,4^{n\Delta}z,4^{n\Delta}z\right),t\right)\leq\nu'\left(p^{n\Delta}\Omega_\nu\left(z,z,z\right),t\right)\end{array}\right\}\tag{4.26}$$

for all $z\in\mathscr{A}$ and all $t>0$ and

$$\left.\begin{array}{l}\lim_{n\to\infty}\mu'\left(\Omega_\mu\left(4^{\Delta n}z_1,4^{\Delta n}z_2,4^{n\Delta}z_3\right),16^{\Delta n}t\right)=1\\ \lim_{n\to\infty}\nu'\left(\Omega_\nu\left(4^{\Delta n}z_1,4^{\Delta n}z_2,4^{n\Delta}z_3\right),16^{\Delta n}t\right)=0\end{array}\right\}\tag{4.27}$$

for all $z_1,z_2,z_3\in\mathscr{A}$ and all $t>0$.

Proof. **Case (1):** Let $\Delta = 1$.

Using evenness of h and transform (z_1, z_2, z_3) by $\left(\frac{z}{2}, \frac{z}{2}, \frac{z}{3}\right)$ in (4.23), we have

$$\left.\begin{array}{l} \mu\left(h_e(4x) - 16h_e(z), t\right) \\ \geq \mu'\left(\Omega_\mu\left(\frac{z}{2}, \frac{z}{2}, \frac{z}{3}\right), t\right) = \mu'\left(\Omega_\mu^E(z), t\right) \\ \nu\left(h_e(4x) - 16h_e(z), t\right) \\ \leq \nu'\left(\Omega_\mu\left(\frac{z}{2}, \frac{z}{2}, \frac{z}{3}\right), t\right) = \nu'\left(\Omega_\nu^E(z), t\right) \end{array}\right\} \quad (4.28)$$

for all $z \in \mathscr{A}$ and all $t > 0$. Using (IFN4) and (IFN10) in (4.28), we arrive

$$\left.\begin{array}{l} \mu\left(\frac{h(4z)}{16} - h(z), \frac{t}{16}\right) \geq \mu'\left(\Omega_\mu^E(z), t\right) \\ \nu\left(\frac{h(4z)}{16} - h(z), \frac{t}{16}\right) \leq \nu'\left(\Omega_\nu^E(z), t\right) \end{array}\right\} \quad (4.29)$$

for all $z \in \mathscr{A}$ and all $t > 0$. The rest of the proof is similar to that of Theorem 3.1. □

The following corollary is an immediate consequence of Theorem 3.3, regarding some stabilities of (1.9)

Corollary 4.4. *Assume an even function $H : \mathscr{A} \longrightarrow \mathscr{B}$ satisfies the double inequality*

$$\begin{array}{l} \mu\left(H(z_1, z_2, z_3), t\right) \geq \left\{\begin{array}{l} \mu'\left(\lambda, t\right), \\ \mu'\left(\lambda \sum_{i=1}^{3} ||z_i||^a, t\right), \\ \mu'\left(\lambda \sum_{i=1}^{3} ||z_i||^{a_i}, t\right), \\ \mu'\left(\lambda \prod_{i=1}^{3} ||z_i||^a, t\right), \\ \mu'\left(\lambda \prod_{i=1}^{3} ||z_i||^{a_i}, t\right), \end{array}\right. \\ \nu\left(H(z_1, z_2, z_3), t\right) \leq \left\{\begin{array}{l} \nu'\left(\lambda, t\right), \\ \nu'\left(\lambda \sum_{i=1}^{3} ||z_i||^a, t\right), \\ \nu'\left(\lambda \sum_{i=1}^{3} ||z_i||^{a_i}, t\right), \\ \nu'\left(\lambda \prod_{i=1}^{3} ||z_i||^a, t\right), \\ \nu'\left(\lambda \prod_{i=1}^{3} ||z_i||^{a_i}, t\right), \end{array}\right. \end{array} \quad (4.30)$$

for all $z_1, z_2, z_3 \in \mathscr{A}$ and all $t > 0$, where $\lambda, a, d_i' s$ are constants with $\lambda > 0$. Then there exists a unique quadratic mapping $\mathscr{H}_2 : \mathscr{A} \longrightarrow \mathscr{B}$ such that

