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Estimating coefficient bounds with respect to a generalized starlike functions on symmetric points

R. Ambrose Prabhu[1]* S. Bhaskaran[2]

Abstract

The purpose of this paper is to estimate the Certain Coefficient for generalized Starlike functions with reference to symmetric points described on the open unit disk for which $R^k_{\lambda,\delta}(\phi)$ of normalized analytic functions $f(z)$ that lies in a region with reference to 1 and symmetric with reference to the real axis.

Keywords

Analytic function, Univalent function, Starlike function, Convex function, Subordination, Hadamard product, Linear operator, Fekete-szegö Inequality.

AMS Subject Classification

Primary 30C45.

[1,2] *Department of Science and Humanities, Saveetha School of Engineering, SIMATS, Tamilnadu, India.*

*Corresponding author: [1] ancyamb@yahoo.com; [2]bhaskaran.hawk@gmail.com;

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1. Introduction

Let $\mathscr{A}$ symbolize the class of all analytic function $f(z)$ as concerns to

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

that are analytic in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and agree the conditions $f(0) = 0, f'(0) = 1$. Also $\mathscr{S}$ the subclass of $\mathscr{A}$ consisting of all functions that are univalent in $\mathbb{U}$. For $f(z)$ and $g(z)$ analytic in $\mathbb{U}$, $f(z)$ is said to subordinate to $g(z)$ when there exist a schwarz function $\omega(z)$, analytic in $\mathbb{U}$ amidst

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}),$$

satisfying

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).$$

This subordination is symbolized as

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

More precisely,, when $g(z)$ is univalent in $\mathbb{U}$, then the subordination be correspondent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Assume $\phi(z)$ an analytic function in $\mathbb{U}$ with $\phi(0) = 1$, $\phi'(0) > 0$ and $Re\{\phi(z)\} > 0$, $z \in \mathbb{U}$ that map $\mathbb{U}$ onto a starlike region with referrence to 1 and symmetric amidst the real axis. We signify $S^*(\phi)$ and $C(\phi)$, respectively, the subclasses of $\mathscr{A}$, that accomplish the relations of subordination:

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \ z \in \mathbb{U}$$

and

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), \ z \in \mathbb{U}.$$

The above functions were put forward and studied by Ma and Minda[9]. Specifically, while

$$\phi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}, \ z \in \mathbb{U}, 0 \leq \alpha < 1,$$

these functions diminish respectively to the established classes $S^*(\alpha)$, $(0 \leq \alpha < 1)$ of α in $\mathbb{U}$ and $C(\alpha)$, $(0 \leq \alpha < 1)$ α in $\mathbb{U}$. Ma and Minda [9], the Fekete-Szegö inequality for $f(z)$

in $C(\phi)$ was acknowledged in relating the classes $S^*(\phi)$ and $C(\phi)$ of Alexander result.
To recollect the Fekete-Szegö problems in consideration to starlike, convex including numerous subclasses in $\mathscr{A}$, the readers are advised to refer the work carried out by Srivatsava et al [20]. Moreover, the primary outcome be oblidged to Fekete and szegö [2] in 1933. Almost 30 years later, Keogh and Merkes [4] derived the problem considering certain subclasses of univalent functions. These articles **[2,4,6,7]** gave remarkable results that are used to solve problems for other extended classes. Thereafter, Shanmugam et al [17] contributed the Fekete-Szegö problem on view of subclasses of starlike functions up on symmetric points. Inspiring by the above work, we derive the Fekete-Szegö inequality in Theorem 2.1 given below, being extended to prevailing class of normalized analytic functions.
While $\lambda, \delta \in N$, $k \in N_0$ Darus [1] put forward the operator $D^k_{\lambda,\delta}$ characterized as

$$D^k_{\lambda,\delta} f(z) = z + \sum_{n=2}^{\infty} [1+(n-1)\lambda]^k C(\delta,n) a_n z^n \qquad (1.2)$$

In this paper, we attain the Fekete-szegö inequality considering the function $f \in \mathscr{A}$ in the class $R^k_{\lambda,\delta}(\phi)$ characterized as follows

