Fuzzy $M$-open sets in Šostak’s fuzzy topological spaces

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Abstract
We introduce the concept of $\mathfrak{i}$-fuzzy $\theta$-open (resp. $\mathfrak{i}$-fuzzy $\theta$-closed), $\mathfrak{i}$-fuzzy $M$-open (resp. $\mathfrak{i}$-fuzzy $M$-closed) sets in fuzzy topological spaces in the sense of Šostak’s. Also, we introduce $\mathfrak{i}$-fuzzy $\theta$-interior (resp. $\mathfrak{i}$-fuzzy $\theta$-closure), $\mathfrak{i}$-fuzzy $M$-interior (resp. $\mathfrak{i}$-fuzzy $M$-closure) and investigate some of their properties. Moreover, we investigate the relationships between $\mathfrak{i}$-fuzzy open, $\mathfrak{i}$-fuzzy $\theta$-semiopen, $\mathfrak{i}$-fuzzy $\theta$-open, $\mathfrak{i}$-fuzzy $\delta$-semiopen, $\mathfrak{i}$-fuzzy $\delta$-preopen, $\mathfrak{i}$-fuzzy $\theta$-open, $\mathfrak{i}$-fuzzy $\theta$-closed, $\mathfrak{i}$-fuzzy $\theta$-preopen, and $\mathfrak{i}$-fuzzy $e^\ast$-open in Šostak’s fuzzy topological spaces.

Keywords
$\mathfrak{i}$-fuzzy $M$-open, $\mathfrak{i}$-fuzzy $M$-closed, $\mathfrak{i}$-fuzzy $M$-interior, $\mathfrak{i}$-fuzzy $M$-closure, $\mathfrak{i}$-fuzzy $\theta$-interior, $\mathfrak{i}$-fuzzy $\theta$-closure, $\mathfrak{i}$-fuzzy $\theta$-semiopen, $\mathfrak{i}$-fuzzy $\theta$-open, $\mathfrak{i}$-fuzzy $\delta$-semiopen, $\mathfrak{i}$-fuzzy $\delta$-preopen, $\mathfrak{i}$-fuzzy $\theta$-open and $\mathfrak{i}$-fuzzy $e^\ast$-open.

AMS Subject Classification
54A40, 54C05, 03E72.

1. Introduction
Šostak [27] introduced the fuzzy topology as an extension of Chang’s fuzzy topology [5]. It has been developed in many directions [12, 13, 25]. Weaker forms of fuzzy continuity between fuzzy topological spaces have been considered by many authors [2, 3, 6, 9, 11, 19, 22] using the concepts of fuzzy semi-open sets [2], fuzzy regular open sets [2], fuzzy preopen sets, fuzzy strongly semiopen sets [3], fuzzy $\gamma$-open sets [11], fuzzy $\delta$-semiopen sets [1], fuzzy $\delta$-preopen sets [1], fuzzy semi $\delta$-preopen sets [29] and fuzzy $e$-open sets [26]. Ganguly and Saha [10] introduced the notions of fuzzy $\delta$-cluster points in Chang’s [5] fuzzy topological spaces. Kim and Park [14] introduced $r$-$\delta$-cluster points and $\delta$-closure operators in Šostak’s fuzzy topological spaces. It is a good extension of the notions of Ganguly and Saha [10]. Park et al. [20] introduced the fuzzy semi-preopen. In 2008, the initiations of $e$-open sets, $e^\ast$-open sets and $a$-open sets in topological spaces are due to Erdal Ekici[7],[8]). Sobana et al. [28] defined $r$-fuzzy $e$-open sets in Šostak’s fuzzy topological space. Vadivel et al. [30] introduced $r$-fuzzy $e^\ast$-open sets in Šostak’s fuzzy topological space. In 1968, Velicko studied $\theta$-open sets [31] and $\delta$-open sets for the purpose of investigating the characerizations of $H$-closed topological spaces. semi-open set [17] were initiated by Levine in 1963. In 1993, Raychaudhuri and Mukherjee defined $\delta$-preopen sets [24]. In 1997, $\delta$-semiopen sets was obtained by Park [21] and $\theta$-semi-open sets were obtained by Caldas in 2008 [4]. Shafei introduced fuzzy $\theta$-closed [32] and fuzzy $\theta$-open sets in 2006. Maghrabi et al.[18] introduced the notion of $M$-open sets in topological spaces in 2011.

In this paper, we introduce the concept of $\mathfrak{i}$-fuzzy $\theta$-
Throughout this article, we denote nonempty sets by $I$. Let $\mu$ denote the set of all fuzzy points in $\mu$. For $\mu \in I^X$ we define an operator $I_{\tau}: I^X \times I_0 \to I^X$ as follows:

\[ I_{\tau}(\mu, t) = \bigvee \{ \nu \in I^X : \mu \geq \nu, \tau(\tau - \nu) \geq t \}. \]

For $\mu, \nu \in I^X$ and $t, s \in I_0$, the operator $C_{\tau}$ satisfies the following conditions:

1. $C_{\tau}(0, t) = 0$,
2. $\mu \leq C_{\tau}(\mu, t)$,
3. $C_{\tau}(\mu, t) \lor C_{\tau}(\nu, t) = C_{\tau}(\mu \lor \nu, t)$,
4. $C_{\tau}(\mu, t) \leq C_{\tau}(\mu, s)$ if $t \leq s$,
5. $C_{\tau}(C_{\tau}(\mu, t), 1) = C_{\tau}(\mu, t)$.