$$\begin{array}{l} \mu\left(h(z) - \mathscr{H}_2(z), t\right) \geq \left\{\begin{array}{l} \mu'\left(\lambda, |16 - p|t\right), \\ \mu'\left(\lambda ||z||^a \left[\frac{2}{2^a} + \frac{1}{3^a}\right], t|16 - p|\right), \\ \mu'\left(\lambda \left[\frac{||z||^{a_1}}{2^{a_1}} + \frac{||z||^{a_2}}{2^{a_2}} + \frac{||z||^{a_3}}{3^{a_3}}\right], t|16 - p|\right), \\ \mu'\left(\lambda ||z||^{3a} \left[\frac{1}{2^{2a} 3^a}\right], t|16 - p|\right), \\ \mu'\left(\lambda ||z||^{a_1 + a_2 + a_3} \left[\frac{1}{2^{a_1 + a_2} \cdot 3^{a_3}}\right], t|16 - p|\right), \end{array}\right. \\ \nu\left(h(z) - \mathscr{H}_2(z), t\right) \leq \left\{\begin{array}{l} \nu'\left(\lambda, |16 - p|t\right), \\ \nu'\left(\lambda ||z||^a \left[\frac{2}{2^a} + \frac{1}{3^a}\right], t|16 - p|\right), \\ \nu'\left(\lambda \left[\frac{||z||^{a_1}}{2^{a_1}} + \frac{||z||^{a_2}}{2^{a_2}} + \frac{||z||^{a_3}}{3^{a_3}}\right], t|16 - p|\right), \\ \nu'\left(\lambda ||z||^{3a} \left[\frac{1}{2^{2a} 3^a}\right], t|16 - p|\right), \\ \nu'\left(\lambda ||z||^{a_1 + a_2 + a_3} \left[\frac{1}{2^{a_1 + a_2} \cdot 3^{a_3}}\right], t|16 - p|\right), \end{array}\right. \end{array} \quad (4.31)$$

for all $z \in \mathscr{A}$ and all $t > 0$.

Theorem 4.5. *Let $\Delta \in \{1, -1\}$. Let $H : \mathscr{A} \to \mathscr{B}$ be a function satisfying the inequality*

$$\left.\begin{array}{l} \mu\left(H(z_1, z_2, z_3), t\right) \geq \mu'\left(\Omega_\mu\left(z_1, z_2, z_3\right), t\right) \\ \nu\left(H(z_1, z_2, z_3), t\right) \leq \nu'\left(\Omega_\nu\left(z_1, z_2, z_3\right), t\right) \end{array}\right\} \quad (4.32)$$

for all $z_1, z_2, z_3 \in \mathscr{A}$ and all $t > 0$. Then there exists a unique additive mapping $\mathscr{H}_1 : \mathscr{A} \longrightarrow \mathscr{B}$ and a unique quadratic mapping $\mathscr{H}_2 : \mathscr{A} \longrightarrow \mathscr{B}$ satisfying (1.9) and

$$\left.\begin{array}{l} \mu\left(h(z) - \mathscr{H}_1(z) - \mathscr{H}_2(z), t\right) \geq \mu'\left(\Omega_\mu^{OE}(z), t\right) \\ \nu\left(h(z) - \mathscr{H}_1(z) - \mathscr{H}_2(z), t\right) \leq \nu'\left(\Omega_\nu^{OE}(z), t\right) \end{array}\right\} \quad (4.33)$$

where

$$\left.\begin{array}{l} \mu'\left(\Omega_\mu^{OE}(z), t\right) = \mu'\left(\Omega_\mu^O(z) + \Omega_\mu^E(z), \frac{t}{2}|3 - p| + t|16 - p|\right) \\ \nu'\left(\Omega_\mu^{OE}(z), t\right) = \nu'\left(\Omega_\nu^O(z) + \Omega_\nu^E(z), \frac{t}{2}|3 - p| + t|16 - p|\right) \end{array}\right\} \quad (4.34)$$

for all $z \in \mathscr{A}$ and all $t > 0$. Here $\Omega_\mu, \Omega_\nu : \mathscr{A} \times \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{C}$ are functions such that for some $0 < \left(\frac{p}{3}\right)^\Delta < 1$; $0 < \left(\frac{p}{4}\right)^\Delta < 1$ with conditions (4.4), (4.26) and (4.5), (4.27) for all $z \in \mathscr{A}$ and all $t > 0$ and for all $z_1, z_2, z_3 \in \mathscr{A}$ and all $t > 0$.