Definition 1.1. *Let $D^k_{\lambda,\delta} : \mathscr{A} \longrightarrow \mathscr{A}$ is a linear operator and $D^k_{\lambda,\delta}$ is analytic in $f(U)$.*
Let

$$D^k_{\lambda,\delta} f(z) = z + \sum_{n=2}^{\infty} [1+(n-1)\lambda]^k C(\delta,n) a_n z^n$$

where

$$C(\delta,n) = \frac{\Gamma(n+\delta)}{\Gamma(n) \quad \Gamma(\delta+1)}$$

while $\lambda = 1$ and $\delta = 0$ sălăgean differential operator is obtained. when $k = 0$ or $\lambda = 0$ leads to Ruscheweyh operator, also $\delta = 0$ leads to Al-oboudi differential operator with order k

$$D^0_{1,0} f(z) = f(z), \;\; D^1_{1,0} f(z) = z f'(z)$$

Definition 1.2. *An univalent starlike function $\phi(z)$ with referrence to 1 that maps $\mathbb{U}$ onto the right half plane that is symmetric with referrence to the real axis $\phi(0) = 1$ and $\phi'(0) > 1$. $f \in \mathscr{A}$ that belongs to the class $R^k_{\lambda,\delta}(\phi)$ if*

$$\frac{(s-t)z\left[D^k_{\lambda,\delta} f(z)\right]'}{D^k_{\lambda,\delta}[f(sz)] - D^k_{\lambda,\delta}[f(tz)]} \prec \phi(z), \quad (\lambda, \delta \in \mathbb{N},\; k \in \mathbb{N}_0).$$

In order to establish our important results, we require the subsequent lemma.

Lemma 1.3. *[9] An analytic function $p_1(z) = 1 + c_1 z + c_2 z^2 + \ldots$ with positive real part in $\mathbb{U}$, then*

$$| c_2 - v c_1^2 | \leq \begin{cases} -4v+2, & \text{if } v \leq 0 \\ 2, & 0 \leq v \leq 1 \\ 4v-2, & v \geq 1. \end{cases}$$

while $v < 0$ or $v > 1$, the equality satisfies iff $p_1(z) = \dfrac{1+z}{1-z}$ or one of its rotations. When $0 < v < 1$, the equality satisfies iff $p_1(z) = \dfrac{1+z^2}{1-z^2}$ or one of its rotation.
When $v = 0$, the equality satisfies iff

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\gamma\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\gamma\right)\frac{1-z}{1+z} \quad (0 \leq \gamma \leq 1),$$

or one of its rotations. When $v = 1$, the equality satisfies iff p_1 is the reciprocal of one of the functions where the equality satisfies if $v = 0$. Moreover, the upper bound is sharp, the same can be improvised as follows, if $0 < v < 1$:

$$| c_2 - v c_1^2 | + v | c_1^2 | \leq 2, \quad 0 < v \leq \frac{1}{2}$$

$$\text{and} \quad | c_2 - v c_1^2 | + (1-v) | c_1^2 | \leq 2, \quad \frac{1}{2} < v \leq 1$$

The following result is more important forour enquiry.

Lemma 1.4. *[15] If $p_1(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is a function with positive real part in $\mathbb{U}$, then*

$$| c_2 - v c_1^2 | \leq 2.max\left(1, | 2v-1 |\right).$$

The conclusion is sharp for $p_1(z)$ given by

$$p_1(z) = \frac{1+z^2}{1-z^2}$$

and

$$p_1(z) = \frac{1+z}{1-z}.$$

2. Fekete-szegö Problem for the Function of the class $R^k_{\lambda,\delta}(\phi)$

Using Lemma 1.2, Fekete-szegö Problem for the class $R^k_{\lambda,\delta}(\phi)$ can be proved.