Theorem 2.5. [25] Let $(X, \tau)$ be a sfts. Then for each $t \in I_0$, $\mu \in I^X$ we define an operator $I_{\tau}: I^X \times I_0 \to I^X$ as follows:

\[ I_{\tau}(\mu, t) = \bigvee \{ \nu \in I^X : \mu \geq \nu, \tau(\tau - \nu) \geq t \}. \]

For $\mu, \nu \in I^X$ and $t, s \in I_0$, the operator $I_{\tau}$ satisfies the following conditions:

1. $I_{\tau}(1, t) = 1$,
2. $\mu \geq I_{\tau}(\mu, t)$,
3. $I_{\tau}(\mu, t) \lor I_{\tau}(\nu, t) = I_{\tau}(\mu \lor \nu, t)$,
4. $I_{\tau}(\mu, t) \leq I_{\tau}(\mu, s)$ if $s \leq t$,
5. $I_{\tau}(I_{\tau}(\mu, t), 1) = I_{\tau}(\mu, t)$.

Definition 2.6. [15] Let $(X, \tau)$ be a sfts. Then for each $\mu \in I^X$, $x \in P_I(X)$ and $t \in I_0$, $\nu$ is called

1. a $t$-open $Q_{\tau}$-neighbourhood of $x$, if $x, \nu$ with $\tau(\tau - \nu) \geq t$.
2. a $t$-open $R_{\tau}$-neighbourhood of $x$, if $x, \nu$ with $\nu = I_{\tau}(C_{\tau}(\mu, t), 1)$.

We denote $Q_{\tau}(x, t) = \{ \nu \in I^X : x, \nu, \tau(\tau - \nu) \geq t \}, R_{\tau}(x, t) = \{ \nu \in I^X : x, \nu = I_{\tau}(C_{\tau}(\mu, t), 1) \}$.

Definition 2.7. [15] Let $(X, \tau)$ be a sfts. Then for each $\mu \in I^X$, $x \in P_I(X)$ and $t \in I_0$, $x$ is called

1. a $t$-cluster point of $\mu$ if for every $\nu \in Q_{\tau}(x, t)$, we have $v \mu$.
2. a $t$-cluster point of $\mu$ if for every $\nu \in R_{\tau}(x, t)$, we have $v \mu$.

(iii) An $\delta$-closure operator is a mapping $D_{\tau}: I^X \times I_0 \to I^X$ defined as follows: $D_{\tau}(C_{\tau}(\mu, t)$ or $D_{\tau}(\mu, t) = \bigvee \{ x \in P_I(X) : x, \tau - \mu \geq \delta - \tau - \mu \}$.

Definition 2.8. Let $(X, \tau)$ be a sfts. For $\mu, \nu \in I^X$ and $t \in I_0$, $\mu$ is called an

1. $t$-fuzzy $\delta$-semiopen (resp. $t$-fuzzy $\delta$-semiclosed)[28] set if $\mu \leq C_{\tau}(\delta I_{\tau}(\mu, t), 1)$
   (resp. $I_{\tau}(\delta I_{\tau}(\mu, t), 1) \leq \mu$).
(ii) t-fuzzy $\delta$-preopen (resp. t-fuzzy $\delta$-preclosed)[28] set if $\mu \leq I_{\tau}(C_{\theta}(\mu, 1), t)$ ($r_{p}C_{\tau}(\mu, t) \leq \mu$).

(iii) t-fuzzy $a$-open (resp. t-fuzzy $a$-closed)[28] set if $\mu \leq I_{\tau}(C_{\theta}(\mu, 1), t)$ ($r_{p}C_{\tau}(\delta C_{\theta}(\mu, 1), 1) \leq \mu$).

(iv) t-fuzzy e-open (resp. t-fuzzy e-closed)[28] set if $\mu \leq C_{\tau}(I_{\tau}(\delta C_{\theta}(\mu, 1), t))$ ($r_{p}C_{\tau}(\delta C_{\theta}(\mu, 1), 1) \leq C_{\tau}(\delta I_{\tau}(\mu, 1), t)$).

(v) t-fuzzy $e'$-open (resp. t-fuzzy $e'$-closed)[30] set if $\mu \leq C_{\tau}(I_{\tau}(\delta C_{\theta}(\mu, 1), t))$ ($r_{p}C_{\tau}(\delta C_{\theta}(\mu, 1), 1) \leq C_{\tau}(\delta I_{\tau}(\mu, 1), t)$).

(vi) t-fuzzy semiopen (resp. t-fuzzy semi-closed) [16] set if $\mu \leq C_{\tau}(I_{\tau}(\delta C_{\theta}(\mu, 1), t))$ ($r_{p}C_{\tau}(\delta C_{\theta}(\mu, 1), 1) \leq C_{\tau}(\delta I_{\tau}(\mu, 1), t)$).

### 3. t-fuzzy $\theta$-open and t-fuzzy $\theta$-closed sets

In the following section, we introduce the concepts of t-fuzzy $\theta$-interior (resp. t-fuzzy $\theta$-closure), t-fuzzy $\theta$-open (resp. t-fuzzy $\theta$-closed), t-fuzzy $\theta$-semipen (resp. t-fuzzy $\theta$-semiclosed), t-fuzzy $\theta$-preopen (resp. t-fuzzy $\theta$-preclosed) sets to the sfts.

