(∈, ∈ ∨qk)-fuzzy soft near-rings and (∈, ∈ ∨qk)-fuzzy soft ideals over near-rings

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Abstract
The concept of quasi-coincidence of a fuzzy point with a fuzzy subset has played a vital role in generating various fuzzy algebraic structures. In this article, we introduce the notions of (∈, ∈ ∨qk)-fuzzy soft near-ring and (∈, ∈ ∨qk)-fuzzy soft ideal over a near-ring which are generalization of (∈, ∈ ∨q)-fuzzy soft near-ring and (∈, ∈ ∨q)-fuzzy soft ideal respectively and study some of their properties with examples. We also introduce the notion of (∈, ∈ ∨qk)-fuzzy soft subnear-ring (resp. ideal) of an (∈, ∈ ∨qk)-fuzzy soft near-ring and obtain some related results.

Keywords

AMS Subject Classification
16Y30, 08A72, 06D72.

In [9], Dheena and Coumaressane defined (∈, ∈ ∨qk)-fuzzy subnear-ring and (∈, ∈ ∨qk)-fuzzy ideals of near-rings. For more details on (∈, ∈ ∨qk)-fuzzy algebraic structures, we refer the readers to [5, 10, 11, 13–15, 23]. In 1999, Russian researcher Molodtsov initiated the concept of a soft set [18] which can be known as an effective mathematical tool to deal with uncertainty. Maji et al. [16] presented the definitions of fuzzy soft set and fuzzy soft set operations. Aygün and Aygun [4] gave the concept of fuzzy soft group and discussed some of its properties. Inan and Oztürk [12] defined the concepts of fuzzy soft ring and (∈, ∈ ∨qk)-fuzzy soft ring over a ring. In [20], Özgür and Inan have extended these concepts to near-rings. In this article, we introduce the notions of (∈, ∈ ∨qk)-fuzzy soft near-ring and (∈, ∈ ∨qk)-fuzzy soft ideal over a near-ring which are generalization of (∈, ∈ ∨qk)-fuzzy soft near-ring and (∈, ∈ ∨qk)-fuzzy soft ideal respectively and study some of their properties with examples. We also introduce the definitions of (∈, ∈ ∨qk)-fuzzy soft subnear-ring (resp. ideal) of an (∈, ∈ ∨qk)-fuzzy soft near-ring and obtain some related results.

1. Introduction

Zadeh [24] initiated the fundamental concept of fuzzy set which provides a framework for generalization of many basic concepts of algebraic structures. The quasi-coincident concept on a fuzzy point with a fuzzy subset was introduced by Ming and Ming in [17]. This concept has played an important role to generate various fuzzy algebraic substructures. In particular, Sandeep Kumar Bhakat and Pratulanada Das [7] initiated the notion of an (∈, ∈ ∨qk)-fuzzy subgroup which is an important generalization of a fuzzy subgroup defined in [6]. In [8], they have extended this notion to rings. Narayanan and Manikantan [19] have introduced the notions of an (∈, ∈ ∨qk)-fuzzy subnear-ring and (∈, ∈ ∨qk)-fuzzy ideals of near-rings.
2. Preliminaries

In this section, the basic concepts which will be used in the subsequent sections are given.

A near-ring [22] is a non-empty set $N$ with two binary operations “$+$” and “$\cdot$” satisfying the axioms: (i) $(N,+)$ is a group, (ii) $(N,\cdot)$ is a semigroup and (iii) $(u+v)\cdot w = u\cdot w + v\cdot w$ for all $u,v,w \in N$. Since it satisfies the right distributive law, we call it a right near-ring. We will use the word “near-ring” to mean “right near-ring”. In what follows, $N$ denotes the near-ring unless otherwise specified.

Let $X$ be a non-empty set. A fuzzy subset $\gamma$ of $X$ is a mapping from $X$ to $[0,1]$ (see [24]).

A fuzzy subset $\gamma$ of $X$ is of the form

$$\gamma(v) = \begin{cases} r \in (0,1] & \text{if } v = u, \\ 0 & \text{if } v \neq u, \end{cases}$$

is called a fuzzy point with support $u$ and value $r$ and is denoted by $u_r$ (see [17]). For a fuzzy point $u_r$ and a fuzzy subset $\gamma$ in $X$, we say that

(i) $u_r \in \gamma$ if $\gamma(u) \geq r$. In this case, $u_r$ is said to belong to $\gamma$ (see [17]).