Theorem 2.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ If $f(z)$ given by (1.1) belongs to the class $R^k_{\lambda,\delta}(\phi)$, then*

$$| a_3 - \mu a_2^2 | \leq \begin{cases} \Lambda, & \mu \leq \sigma_1 \\ \eta, & \sigma_1 \leq \mu \leq \sigma_2 \\ -\Lambda, & \mu \geq \sigma_3. \end{cases}$$

where

$$\sigma_1 = \frac{(\delta+1)(s+t-2)(1+\lambda)^{2k}}{2(\delta+2)(1+2\lambda)^k[(s^2+st+t^2)-3]} \left[\frac{2(B_2-B_1)(s+t-2)-B_1^2(s+t)}{B_1^2}\right],$$

$$\sigma_2 = \frac{(\delta+1)(s+t-2)(1+\lambda)^{2k}}{2(\delta+2)(1+2\lambda)^k[(s^2+st+t^2)-3]} \left[\frac{2B_2(s+t-2)-(s+t)B_1^2}{B_1^2}\right],$$

$$\sigma_3 = \frac{(\delta+1)(s+t-2)(1+\lambda)^{2k}}{2(\delta+2)(1+2\lambda)^k[(s^2+st+t^2)-3]} \left[\frac{(2B_2-B_1)(s+t-2)-B_1^2(s+t)}{B_1^2}\right],$$

$$\Lambda = \frac{4(\delta+1)(\delta+2)(1+2\lambda)^k[3-(s^2+st+t^2)]}{B_1^2} \times\left[(B_2-B_1^2)-\frac{B_1^2}{2}\left(\frac{s+t}{s+t-2} +\frac{2\mu(1+2\lambda)^k(\delta+2)[(s^2+st+t^2)-3]}{(\delta+1)(1+\lambda)^{2k}(2-s-t)^2}\right)\right],$$

$$\eta = \frac{2B_1}{(1+2\lambda)^k(\delta+1)(\delta+2)(3-[s^2+st+t^2])}.$$

Further,
If $\sigma_1 \le \mu \le \sigma_3$, then

$$|a_3-\mu a_2^2| + \frac{(\delta+1)(s+t-2)(1+\lambda)^{2k}}{(\delta+2)(1+2\lambda)^k[(s^2+st+t^2)-3]B_1^2} \left\{(B_1-B_2)(s+t-2)+\sigma_4 B_1^2\right\} |a_2|^2 \le \eta.$$

If $\sigma_3 \le \mu \le \sigma_2$, then

$$|a_3-\mu a_2^2| + \frac{(\delta+1)(s+t-2)(1+\lambda)^{2k}}{(\delta+2)(1+2\lambda)^k[(s^2+st+t^2)-3]B_1^2} \left\{B_2(s+t-2)-\sigma_4 B_1^2\right\} |a_2|^2 \le \eta.$$

Where

$$\sigma_4 = \left[\frac{(s+t)(s+t-2)(\delta+1)(1+\lambda)^{2k}}{(\delta+1)(s+t-2)(1+\lambda)^{2k}} + \frac{\mu(\delta+2)(1+2\lambda)^k[(s^2+st+t^2)-3]}{(\delta+1)(s+t-2)(1+\lambda)^{2k}}\right].$$

The result is sharp.

Proof. When $f \in R^k_{\lambda,\delta}(\phi)$, there shall exist a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$ in $\mathbb{U}$ satisfying

$$\frac{(s-t)z\left[D^k_{\lambda,\delta}f(z)\right]'}{D^k_{\lambda,\delta}[f(sz)]-D^k_{\lambda,\delta}[f(tz)]} = \phi(w(z))$$

A function $p_1(z)$ is defined as

$$p_1(z) = \frac{1+w(z)}{1-w(z)}.$$

since $w(z)$ is a Schwarz function, it is known that $Re\{p_1(z)\} > 0$ and $p_1(0)=0$. A function $p(z)$ is defined by

$$p(z) = \frac{(s-t)z\left[D^k_{\lambda,\delta}f(z)\right]'}{D^k_{\lambda,\delta}[f(sz)]-D^k_{\lambda,\delta}[f(tz)]} \tag{2.1}$$

$$= \phi(w(z)) = 1+b_1z+b_2z^2+\cdots \tag{2.2}$$

From (2.1), we obtain

$$a_2 = \frac{b_1}{(1+\lambda)^k(\delta+1)(2-s-t)} \tag{2.3}$$

and

$$a_3 = 2\frac{b_2-(s+t)(s+t-2)(1+\lambda)^{2k}(\delta+1)^2a_2^2}{[3-(s^2+st+t^2)](1+2\lambda)^k(\delta+1)(\delta+2)}$$