**Definition 3.1.**

(i) t-fuzzy $\theta$-interior (resp. t-fuzzy $\theta$-semi-interior and t-fuzzy $\theta$-pre-interior) of a subset $\mu$ in a sfts $(X, \tau)$, $\forall \; t \in I_{0}$, denoted by $\theta I_{\tau}(\mu, 1)$ (resp. $\theta sI_{\tau}(\mu, 1)$ and $\theta pI_{\tau}(\mu, 1)$) defined as $\theta I_{\tau}(\mu, 1) = \bigvee\{I_{\tau}(v) : \mu \geq v, \forall \tau(v) \geq 1\} \land sI_{\tau}(\mu, 1) = \bigvee\{s_{\tau}(v) : \mu \geq v, \forall \tau(v) \geq 1\} \land pI_{\tau}(\mu, 1) = \bigvee\{p_{\tau}(v) : \mu \geq v, \forall \tau(v) \geq 1\}$.

(ii) t-fuzzy $\theta$-closure (resp. t-fuzzy $\theta$-semi-closure and t-fuzzy $\theta$-pre-closure) of a subset $\mu$ in a sfts $(X, \tau)$, $\forall \; t \in I_{0}$, denoted by $\theta C_{\tau}(\mu, 1)$ (resp. $\theta sC_{\tau}(\mu, 1)$ and $\theta pC_{\tau}(\mu, 1)$) defined as $\theta C_{\tau}(\mu, 1) = \bigwedge\{C_{\tau}(v) : \mu \leq v, \forall \tau(v) \geq 1\} \land sC_{\tau}(\mu, 1) = \bigwedge\{s_{\tau}(v) : \mu \leq v, \forall \tau(v) \geq 1\} \land pC_{\tau}(\mu, 1) = \bigwedge\{p_{\tau}(v) : \mu \leq v, \forall \tau(v) \geq 1\}$.

**Definition 3.2.** Let $(X, \tau)$ be a sfts. For $\mu, v \in I^{X}$ and $t \in I_{0}$, $\mu$ is called an

(i) t-fuzzy $\theta$-open (resp. t-fuzzy $\theta$-closed) set if $\mu = \theta I_{\tau}(\mu, 1)$ (resp. $\mu = \theta C_{\tau}(\mu, 1)$).

(ii) t-fuzzy $\theta$-semipen (resp. t-fuzzy $\theta$-semiclosed) set if $\mu \leq C_{\tau}(\theta I_{\tau}(\mu, 1), t)$ (resp. $I_{\tau}(\theta C_{\tau}(\mu, 1), t) \leq \mu$).

(iii) t-fuzzy $\theta$-preopen (resp. t-fuzzy $\theta$-preclosed) set if $\mu \leq I_{\tau}(\theta C_{\tau}(\mu, 1), t)$ (resp. $C_{\tau}(\theta I_{\tau}(\mu, 1), t) \leq \mu$).

The family of all t-fuzzy $\theta$-open (resp. t-fuzzy $\theta$-closed), t-fuzzy $\theta$-semipen (resp. t-fuzzy $\theta$-semiclosed), t-fuzzy $\theta$-preopen (resp. t-fuzzy $\theta$-preclosed) sets will be denoted by $t-f\theta o$ (resp. $t-f\theta c$, $t-f\theta so$ (resp. $t-f\theta sc$, $t-f\theta po$ (resp. $t-f\theta pc$) sets.

**Lemma 3.3.** Let $\mu, v \in I^{X}$ and $t \in I_{0}$ in a sfts $(X, \tau)$, then

(i) $\mu$ is t-fuzzy $\theta$-open iff $\mu = \theta I_{\tau}(\mu, 1)$.

(ii) If $\mu < v$, then $\theta I_{\tau}(\mu, 1) < \theta I_{\tau}(v, 1)$.

(iii) $\theta I_{\tau}(\theta I_{\tau}(\mu, 1), t) \leq \theta I_{\tau}(\mu, t)$.

(iv) For any subset $\mu$ of $X, \mu \leq c_{\tau}(\mu, 1) \leq \delta c_{\tau}(\mu, 1) \leq \delta c_{\tau}(\mu, 1) \leq \delta I_{\tau}(\mu, 1) \leq I_{\tau}(\mu, 1) \leq 1$.

(v) $\cap (\theta I_{\tau}(\mu, 1)) = \cap c_{\tau}(\mu, 1) = \cap c_{\tau}(\mu, 1) \cap I_{\tau}(\mu, 1) = I_{\tau}(\mu, 1)$.

(vii) $\cap (\theta C_{\tau}(\mu, 1)) = \cap c_{\tau}(\mu, 1) \cap \cap c_{\tau}(\mu, 1) \cap \cap c_{\tau}(\mu, 1)$.

Proposition 3.4. Let $\mu \in I^{X}$ and $t \in I_{0}$ in a sfts $(X, \tau)$, then

(i) $\theta sC_{\tau}(\mu, 1) = \mu \land \theta sC_{\tau}(\mu, 1) \land \theta sC_{\tau}(\mu, 1) = \mu \land I_{\tau}(\theta I_{\tau}(\mu, 1), t)$.

(ii) $\delta pC_{\tau}(\mu, 1) \land \delta pC_{\tau}(\mu, 1) \land \delta pC_{\tau}(\mu, 1) = \mu \land I_{\tau}(\theta I_{\tau}(\mu, 1), t)$.

(iii) $\cap (\theta I_{\tau}(\mu, 1)) = \cap c_{\tau}(\mu, 1) \cap \cap c_{\tau}(\mu, 1) \cap c_{\tau}(\mu, 1)$.

**Lemma 3.5.** Let $v \in I^{X}$ and $t \in I_{0}$ in a sfts $(X, \tau)$, then

(i) $\delta pC_{\tau}(v, 1) = v \lor \delta c_{\tau}(\delta I_{\tau}(v, 1), t)$ and $\delta pC_{\tau}(v, 1) = v \lor \delta c_{\tau}(\delta I_{\tau}(v, 1), t)$.