Proof. Let $h_O(z) = \frac{h(z) - h(-z)}{2}$ for all $z \in \mathscr{A}$. It is easy to verify that $h_O(0) = 0$ and $h_O(-z) = -h_O(z)$ for all $z \in \mathscr{A}$. By definition of $h_O(z)$, we have

$$\left.\begin{array}{l}\mu\left(h_O(z_1,z_2,z_3),t\right)\\=\mu\left(h(z_1,z_2,z_3)-h(-z_1,-z_2,-z_3),2t\right)\\\geq\mu\left(h(z_1,z_2,z_3),t\right)*\mu\left(h(-z_1,-z_2,-z_3),t\right)\\\geq\mu'\left(\Omega_\mu\left(z_1,z_2,z_3\right),t\right)*\mu'\left(\Omega_\mu\left(-z_1,-z_2,-z_3\right),t\right)\\\nu\left(h_O(z_1,z_2,z_3),t\right)\\=\nu\left(h(z_1,z_2,z_3)-h(-z_1,-z_2,-z_3),2t\right)\\\leq\nu\left(h(z_1,z_2,z_3),t\right)\diamond\nu\left(h(-z_1,-z_2,-z_3),t\right)\\\leq\nu'\left(\Omega_\nu\left(z_1,z_2,z_3\right),t\right)\diamond\nu'\left(\Omega_\nu\left(-z_1,-z_2,-z_3\right),t\right)\end{array}\right\} \tag{4.35}$$

for all $z_1,z_2,z_3\in\mathscr{A}$ and all $t>0$. By Theorem 4.1, there exists a unique additive mapping $\mathscr{H}_1:\mathscr{A}\longrightarrow\mathscr{B}$ satisfying (1.9) and

$$\left.\begin{array}{l}\mu\Big(h_O(z)-\mathscr{H}_1(z),t\Big)\\\geq\mu'\left(\Omega^O_\mu(z),\frac{t}{2}|3-p|\right)*\mu'\left(\Omega^O_\mu(z),\frac{t}{2}|3-p|\right)\\\geq\mu'\left(\Omega^O_\mu(z),\frac{t}{2}|3-p|\right)\\\nu\Big(h_O(z)-\mathscr{H}_1(z),t\Big)\\\leq\nu'\left(\Omega^O_\nu(z),\frac{t}{2}|3-p|\right)\diamond\nu'\left(\Omega^O_\nu(z),\frac{t}{2}|3-p|\right)\\\leq\nu'\left(\Omega^O_\nu(z),\frac{t}{2}|3-p|\right)\end{array}\right\} \tag{4.36}$$

where $\mu'\left(\Omega^O_\mu(z),t\right)$ and $\nu'\left(\Omega^O_\nu(z),t\right)$ are defined in (4.3) for all $z\in\mathscr{A}$ and all $t>0$.

Also, let $h_E(z)=\frac{h(z)+h(-z)}{2}$ for all $z\in\mathscr{A}$. It is easy to verify that $h_E(0)=0$ and $h_E(-z)=h_E(z)$ for all $z\in\mathscr{A}$. By definition of $h_E(z)$, we have

$$\left.\begin{array}{l}\mu\left(h_E(z_1,z_2,z_3),t\right)\\=\mu\left(h(z_1,z_2,z_3)+h(-z_1,-z_2,-z_3),2t\right)\\\geq\mu\left(h(z_1,z_2,z_3),t\right)*\mu\left(h(-z_1,-z_2,-z_3),t\right)\\\geq\mu'\left(\Omega_\mu\left(z_1,z_2,z_3\right),t\right)*\mu'\left(\Omega_\mu\left(-z_1,-z_2,-z_3\right),t\right)\\\nu\left(h_E(z_1,z_2,z_3),t\right)\\=\nu\left(h(z_1,z_2,z_3)+h(-z_1,-z_2,-z_3),2t\right)\\\leq\nu\left(h(z_1,z_2,z_3),t\right)\diamond\nu\left(h(-z_1,-z_2,-z_3),t\right)\\\leq\nu'\left(\Omega_\nu\left(z_1,z_2,z_3\right),t\right)\diamond\nu'\left(\Omega_\nu\left(-z_1,-z_2,-z_3\right),t\right)\end{array}\right\} \tag{4.37}$$