since

$$p_1(z) = \frac{1+\phi^{-1}(p(z))}{1-\phi^{-1}(p(z))}$$

then

$$p(z) = \phi\left[\frac{p_1(z)-1}{p_1(z)+1}\right],$$

and

$$1+b_1z+b_2z^2+\cdots = \phi\left(\frac{c_1z+c_2z^2+\cdots}{2+c_1z+c_2z^2+\cdots}\right)$$

$$= \phi\left[\frac{1}{2}c_1z+\frac{1}{2}(c_2-\frac{1}{2}c_1^2)z^2+\cdots\right] \tag{2.4}$$

while equating the coefficients of z and z^2, we infer

$$b_1 = \frac{1}{2}B_1c_1 \quad \textit{and} \quad b_2 = \frac{1}{2}B_1(c_2-\frac{1}{2}c_1^2)+\frac{1}{4}B_2c_1^2 \tag{2.5}$$

From (2.2) and (2.4), we get

$$a_2 = \frac{B_1c_1}{2(1+\lambda)^k(\delta+1)(2-s-t)}$$

$$\textit{and} \quad a_3 = \frac{B_1}{(1+2\lambda)^k(\delta+1)(\delta+2)[3-(s^2+st+t^2)]} \left\{c_2-c_1^2\left(1-\frac{1}{2}\frac{B_2}{B_1}+\frac{1}{2}B_1(\frac{s+t}{s+t-2})\right)\right\}$$

Therefore we have

$$|a_3-\mu a_2^2| = \frac{B_1}{[3-(s^2+st+t^2)](1+2\lambda)^k(\delta+1)(\delta+2)} \left\{c_2-c_1^2\left(1-\frac{1}{2}\frac{B_2}{B_1}+\frac{1}{2}B_1(\frac{s+t}{s+t-2})\right)\right\} - \frac{\mu B_1^2c_1^2}{(\delta+1)^2(1+\lambda)^{2k}(2-s-t)}$$

$$= \frac{B_1}{[3-(s^2+st+t^2)](1+2\lambda)^k(\delta+1)(\delta+2)}\{c_2-vc_1^2\}$$

where

$$v=\Bigg[1-\frac{1}{2}\frac{B_2}{B_1}+\frac{B_1}{2}(\frac{s+t}{s+t-2}) -\frac{\mu B_1(\delta+2)(1+2\lambda)^k[3-(s^2+st+t^2)]}{(\delta+1)(1+\lambda)^{2k}(2-s-t)^2}\Bigg]$$

If $\mu \le \sigma_1$, using Lemmas 1.1 and 1.2, the following is obtained.

$| a_3-\mu a_2^2 |$

$$\le \frac{4(\delta+1)(\delta+2)(1+2\lambda)^k[3-(s^2+st+t^2)]}{B_1^2}\Bigg[(B_2-B_1)-\frac{B_1^2}{2}\Bigg(\frac{s+t}{s+t-2}+\frac{2\mu(\delta+2)(1+2\lambda)^k[(s^2+st+t^2)-3]}{(\delta+1)(1+\lambda)^{2k}(2-s-t)^2}\Bigg)\Bigg]$$

which is the first part of Theorem 1.1.

Similarly, if $\mu \ge \sigma_2$, using Lemmas 1.1 and 1.2, we get

$| a_3-\mu a_2^2 |$

$$\le \frac{4(\delta+1)(\delta+2)(1+2\lambda)^k[3-(s^2+st+t^2)]}{B_1^2}\Bigg[(B_1-B_2)+\frac{B_1^2}{2}\Bigg(\frac{s+t}{s+t-2}+\frac{2\mu(\delta+2)(1+2\lambda)^k[(s^2+st+t^2)-3]}{(\delta+1)(1+\lambda)^{2k}(2-s-t)^2}\Bigg)\Bigg]$$

when $\sigma_1 \le \mu \le \sigma_2$, we see that

$$| a_3-\mu a_2^2 | = \frac{2B_1\{c_2-vc_1^2\}}{2(1+2\lambda)^k(\delta+1)(\delta+2)(3-[s^2+st+t^2])} \le \frac{2B_1}{\delta+1(\delta+2)(1+2\lambda)^k[3-(s^2+st+t^2)]}$$