(ii) $\delta pC_{\tau}(\delta pC_{\tau}(v, 1), t) = \delta pC_{\tau}(v, 1) \lor \delta c_{\tau}(\delta I_{\tau}(v, 1), t)$.

(iii) $\delta pC_{\tau}(\delta pC_{\tau}(v, 1), t) = \delta pC_{\tau}(v, 1) \lor \delta c_{\tau}(\delta I_{\tau}(v, 1), t)$.

(iv) $\delta I_{\tau}(v, 1) = v \lor \delta c_{\tau}(\delta I_{\tau}(v, 1), t)$ and $\delta sI_{\tau}(v, 1) = v \lor \delta c_{\tau}(\delta I_{\tau}(v, 1), t)$.

**Lemma 3.6.** Let $v \in I^{X}$ and $t \in I_{0}$ in a sfts $(X, \tau)$, then

(i) $C_{\tau}(\delta I_{\tau}(v, 1), t) = \delta I_{\tau}(\delta C_{\tau}(v, 1), t)$.

(ii) $\delta pC_{\tau}(\delta pC_{\tau}(v, 1), t) = \delta pC_{\tau}(v, 1) \lor \delta c_{\tau}(\delta I_{\tau}(v, 1), t)$.

**Definition 3.7.** A subset $\mu$ in a sfts $(X, \tau)$ is called a t-fuzzy locally closed set $\forall \; t \in I_{0}$, if $\mu = v \cap \alpha$, where $\tau(v) \geq 1$, $\alpha$ is t-fuzzy closed in $X$.

**Definition 3.8.** Let $(X, \tau)$ be a sfts. For $\mu \in I^{X}$ and $t \in I_{0}$, then A sfts $(X, \tau)$ is t-fuzzy extremely disconnected (briefly, t-FED) if the t-fuzzy closure of every t-fuzzy open set of $X$ is t-fuzzy open.
4. 1-fuzzy $M$-open and 1-fuzzy $M$-closed sets

**Definition 4.1.** A subset $\mu \in I^X$, $\forall t \in I_0$ in a sfts $(X, \tau)$ is called an 1-fuzzy

(i) $M$-open set if $\mu \leq C_1(\Theta_1(\mu, t), t) \vee I_1(\delta C_1(\mu, t), t)$.

(ii) $M$-closed set if $\mu \geq I_1(\Theta_1(\mu, t), t) \wedge C_1(I_1(\delta C_1(\mu, t), t)$.

**Definition 4.2.** 1-fuzzy $M$-interior (resp. 1-fuzzy $M$-closure) of $\mu$ in a sfts $(X, \tau)$, $\forall t \in I_0$, denoted by $M_1(\mu, t)$ (resp. $MC_1(\mu, t)$) defined as $M_1(\mu, t) = \{v \in I^X : \mu \geq v, v$ is a 1-fMo set $\}$; $MC_1(\mu, t) = \{v \in I^X : \mu \leq v, v$ is a 1-fMo set $\}$.

**Remark 4.3.** The implications are true $\forall \mu$ in a sfts $X$ and $t \in I_0$.

\[
\begin{align*}
\text{where } t\text{-fo, } t\text{-fo so, } t\text{-fo o, } t\text{-fo c, } t\text{-fo so, } t\text{-fo sc, } t\text{-fo po, } t\text{-fo pc, } t\text{-fao, } t\text{-fac, } t\text{-fMo, } t\text{-feo, } t\text{-feco, } t\text{-fe o'c, and } t\text{-fe o'c are abbreviated by t-fMo open, t-fMo \theta-semiopen, t-fMo \theta-semiclosed, t-fMo \theta-open, t-fMo \theta-\delta-\theta-semiopen, t-fMo \theta-\delta-\theta-semiclosed, t-fMo \theta-\delta-preopen, t-fMo \theta-\delta-preclosed, t-fMo a-open, t-fMo a-closed, t-fMo M-open, t-fMo M-closed, t-fMo e-open, t-fMo e-closed, t-fMo e'-open and t-fMo e'-closed respectively.}
\end{align*}
\]

From the above definitions, it is clear that every 1-fMo po is t-fMo set and every 1-fMo so is t-fMo set. Also, it is clear that every 1-fMo set is 1-fc set and 1-fMo set. Also, every 1-fMo, 1-fso, t-fao set is t-fMo set. The converses need not be true in general.

**Example 4.4.** Let $X = \{x, y, z\}$ and $\mu, v \in I^X$ given by $\mu(x) = 0.4, \mu(y) = 0.5, \mu(z) = 0.2; v(x) = 0.5, v(y) = 0.4, v(z) = 0.7$. and $\tau : I^X \rightarrow I$ is

\[
\tau(\mu) = \begin{cases} 
1, & \mu \in \{0, 1\}, \\
\frac{1}{2}, & \mu = \mu,
\end{cases}
\]

is a fuzzy topology on $X$. For $t = \frac{1}{2}$, then $v$ is $\frac{1}{2}$-fe o set but $v$ is not $\frac{1}{2}$-fMo set.

**Example 4.5.** Let $X = \{x, y, z\}$ and $\mu, v \in I^X$ given by $\mu(x) = 0.5, \mu(y) = 0.3, \mu(z) = 0.2; v(x) = 0.5, v(y) = 0.4, v(z) = 0.4$. and $\tau : I^X \rightarrow I$ is

\[
\tau(\mu) = \begin{cases} 
1, & \mu \in \{0, 1\}, \\
\frac{1}{2}, & \mu = \mu,
\end{cases}
\]

is a fuzzy topology on $X$. For $t = \frac{1}{2}$, then $v$ is $\frac{1}{2}$-fe o set but $v$ is not $\frac{1}{2}$-fMo set.