(ii) $u_r \gamma$ if $\gamma(u) + r > 1$. In this case, $u_r$ is said to be quasi-coincident with $\gamma$ (see [17]).

(iii) $u_r \gamma$ if $\gamma(u) + r > 1 - k$, where $k \in [0,1]$. In this case, $u_r$ is said to be k-quasi-coincident with $\gamma$ (see [9]).

We say that, $u_r \in \gamma \gamma \gamma$ (resp. $x_i \in \gamma \gamma \gamma$) means $x_i \in \gamma$ or $x_i \gamma \gamma \gamma$ (resp. $x_i \in \gamma$ and $x_i \gamma \gamma \gamma$) and $x_i \gamma \gamma \gamma$ will respectively mean $x_i \in \gamma$ and $x_i \in \gamma \gamma \gamma$ does not hold.

**Definition 2.1.** [1] A fuzzy subset $\gamma$ of $N$ is called a fuzzy subnear-ring of $N$ if for all $u,v \in N$,

(i) $\gamma(u - v) \geq \min\{\gamma(u), \gamma(v)\}$,

(ii) $\gamma(uv) \geq \min\{\gamma(u), \gamma(v)\}$.

**Definition 2.2.** [1] A fuzzy subset $\gamma$ of $N$ is called a fuzzy ideal of $N$ if for all $u,v,w \in N$,

(i) $\gamma(u - v) \geq \min\{\gamma(u), \gamma(v)\}$,

(ii) $\gamma(v + u - v) \geq \gamma(u)$,

(iii) $\gamma(uv) \geq \gamma(u)$,

(iv) $\gamma(u(v + w) - uv) \geq \gamma(w)$.

**Definition 2.3.** [9] A fuzzy subset $\gamma$ of $N$ is called an $(\in, \in \cup q_k)$-fuzzy subnear-ring (briefly, $(\in, \in \cup q_k)$-FN) of $N$ if

(i) $u_r \in \gamma, v_s \in \gamma \Rightarrow (u + v)_{\min\{r,s\}} \in \gamma_k \gamma$,

(ii) $u_r \in \gamma \Rightarrow (-u)_r \in \gamma_k \gamma$,

(iii) $u_r \in \gamma, v_s \in \gamma \Rightarrow (uv)_{\min\{r,s\}} \in \gamma_k \gamma$

for all $u,v \in N$ and $r,s \in (0,1]$. An $(\in, \in \cup q_k)$-fuzzy subnear-ring of $N$ with $k = 0$ is an $(\in, \in \cup q_k)$-fuzzy subnear-ring of $N$ (see [19]).

**Definition 2.4.** [9] A fuzzy subset $\gamma$ of $N$ is called an $(\in, \in \cup q_k)$-fuzzy ideal (briefly, $(\in, \in \cup q_k)$-FI) of $N$ if

(i) $u_r \gamma, v_t \gamma \Rightarrow (u - v)_{\min\{r,t\}} \in \gamma_k \gamma$,

(ii) $u_r \gamma, v_t \gamma \Rightarrow (v + u - v)_{\min\{r,t\}} \in \gamma_k \gamma$,

(iii) $u_r \gamma, v_t \gamma \Rightarrow (uv)_{\min\{r,t\}} \in \gamma_k \gamma$,

(iv) $w_r \gamma, u_t \gamma, v_s \gamma \Rightarrow (uv + w - uv)_{r,t,s} \in \gamma_k \gamma$

for all $u,v,w \in N$ and $r,s \in (0,1]$. An $(\in, \in \cup q_k)$-fuzzy ideal of $N$ with $k = 0$ is an $(\in, \in \cup q_k)$-fuzzy ideal of $N$ (see [19]).

In what follows, $U$ is a basic universe set, $E$ is a set of parameters, $P \subseteq E$ and $\mathcal{P}(U)$ is the power set of $U$ unless otherwise specified.

**Definition 2.5.** [18] A pair $(H, P)$ is called a soft set over $U$, where $H : P \rightarrow \mathcal{P}(U)$ is a mapping and $P \subseteq E$.

**Definition 2.6.** [16] Let $\mathcal{F}(U)$ be a set of all fuzzy subsets of $U$ and $P \subseteq E$. A pair $(\Delta, P)$ is called a fuzzy soft set (FSS, in short) over $U$, where $\Delta : P \rightarrow \mathcal{F}(U)$ is a mapping. That is, for each $p \in P$, $\Delta(p) = \Delta_p : U \rightarrow [0,1]$ is a fuzzy subset of $U$.