Further, if $\sigma_1 \le \mu \le \sigma_3$, then

$$| a_3-\mu a_2^2 | +(\mu-\sigma_1)\, | a_2^2 | \le \frac{2B_1}{\delta+1(\delta+2)(1+2\lambda)^k[3-(s^2+st+t^2)]}$$

Finally,we see that

If $\sigma_3 \le \mu \le \sigma_2$, then

$$| a_3-\mu a_2^2 | +(\sigma_2-\mu)\, | a_2^2 | \le \frac{2B_1}{\delta+1(\delta+2)(1+2\lambda)^k[3-(s^2+st+t^2)]}$$

In order to express that the bounds are sharp, the function $k_n^{\phi}(n=2,3,...)$ defined by

$$\frac{z(D_{\lambda,\delta}^k k_n^{\phi}(z))'}{D_{\lambda,\delta}^k k_n^{\phi}(z)}=\phi(z^{n-1}),\ k_n^{\phi}(0)=0=(k_n^{\phi}(0))'-1,$$

and the function F_γ and $G_\gamma(0\le\gamma\le 1)$ by

$$\frac{z(D_{\lambda,\delta}^k F_\gamma(z))'}{D_{\lambda,\delta}^k F_\gamma(z)}=\phi(\frac{z(z+\gamma)}{1+\gamma z}),\ F_\gamma(0)=0=(F_\gamma(0))'-1$$

and

$$\frac{z(D_{\lambda,\delta}^k G_\gamma(z))'}{D_{\lambda,\delta}^k G_\gamma(z)}=\phi(-\frac{z(z+\gamma)}{1+\gamma z}),\ G_\gamma(0)=0=(G_\gamma(0))'-1$$

Clearly the functions k_n^{ϕ}, F_γ and $G_\gamma \in R_{\lambda,\delta}^k(\phi)$. It can also be denoted as $K^{\phi}=K_2^{\phi}$. If $\mu<\sigma_1$ or $\mu>\sigma_2$, then the equality in Theorem 2.1 satisfies iff f is K^{ϕ} or one of its rotations.

If $\sigma_1<\mu<\sigma_2$, then the equality satisfies iff f is K_3^{ϕ} or one of its rotations.

When $\mu=\sigma_1$, the equality satisfies iff f is F_γ or one of its rotations. When $\mu=\sigma_2$, the equality satisfies iff f is G_γ or one of its rotations.

□

Using Lemma 1.2,the following theorem can easily be obtained.

Theorem 2.2. *Let $\phi(z)=1+B_1z+B_2z^2+\cdots$, where B_n are real with $B_1>0$ and $B_2\ge 0$. If $f(z)$ given by (1.1) belongs to $R_{\lambda,\delta}^k(\phi)$, then*

$| a_3-\mu a_2^2 |$

$$\le \Bigg(\frac{4B_1}{(1+2\lambda)^k(\delta+1)(\delta+2)[3-(s^2+st+t^2)]}\Bigg) \times \max\Bigg\{1,\Bigg|1-\frac{2B_2}{B_1}+B_1\Bigg(\frac{s+t}{s+t-2} -\frac{2\mu(\delta+2)(1+2\lambda)^k[3-(s^2+st+t^2)]}{(\delta+1)(2-s-t)^2(1+\lambda)^{2k}}\Bigg)\Bigg|\Bigg\},$$

$\mu\in\mathbb{C}$. The result is Sharp.

Remark 2.3. *The coefficient bounds for $| a_2 |$ and $| a_3 |$ are special cases of those claimed by Theorem 2.1*

Remark 2.4. *In its distinctive case when $\lambda=1$, $\delta=0$ and $k=0$, a known result of Ma and Minda [9] was arrived.*

Remark 2.5. *In its distinctive case, when $\lambda=1$, $\delta=0$ and $k=0$,$s=1$, $t=-1$, a known result due to T.N Shanmugam et al [17] was arrived.*

3. Applications to Analytic Function Defined by Fractional Calculus

The dependece of fractional calculus has earned appreciable demand upon early decades. Two of the current contributions on this area of deeprooted investigations include comprehensive treatises on the theory and applications of fractional differential equations by Podlubny [13] and Kilbas et al.[5]

We first introduce the class $\mathscr{M}_{\alpha,\beta,\lambda}^{\delta}(\phi)$, that is defined using Hadamard product and a certain operator Owa-Srivatsava operator (see for details,[18] and [8] ; see also [11], [12] and [21]) in fractional calculus.