**Example 4.6.** Let $X = \{x, y, z\}$ and $\mu, v \in I^X$ given by $\mu(x) = 0.1, \mu(y) = 0.1, \mu(z) = 0.1; v(x) = 0.9, v(y) = 0.9, v(z) = 0.9$. and $\tau : I^X \rightarrow I$ is

\[
\tau(\mu) = \begin{cases} 
1, & \mu \in \{0, 1\}, \\
\frac{1}{2}, & \mu = \mu,
\end{cases}
\]

is a fuzzy topology on $X$. For $t = \frac{1}{2}$, then $v$ is $\frac{1}{2}$-fe o set but $v$ is not $\frac{1}{2}$-fMo set.

**Example 4.7.** Let $X = \{x, y, z\}$ and $\mu, v \in I^X$ given by $\mu(x) = 0.1, \mu(y) = 0.1, \mu(z) = 0.1; v(x) = 0.9, v(y) = 0.9, v(z) = 0.9$. and $\tau : I^X \rightarrow I$ is

\[
\tau(\mu) = \begin{cases} 
1, & \mu \in \{0, 1\}, \\
\frac{1}{2}, & \mu = \mu,
\end{cases}
\]

is a fuzzy topology on $X$. For $t = \frac{1}{2}$, then $v$ is $\frac{1}{2}$-fMo set but $v$ is neither $\frac{1}{2}$-fMo nor $\frac{1}{2}$-fMo set.

**Example 4.8.** Let $X = \{x, y, z\}$ and $\mu, v, \omega \in I^X$ given by $\mu(x) = 0.3, \mu(y) = 0.4, \mu(z) = 0.5; v(x) = 0.6, v(y) = 0.9, v(z) = 0.5; \omega(x) = 0.7, \omega(y) = 1, \omega(z) = 0.5$. and $\tau : I^X \rightarrow I$ is

\[
\tau(\mu) = \begin{cases} 
1, & \mu \in \{0, 1\}, \\
\frac{1}{2}, & \mu = \mu, \omega,
\end{cases}
\]

is a fuzzy topology on $X$. For $t = \frac{1}{2}$, then $\omega$ is $\frac{1}{2}$-fMo but $\omega$ is neither $\frac{1}{2}$-fMo nor $\frac{1}{2}$-fMo set.

**Example 4.9.** Let $X = \{x, y, z\}$ and $\mu, v, \omega \in I^X$ given by $\mu(x) = 0.3, \mu(y) = 0.4, \mu(z) = 0.5; v(x) = 0.6, v(y) = 0.5, v(z) = 0.5; \omega(x) = 0.7, \omega(y) = 0.6, \omega(z) = 0.5$. and $\tau : I^X \rightarrow I$ is

\[
\tau(\mu) = \begin{cases} 
1, & \mu \in \{0, 1\}, \\
\frac{1}{2}, & \mu = \mu, \omega,
\end{cases}
\]

is a fuzzy topology on $X$. For $t = \frac{1}{2}$, then $\omega$ is $\frac{1}{2}$-fMo and $\frac{1}{2}$-fMo set but $\omega$ is not $\frac{1}{2}$-fMo set.
Example 4.10. Let $X = \{x, y, z\}$ and $\mu, \nu, \omega \in I^X$ given by $\mu(x) = 0.3, \mu(y) = 0.5, \mu(z) = 0.5$; $\nu(x) = 0.5, \nu(y) = 0.5, \nu(z) = 0.5$; $\omega(x) = 0.7, \omega(y) = 0.6, \omega(z) = 0.5$. and $\tau : I^X \to I$ is

$$
\tau(\mu) = \begin{cases} 
1, & \text{if } \mu = \overline{0}, \\
\frac{1}{2}, & \text{if } \mu = \nu, \\
0, & \text{otherwise},
\end{cases}
$$

is a fuzzy topology on $X$. For $t = \frac{1}{2}$, then $\mu$ is a $\frac{1}{2}$-fuzzy open set but $\mu$ is not a $\frac{1}{2}$-fuzzy open set set nor $\frac{1}{2}$-fuzzy $\delta$-open set.

Proposition 4.11. If $\mu$ is an $t$-fuzzy $M$-open subset of a sfts $(X, \tau)$ and $\theta I_t(\mu, t) = \emptyset$, then $\mu$ is an $t$-fuzzy $\delta$-preopen.

Proof. Obvious.

Theorem 4.12. Arbitrary union (resp. intersection) of two $t$-fMo (resp. $t$-fMc) sets is an $t$-fMo (resp. $t$-fMc) set.

Proof. We prove only for $t$-fMo sets. Let $\{\mu_\alpha : \alpha \in \Gamma\}$ be a family of $t$-fMo sets. For each $\alpha \in \Gamma$,

$$\mu_\alpha \leq C_\tau(\theta I_t(\mu_\alpha, t) \cup I_t(\delta C_\tau(\mu_\alpha, t), t)).$$

Thus, $\bigcup_{\alpha \in \Gamma} \mu_\alpha \leq \bigcup_{\alpha \in \Gamma} (C_\tau(\theta I_t(\mu_\alpha, t) \cup I_t(\delta C_\tau(\mu_\alpha, t), t)))$.

Remark 4.13. The intersection (resp. union) of two $t$-fMo (resp. $t$-fMc) sets need not be $t$-fMo (resp. $t$-fMc) as illustrated in the forthcoming example.