for all $z_1,z_2,z_3\in\mathscr{A}$ and all $t>0$. Also, by Theorem 4.3, there exists a unique quadratic mapping $\mathscr{H}_2:\mathscr{A}\longrightarrow\mathscr{B}$ satisfying (1.9) and

$$\left.\begin{array}{l}\mu\Big(h_E(z)-\mathscr{H}_2(z),t\Big)\\\geq\mu'\left(\Omega^E_\mu(z),t|16-p|\right)*\mu'\left(\Omega^E_\mu(z),t|16-p|\right)\\\geq\mu'\left(\Omega^E_\mu(z),t|16-p|\right)\\\nu\Big(h_E(z)-\mathscr{H}_2(z),t\Big)\\\leq\nu'\left(\Omega^E_\nu(z),t|16-p|\right)\diamond\nu'\left(\Omega^E_\nu(z),t|16-p|\right)\\\leq\nu'\left(\Omega^E_\nu(z),t|16-p|\right)\end{array}\right\} \tag{4.38}$$

where $\mu'\left(\Omega^E_\mu(z),t\right)$ and $\nu'\left(\Omega^E_\nu(z),t\right)$ are defined in (4.25) for all $z\in\mathscr{A}$ and all $t>0$.

Suppose if we define a function $h(z)$ by

$$h(z)=h_O(z)+h_E(z) \tag{4.39}$$

for all $z\in\mathscr{A}$. It follows from (4.36), (4.38), (4.39), we arrive

$$\left.\begin{array}{l}\mu\Big(h(z)-\mathscr{H}_1(z)-\mathscr{H}_2(z),2t\Big)\\=\mu\Big(h_O(z)+h_E(z)-\mathscr{H}_1(z)-\mathscr{H}_2(z),2t\Big)\\\geq\mu\Big(h_O(z)-\mathscr{H}_1(z),t\Big)*\mu\Big(h_E(z)-\mathscr{H}_2(z),t\Big)\\\geq\mu'\left(\Omega^O_\mu(z),\frac{t}{2}|3-p|\right)*\mu'\left(\Omega^E_\mu(z),t|16-p|\right)\\=\mu'\left(\Omega^O_\mu(z)+\Omega^E_\mu(z),\frac{t}{2}|3-p|+t|16-p|\right)\\=\mu'\left(\Omega^{OE}_\mu(z),t\right)\\\nu\Big(h(z)-\mathscr{H}_1(z)-\mathscr{H}_2(z),2t\Big)\\=\nu\Big(h_O(z)+h_E(z)-\mathscr{H}_1(z)-\mathscr{H}_2(z),2t\Big)\\\geq\nu\Big(h_O(z)-\mathscr{H}_1(z),t\Big)\diamond\nu\Big(h_E(z)-\mathscr{H}_2(z),t\Big)\\\geq\nu'\left(\Omega^O_\nu(z),\frac{t}{2}|3-p|\right)\diamond\nu'\left(\Omega^E_\nu(z),t|16-p|\right)\\=\nu'\left(\Omega^O_\nu(z)+\Omega^E_\nu(z),\frac{t}{2}|3-p|+t|16-p|\right)\\=\nu'\left(\Omega^{OE}_\nu(z),t\right)\end{array}\right\}$$

for all $z\in\mathscr{A}$ and all $t>0$. □

The following corollary is an immediate consequence of Theorem 4.5, regarding some stabilities of (1.9).

Corollary 4.6. *Assume a function $H:\mathscr{A}\longrightarrow\mathscr{B}$ satisfies the double inequality*

$$\left.\begin{array}{l}\mu\left(H(z_1,z_2,z_3),t\right)\geq\left\{\begin{array}{l}\mu'\left(\lambda,t\right),\\\mu'\left(\lambda\sum_{i=1}^3||z_i||^a,t\right),\\\mu'\left(\lambda\sum_{i=1}^3||z_i||^{a_i},t\right),\\\mu'\left(\lambda\prod_{i=1}^3||z_i||^a,t\right),\\\mu'\left(\lambda\prod_{i=1}^3||z_i||^{a_i},t\right),\end{array}\right.\\\nu\left(H_e(z_1,z_2,z_3),t\right)\leq\left\{\begin{array}{l}\nu'\left(\lambda,t\right),\\\nu'\left(\lambda\sum_{i=1}^3||z_i||^a,t\right),\\\nu'\left(\lambda\sum_{i=1}^3||z_i||^{a_i},t\right),\\\nu'\left(\lambda\prod_{i=1}^3||z_i||^a,t\right),\\\nu'\left(\lambda\prod_{i=1}^3||z_i||^{a_i},t\right),\end{array}\right.\end{array}\right\} \tag{4.40}$$