Definition 3.1. *The fractional integral of order δ is elucidated for $f(z)$,*

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_o^z \frac{f(\zeta)}{(z-\zeta)^{1-\delta}} d\zeta \quad (\delta > 0) \qquad (3.1)$$

where $f(z)$ is analytic in a simply-connected domain of the complex z-plane that contains the origin and the multiplicity of $(z-\zeta)^{\delta-1}$ is expelled by including $log(z-\zeta)$ to be real when $z-\zeta > 0$

Definition 3.2. *The fractional derivative of order δ is elucidated, for $f(z)$*

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \int_o^z \frac{f(\zeta)}{(z-\zeta)^{\delta}} d\zeta \quad (0 \le \delta < 1) \qquad (3.2)$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\delta}$ is expelled by requiring $log(z-\zeta)$ to be real when $z-\zeta > 0$

Definition 3.3. *Using Definition 3.2, the fractional derivative of order $n+\delta$ is elucidated, for$f(z)$*

$$D_z^{n+\delta} f(z) = \frac{d^n}{dz^n}\left\{D_z^{\delta} f(z)\right\} \quad (0 \le \delta < 1; n \in \mathbb{N}_0) \qquad (3.3)$$

Using Definitions 3.1, 3.2 and 3.3 of fractional derivatives and fractional integrals, Owa and Srivatsava put forward the Owa-Srivatsava operator

$$(\Omega^{\delta} f)(z) = \Gamma(2-\delta) z^{\delta} D_z^{\delta} f(z), \quad (\delta \ne 2,3,4,\dots) \qquad (3.4)$$

In terms of the Owa-Srivatsava operator Ω^{δ} defined by 3.4, we now introduce the function class $\mathscr{M}^{\delta}_{\alpha,\beta,\lambda}(\phi)$ in the following way:

$$\mathscr{M}^{\delta}_{\alpha,\beta,\lambda}(\phi) = \left\{ f : f \in \mathscr{A} \text{ and } \Omega^{\delta} f \in \mathscr{M}^{\delta}_{\alpha,\beta,\lambda}(\phi)\right\}. \qquad (3.5)$$

Evidently, the function class $\mathscr{M}^{\delta}_{\alpha,\beta,\lambda}(\phi)$ is a special case of the function class $\mathscr{M}^{g}_{\alpha,\beta,\lambda}(\phi)$ if

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} z^n. \qquad (3.6)$$

Suppose

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \quad (g_n > 0).$$

since

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{M}^{\delta}_{\alpha,\beta,\lambda}(\phi) \qquad (3.7)$$

if and only if

$$(f*g)(z) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in \mathscr{M}_{\alpha,\beta,\lambda}(\phi)$$

the coefficient estimates are obtained for the functions in the class $\mathscr{M}^{g}_{\alpha,\beta,\lambda}(\phi)$ with respect to the corresponding estimates for functions in the class $\mathscr{M}_{\alpha,\beta,\lambda}(\phi)$. Using Theorem 2.1 to the following Hadamard product:

$$(f*g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \dots,$$

we infer the Theorem 3.4 defined below after an evident change of the parameter μ.

Theorem 3.4. *Let*

$$0 \le \mu \le 1, 0 \le \alpha \le 1, 0 < \beta \le 1 \text{ and } 0 \le \lambda \le 1.$$

Suppose

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots,$$

where the coefficients B_n are real with

$$B_1 > 0, B_2 > 0, \text{ and } B_n > 0 \quad (n \in \mathbb{N}\backslash\{1,2\}).$$

When $f(z)$ given by (1.1) be the part of the class $\mathscr{M}^{g}_{\alpha,\beta,\lambda}(\phi)$, then