Example 4.14. Let $X = \{x, y, z\}$ and $\mu, \nu, \omega, \delta$ be fuzzy subsets of $X = \{x, y, z\}$ defined as follows

$$
\mu(x) = 0.4, \mu(y) = 0.5, \mu(z) = 0.2; \nu(x) = 0.6, \nu(y) = 0.6, \nu(z) = 0.8; \omega(x) = 0.7, \omega(y) = 0.4, \omega(z) = 0.8; \delta(x) = 0.6, \delta(y) = 0.4, \delta(z) = 0.8;
$$

and $\tau : I^X \to I$ is

$$
\tau(\mu) = \begin{cases} 
1, & \text{if } \mu = \overline{0} \text{ or } \overline{1}, \\
\frac{1}{2}, & \text{if } \mu = \nu, \\
0, & \text{otherwise},
\end{cases}
$$

is a fuzzy topology on $X$. For $t = \frac{1}{2}$, then $\nu$ and $\omega$ is a $\frac{1}{2}$-fMo but $\nu \land \omega = \delta$ is not a $\frac{1}{2}$-fMo set.

Theorem 4.15. In a sfts $(X, \tau)$ $\mu$ is an $t$-fMo set iff $\mu = \theta I_t(\mu, t) \cup \delta p I_t(\mu, t)$.

Proof. Let $\mu$ be an $t$-fMo set. Then

$$
\mu \leq C_\tau(\theta I_t(\mu, t) \cup I_t(\delta C_\tau(\mu, t), t)). \quad (4.1)
$$

Hence by Proposition 3.4, Lemma 3.5 and by using (4.1), we have,

$$
\theta s I_t(\mu, t) \cup \delta p I_t(\mu, t) = (\mu \land C_\tau(\theta I_t(\mu, t), t)) \cup (\mu \land I_t(\delta C_\tau(\mu, t), t))
= \mu \land (\theta I_t(\mu, t) \cup I_t(\delta C_\tau(\mu, t), t))
= \mu.
$$

Conversely, suppose that $\mu = \theta s I_t(\mu, t) \cup \delta p I_t(\mu, t)$. Then by Proposition 3.4 and Lemma 3.5,

$$
\mu = (\mu \land C_\tau(\theta I_t(\mu, t), t)) \cup (\mu \land I_t(\delta C_\tau(\mu, t), t)) \leq C_\tau(\theta I_t(\mu, t), t) \cup I_t(\delta C_\tau(\mu, t), t).
$$

Therefore, $\mu$ is an $t$-fMo set.

Proposition 4.16. In a sfts $(X, \tau)$ $\mu$ is an $t$-fMc set iff $\mu = \theta s C_\tau(\mu, t) \land \delta p C_\tau(\mu, t)$.

Theorem 4.17. In a sfts $(X, \tau)$, $\forall \mu \in I^X$ and $t \in I_0$, (i) & (ii) are hold.

(i) $MC_\tau(\mu, t) = \theta s C_\tau(\mu, t) \land \delta p C_\tau(\mu, t)$.

(ii) $MI_\tau(\mu, t) = \theta s I_t(\mu, t) \cup \delta p I_t(\mu, t)$.

Proof. (i) Since $MC_\tau(\mu, t) \leq \theta s C_\tau(\mu, t) \land \delta p C_\tau(\mu, t)$.

Also, $\theta s C_\tau(\mu, t) \land \delta p C_\tau(\mu, t)$

$$
= (\mu \land I_t(\theta C_\tau(\mu, t), t)) \land (\mu \land I_t(\delta C_\tau(\mu, t), t))
= \mu \land (I_t(\theta C_\tau(\mu, t), t) \land I_t(\delta C_\tau(\mu, t), t)).
$$

But, $MC_\tau(\mu, t)$ is $t$-Mc, hence, $MC_\tau(\mu, t)$

$$
> I_t(\theta C_\tau(MC_\tau(\mu, t), t), t) \land I_t(\delta C_\tau(MC_\tau(\mu, t), t), t)
> I_t(\theta C_\tau(\mu, t), t) \land I_t(\delta C_\tau(\mu, t), t).
$$

Thus $\mu \land I_t(\theta C_\tau(\mu, t), t) \land I_t(\delta C_\tau(\mu, t), t) < \mu \land MC_\tau(\mu, t) = MC_\tau(\mu, t)$

Therefore, $\theta s C_\tau(\mu, t) \land \delta p C_\tau(\mu, t) < MC_\tau(\mu, t)$. So,

$$
MC_\tau(\mu, t) = \theta s C_\tau(\mu, t) \land \delta p C_\tau(\mu, t).
$$

(ii) Obvious.

Theorem 4.18. In a sfts $(X, \tau)$, $\forall \mu \in I^X$ and $t \in I_0$, (i) & (ii) are hold.

(i) $\mu$ is $t$-Mo iff $\mu = MI_t(\mu, t)$.

(ii) $\mu$ is $t$-Mc iff $\mu = MC_\tau(\mu, t)$. 


\textbf{Proof.}  (i) Let $\mu$ be an $t$-fMo set. Then by Theorem 4.15,
\[ \mu = \theta sI_t(\mu, t) \vee \delta pI_t(\mu, t) \]
and by Theorem 4.17, we have $\mu = MI_t(\mu, t)$.
Conversely, let $\mu = MI_t(\mu, t)$. Then by Theorem 4.17, $\mu = \theta sI_t(\mu, t) \vee \delta pI_t(\mu, t)$ and by Theorem 4.15, $\mu$ is an $t$-fMo set.
(ii) Obvious. \hfill \square

\textbf{Theorem 4.19.} In a sfts $(X, \tau)$, $\forall \mu \in I^X$ and $t \in I_o$, (i)$&$(ii) are hold.