for all $z_1,z_2,z_3\in\mathscr{A}$ and all $t>0$, where $\lambda,a,a_i's$ are constants with $\lambda>0$. Then there exists a unique additive mapping $\mathscr{H}_1:\mathscr{A}\longrightarrow\mathscr{B}$ and unique quadratic mapping $\mathscr{H}_2:\mathscr{A}\longrightarrow\mathscr{B}$

such that

$$\mu\left(h(z)-\mathscr{H}_1(z)-\mathscr{H}_2(z),2t\right) \geq \begin{cases} \mu'\left(3\lambda,|3-p|t+|16-p|t\right), \\ \mu'\Big(\lambda||z||^a\left\{2\left[\frac{5}{2^a}+\frac{1}{6^a}\right]+\left[\frac{2}{2^a}+\frac{1}{3^a}\right]\right\}, \\ \qquad t|3-p|+t|16-p|\Big), \\ \mu'\Big(\lambda\Big[\frac{5||z||^{a_1}}{2^{a_1}}+\frac{5||z||^{a_2}}{2^{a_2}}+\frac{2||z||^{a_3}}{2^{a_3}}+\frac{||z||^{a_3}}{3^{a_3}} \\ \qquad +\frac{||z||^{a_3}}{6^{a_3}}\Big],t|3-p|+t|16-p|\Big), \\ \mu'\Big(\left\{2\left[\frac{1}{2^{3a}}+\frac{1}{2^{2a}6^a}\right]+\left[\frac{1}{2^{2a}3^a}\right]\right\}, \\ \qquad t|3-p|+t|16-p|\Big), \\ \mu'\Big(\lambda||z||^{a_1+a_2+a_3} \\ \left\{2\left[\frac{1}{2^{a_1+a_2+a_3}}+\frac{1}{2^{a_1+a_2}.6^{a_3}}\right]+\left[\frac{1}{2^{a_1+a_2}.3^{a_3}}\right]\right\}, \\ \qquad t|3-p|+t|16-p|\Big), \end{cases}$$

$$\nu\left(h(z)-\mathscr{H}_1(z)-\mathscr{H}_2(z),2t\right) \leq \begin{cases} \nu'\left(3\lambda,|3-p|t+|16-p|t\right), \\ \nu'\Big(\lambda||z||^a\left\{2\left[\frac{5}{2^a}+\frac{1}{6^a}\right]+\left[\frac{2}{2^a}+\frac{1}{3^a}\right]\right\}, \\ \qquad t|3-p|+t|16-p|\Big), \\ \mu'\Big(\lambda||z||^a\left\{2\left[\frac{5}{2^a}+\frac{1}{6^a}\right]+\left[\frac{2}{2^a}+\frac{1}{3^a}\right]\right\}, \\ \qquad t|3-p|+t|16-p|\Big), \\ \nu'\Big(\lambda\Big[\frac{5||z||^{a_1}}{2^{a_1}}+\frac{5||z||^{a_2}}{2^{a_2}}+\frac{2||z||^{a_3}}{2^{a_3}}+\frac{||z||^{a_3}}{3^{a_3}} \\ \qquad +\frac{||z||^{a_3}}{6^{a_3}}\Big],t|3-p|+t|16-p|\Big), \\ \nu'\Big(\left\{2\left[\frac{1}{2^{3a}}+\frac{1}{2^{2a}6^a}\right]+\left[\frac{1}{2^{2a}3^a}\right]\right\}, \\ \qquad t|3-p|+t|16-p|\Big), \\ \nu'\Big(\lambda||z||^{a_1+a_2+a_3} \\ \left\{2\left[\frac{1}{2^{a_1+a_2+a_3}}+\frac{1}{2^{a_1+a_2}.6^{a_3}}\right]+\left[\frac{1}{2^{a_1+a_2}.3^{a_3}}\right]\right\}, \\ \qquad t|3-p|+t|16-p|\Big), \end{cases} \tag{4.41}$$

all $z \in \mathscr{A}$ and all $t > 0$.

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