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{\Lambda}{g_3}, & \text{if } \mu \le \sigma_5 \\ \dfrac{\eta}{g_3}, & \sigma_5 \le \mu \le \sigma_6 \\ -\dfrac{\Lambda}{g_3}, & \mu \ge \sigma_6. \end{cases}$$

where

$$\sigma_5 = \frac{g_3}{g_2^2} \frac{(\delta+1)(s+t-2)(1+\lambda)^{2k}}{2(\delta+2)(1+2\lambda)^k[(s^2+st+t^2)-3]} \left[\frac{2(B_2-B_1)(s+t-2) - B_1^2(s+t)}{B_1^2}\right]$$

$$\sigma_6 = \frac{g_3}{g_2^2} \frac{(\delta+1)(s+t-2)(1+\lambda)^{2k}}{2(\delta+2)(1+2\lambda)^k[(s^2+st+t^2)-3]} \left[\frac{2B_2(s+t-2) - (s+t)B_1^2}{B_1^2}\right]$$

where Λ and η are elucidated in Theorem 2.1, respectively. These results are sharp. Since, by (1.1) and 3.4,

$$(\Omega^{\delta} f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} a_n z^n, \qquad (3.8)$$

we readily obtain

$$g_2 = \frac{\Gamma(3)\Gamma(2-\delta)}{\Gamma(3-\delta)} = \frac{2}{2-\delta} \qquad (3.9)$$

and

$$g_3 = \frac{\Gamma(4)\Gamma(2-\delta)}{\Gamma(4-\delta)} = \frac{6}{(2-\delta)(3-\delta)}. \qquad (3.10)$$

For g_2 and g_3 obtained by (3.9) and (3.10), respectively, Theorem 3.4 diminishes to the interesting result.

Theorem 3.5. *Let*

$$0 \leq \mu \leq 1, 0 \leq \alpha \leq 1, 0 < \beta \leq 1 \text{ and } 0 \leq \lambda \leq 1.$$

Suppose also that

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots,$$

where the coefficients B_n are real with

$$B_1 > 0, B_2 > 0, \text{and} B_n > 0 \ \ (n \in \mathbb{N} \backslash \{1,2\}).$$

If $f(z)$ given by (1.1) belongs to the class $\mathscr{M}^g_{\alpha,\beta,\lambda}(\phi)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{(2-\delta)(3-\delta)}{6}\Lambda, & \text{if } \mu \leq \sigma_7 \\ \dfrac{(2-\delta)(3-\delta)}{6}\eta, & \sigma_7 \leq \mu \leq \sigma_8 \\ -\dfrac{(2-\delta)(3-\delta)}{6}\Lambda, & \mu \geq \sigma_8. \end{cases}$$

where, for convenience,

$$\sigma_7 = \frac{2(3-\delta)}{3(2-\delta)} \frac{g_3}{g_2^2} \frac{(\delta+1)(s+t-2)(1+\lambda)^{2k}}{2(\delta+2)(1+2\lambda)^k[(s^2+st+t^2)-3]} \left[\frac{2(B_2-B_1)(s+t-2) - B_1^2(s+t)}{B_1^2}\right]$$

$$\sigma_8 = \frac{2(3-\delta)}{3(2-\delta)} \frac{(\delta+1)(s+t-2)(1+\lambda)^{2k}}{2(\delta+2)(1+2\lambda)^k[(s^2+st+t^2)-3]} \left[\frac{2B_2(s+t-2) - (s+t)B_1^2}{B_1^2}\right]$$

and Λ and η are defined as in Theorem 2.1, respectively.

Remark 3.6. *In its special case, when*

$$\lambda = 0,\ \beta = 1\ \alpha = 0,\ B_1 = \frac{8}{\pi^2} \text{ and } B_2 = \frac{16}{3\pi^2}$$

Theorem 3.5 coincides with the following result due to Srivatsava et al. [19] for which $\Omega^\lambda f(z)$ is a parabolic starlike function ([3] and [16]).

Remark 3.7. *When*

$$\lambda = 0,\ \beta = 1\ \alpha = 0,\ \delta = 1,\ B_1 = \frac{8}{\pi^2} \text{ and } B_2 = \frac{16}{3\pi^2}$$

Theorem 3.5 would coincide with the result obtained earlier by Ma and Minda [10]

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