(i) $MI_t(1 - \mu, t) = 1 - (MC_t(\mu, t))$.
(ii) $MC_t(1 - \mu, t) = 1 - (MI_t(\mu, t))$.

\textbf{Proof.}  (i) $\forall \mu \in I^X$, $t \in I_0$
\[ 1 - (MC_t(\mu, t)) = 1 - \big\{ v : v \geq \mu, \text{v is r-fMo} \big\} \]
\[ = \big\{ 1 - v : 1 - v \leq 1 - \mu, 1 - v \text{is r-fMo} \big\} \]
\[ = MI_t(1 - \mu, t). \]
(ii) same as (i). \hfill \square

\textbf{Theorem 4.20.} In a sfts $(X, \tau)$, $\forall \mu \in I^X$ and $t \in I_o$, (i)–(vi) are hold.

(i) $MC_t(0, t) = 0$ and $MI_t(1, t) = 1$.
(ii) $I_t(\mu, t) \leq MI_t(\mu, t) \leq \mu \leq MC_t(\mu, t)$.
(iii) $\mu \leq v \Rightarrow MI_t(\mu, t) \leq MI_t(v, t)$ and $MC_t(\mu, t) \leq MC_t(v, t)$.
(iv) $MC_t(MC_t(\mu, t), t) = MC_t(\mu, t)$ and $MI_t(MI_t(\mu, t), t) = MI_t(\mu, t)$.
(v) $MC_t(\mu, t) \vee MC_t(v, t) < MC_t(\mu \vee v, t)$ and $MI_t(\mu, t) \vee MI_t(v, t) < MI_t(\mu \vee v, t)$.
(vi) $MC_t(\mu \vee v, t) < MC_t(\mu, t) \wedge MC_t(v, t)$ and $MI_t(\mu \vee v, t) < MI_t(\mu, t) \wedge MI_t(v, t)$.

\textbf{Proof.}  (i) It is trivial from the definitions of $MC_t$ and $MI_t$.
(ii) Since, by Theorem 4.15,
\[ MC_t(\mu, t) = \theta sC_t(\mu, t) \wedge \delta pC_t(\mu, t) \] and $\mu \leq v$,
\[ MC_t(\mu, t) = \theta sC_t(\mu, t) \wedge \delta pC_t(\mu, t) \]
\[ \leq \theta sC_t(v, t) \wedge \delta pC_t(v, t) \]
\[ = MC_t(v, t). \]
Similarly, by Theorem 4.15,
\[ MI_t(\mu, t) = \theta sI_t(\mu, t) \vee \delta pI_t(\mu, t) \] and $\mu \leq v$,
\[ MI_t(\mu, t) = \theta sI_t(\mu, t) \vee \delta pI_t(\mu, t) \]
\[ \leq \theta sI_t(v, t) \vee \delta pI_t(v, t) \]
\[ = MC_t(v, t). \]

(iv) Since, by Theorem 4.18,
\[ MC_t(MC_t(\mu, t), t) \]
\[ = \theta sC_t(MC_t(\mu, t), t) \wedge \delta pC_t(MC_t(\mu, t), t) \]
\[ = \theta sC_t(\theta sC_t(\mu, t) \wedge \delta pC_t(\mu, t), t) \wedge \]
\[ \delta pC_t(\theta sC_t(\mu, t) \wedge \delta pC_t(\mu, t), t) \]
\[ \leq \theta sC_t(\theta sC_t(\mu, t) \wedge \delta pC_t(\mu, t), t) \wedge \]
\[ \delta pC_t(\theta sC_t(\mu, t), t) \wedge \delta pC_t(\mu, t) \]
\[ = \theta sC_t(\mu, t) \wedge \delta pC_t(\mu, t) \]
\[ = MC_t(\mu, t). \]

Hence, $MC_t(MC_t(\mu, t), t) \leq MC_t(\mu, t)$.
But $MC_t(\mu, t) \leq MC_t(MC_t(\mu, t), t)$. Therefore,
\[ MC_t(MC_t(\mu, t), t) = MC_t(\mu, t). \]

(v) and (vi): Obvious. \hfill \square

\textbf{Lemma 4.21.} In a sfts $(X, \tau)$, $\forall \mu \in I^X$ and $t \in I_o$, (i)–(iv) are hold.

(i) $pI_t(\mu, t) = \delta pC_t(\mu, t) \wedge I_t(\theta C_t(\mu, t), t)$ and $\theta pC_t(\delta pI_t(\mu, t), t) = \mu \wedge pC_t(\mu, t) \wedge \theta C_t(\mu, t), t)$.
(ii) $\theta pC_t(\theta pC_t(\mu, t), t) = \delta pI_t(\mu, t) \wedge I_t(\theta C_t(\mu, t), t)$ and $\theta I_t(\theta pC_t(\mu, t), t) = \theta pC_t(\delta pI_t(\mu, t), t) \wedge \theta C_t(\mu, t), t)$.
(iii) $\theta sC_t(\theta I_t(\mu, t), t) = sC_t(\theta I_t(\mu, t), t)$
\[ = I_t(\theta C_t(\theta I_t(\mu, t), t), t). \]
(iv) $\theta sI_t(\mu, t) = sI_t(\theta C_t(\mu, t), t)$
\[ = C_t(\theta I_t(\theta C_t(\mu, t), t), t). \]

\textbf{Proof.}  Obvious. \hfill \square

\textbf{Proposition 4.22.} In a sfts $(X, \tau)$, $\forall \mu \in I^X$ and $t \in I_o$, (i)–(iii) are hold.

(i) $MC_t(\mu, t) = \mu \vee pI_t(\delta pC_t(\mu, t), t)$.
(ii) $MI_t(\mu, t) = \mu \wedge pC_t(\delta pI_t(\mu, t), t)$.

\textbf{Proof.}  (i) By Lemma 4.21,
\[ \mu \vee pI_t(\delta pC_t(\mu, t), t) \]
\[ = \mu \vee (\delta pC_t(\mu, t) \wedge I_t(\theta C_t(\mu, t), t)) \]
\[ = (\mu \vee \delta pC_t(\mu, t)) \wedge (\mu \vee I_t(\theta C_t(\mu, t), t)) \]
\[ = \delta pC_t(\mu, t) \wedge \theta sC_t(\mu, t) \]
\[ = MC_t(\mu, t). \]
(ii) Obvious. \hfill \square

\textbf{Theorem 4.23.} In a sfts $(X, \tau)$, $\forall \mu \in I^X$ and $t \in I_o$, (i)–(iii) are equivalent.

(i) $\mu$ is an $t$-fMo set.
(ii) $\mu \leq \theta pC_t(\delta pI_t(\mu, t), t)$.
(iii) $\theta pC_t(\mu, t) = \theta pC_t(\delta pI_t(\mu, t), t)$. 


Proof. (i) ⇒ (ii): Let \( \mu \) be an \( \text{fMo} \) set. Then by Theorem 4.18, \( M = MI_\tau(\mu, t) \) and by Proposition 4.22,

\[
\mu = \mu \land \theta \mu C_\tau(\delta pI_\tau(\mu, t), t)
\]

and hence \( \mu \leq \theta \mu C_\tau(\delta pI_\tau(\mu, t), t) \).

(ii) ⇒ (i): Let \( \mu \leq \theta \mu C_\tau(\delta pI_\tau(\mu, t), t) \). Then by Proposition 4.22,

\[
\mu \leq \mu \land \theta \mu C_\tau(\delta pI_\tau(\mu, t), t) = MI_\tau(\mu, t).
\]

So \( M \leq MI_\tau(\mu, t) \). Then \( M = MI_\tau(\mu, t) \) and hence, \( \mu \) is an t-fMo set.

(iii) ⇒ (ii): Obvious.

\[\square\]

Theorem 4.24. In a sfts \((X, \tau), \forall \mu \in I^X \) and \( t \in I_0 \), (i)–(iii) are equivalent.

(i) \( \mu \) is an t-fMo set.

(ii) \( \theta pI_\tau(\delta pC_\tau, \mu, t), t) \leq \mu \).

(iii) \( \theta pI_\tau(\delta pC_\tau, \mu, t) = \mu pC_\tau \delta pI_\tau(\mu, t), t) \).

Theorem 4.25. If \( \mu \) is a fuzzy subset of an extremely disconnected sfts \((X, \tau) \) and \( t \in I_0 \). Then (i) & (ii) are equivalent:

(i) \( \mu \) is an t-fuzzy open set.

(ii) \( \mu \) is an t-fMo set and t-fuzzy locally closed.

Proof. (i) ⇒ (ii): Obvious from Definitions 3.8 and 4.1.

(ii) ⇒ (i): Let \( \mu \) be an t-fuzzy open and a t-fuzzy locally closed subset of \( X \). Then \( \mu = v \land C_\tau(\mu, t) \) and

\[
\mu \leq C_\tau(\theta I_\tau(\mu, t), t) \lor I_\tau(\delta C_\tau(\mu, t), t).
\]

Hence,

\[
\mu \leq v \land [C_\tau(\theta I_\tau(\mu, t), t) \lor I_\tau(\delta C_\tau(\mu, t), t)]
\]

\[
\leq (v \land C_\tau(\theta I_\tau(\mu, t), t)) \lor (I_\tau(v \land C_\tau(\mu, t), t))
\]

\[
\leq (v \land C_\tau(I_\tau(\mu, t), t)) \lor (I_\tau(v \land C_\tau(\mu, t), t))
\]

\[
\leq (v \land I_\tau(C_\tau(\mu, t), t)) \lor (I_\tau(v \land C_\tau(\mu, t), t))
\]

\[
\leq I_\tau(v \land C_\tau(\mu, t), t) \lor I_\tau(v \land C_\tau(\mu, t), t)
\]

\[
= I_\tau(\mu, t) \lor I_\tau(\mu, t)
\]

\[
= I_\tau(\mu, t).
\]

Hence \( \mu \) is t-fuzzy open.

\[\square\]

Theorem 4.26. In a sfts \((X, \tau), \forall \mu, v \in I^X \) and \( t \in I_0 \), (i) & (ii) are hold.
5. Conclusion

We have introduced the concept of t-fuzzy θ-open (resp. t-fuzzy θ-closed), t-fuzzy M-open (resp. t-fuzzy M-closed) sets in fuzzy topological spaces in the sense of Šostak’s. Also, we have introduced t-fuzzy θ-interior (resp. t-fuzzy θ-closure), t-fuzzy M-interior (resp. t-fuzzy M-closure) and investigated some of their properties. Moreover, we investigated the relationships between t-fuzzy open, t-fuzzy θ-semiopen, t-fuzzy θ-open, t-fuzzy δ-semiopen, t-fuzzy δ-preopen, t-fuzzy a-open, t-fuzzy e-open and t-fuzzy e*-open in Šostak’s fuzzy topological spaces.

References

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