**Definition 2.7.** [21] Let $(\Delta, P)$ be a FSS over $U$. Then the set $\text{Supp}(\Delta, P) = \{p \in P | \Delta(p) \neq \emptyset\}$ is called the support of $(\Delta, P)$. A FSS $(\Delta, P)$ is called non-null if $\text{Supp}(\Delta, P) \neq \emptyset$. It is denoted by $(\Delta, P) \neq \emptyset$.

**Definition 2.8.** [16] Let $(\Delta, P_1)$ and $(\Theta, P_2)$ be FSSs over $U$. Then,

(i) “$(\Delta, P_1)$ AND $(\Theta, P_2)$”, shown by $(\Delta, P_1) \wedge (\Theta, P_2)$, is defined as $(\Delta, P_1) \wedge (\Theta, P_2) = (\Psi, P)$, where $P = P_1 \times P_2$ and $\Psi(p, \sigma) = \Delta(p) \cap \Theta(\sigma)$ for all $(p, \sigma) \in P_1 \times P_2$,

(ii) “$(\Delta, P_1)$ OR $(\Theta, P_2)$”, denoted by $(\Delta, P_1) \vee (\Theta, P_2)$, is defined as $(\Delta, P_1) \vee (\Theta, P_2) = (\Psi, P)$, where $P = P_1 \times P_2$ and $\Psi(p) = \Delta(p) \cup \Theta(p)$ for all $(p, \sigma) \in P_1 \times P_2$.

**Definition 2.9.** [16] The union of FSSs $(\Delta, P_1)$ and $(\Theta, P_2)$ over $U$ is expressed as a FSS $(\Psi, P)$, where $P = P_1 \cup P_2$ and for all $p \in P$,

$$\Psi(p) = \begin{cases} \Delta(p) & \text{if } p \in P_1 \setminus P_2, \\ \Theta(p) & \text{if } p \in P_2 \setminus P_1, \\ \Delta(p) \cup \Theta(p) & \text{if } p \in P_1 \cap P_2. \end{cases}$$

This is denoted by $(\Delta, P_1) \cup (\Theta, P_2) = (\Psi, P)$.

**Definition 2.10.** [3] The restricted intersection of FSSs $(\Delta, P_1)$ and $(\Theta, P_2)$ over $U$, shown by $(\Delta, P_1) \cap (\Theta, P_2)$, is defined as $(\Delta, P_1) \cap (\Theta, P_2) = (\Psi, P)$, where $P = P_1 \cap P_2 \neq \emptyset$ and $\Psi(p) = \Delta(p) \cap \Theta(p)$ for all $p \in P$. 

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Definition 2.11. [3] The extended intersection of FSSs \((\Delta, P_1)\) and \((\Theta, P_2)\) over \(U\) is expressed as a FSS \((\Psi, P)\), where \(P = P_1 \cup P_2\) and \(\Psi(p) = \begin{cases} \Delta(p) \quad \text{if } p \in P_1 \setminus P_2, \\ \Theta(p) \quad \text{if } p \in P_2 \setminus P_1, \\ \Delta(p) \cup \Theta(p) \quad \text{if } p \in P_1 \cap P_2. \end{cases}\)

This is denoted by \((\Delta, P_1) \cap_{\Delta} (\Theta, P_2) = (\Psi, P)\).

Definition 2.12. [2] For the family of FSSs \(\{(\Delta_i, P_i)|i \in \Omega\}\) over \(U\), \(\Omega\) is an index set.

(i) the restricted intersection of these FSSs is a FSS \(\bigcap_{i \in \Omega} (\Delta_i, P_i)|i \in \Omega\rangle = (\Delta, P)\), where \(P = \bigcap_{i \in \Omega} P_i \neq \emptyset\) and \(\Delta(p) = \bigcap_{i \in \Omega} \Delta_i(p)\) for all \(p \in P\).

(ii) the union of these FSSs is a FSS \(\bigcup_{i \in \Omega} (\Delta_i, P_i)|i \in \Omega\rangle = (\Delta, P)\), where \(P = \bigcup_{i \in \Omega} P_i\) and \(\Delta(p) = \bigcup_{i \in \Omega} \Delta_i(p)\) for all \(p \in P\) and \(\Omega(p) = \{i \in \Omega : p \in P_i\}\).

(iii) the AND of these FSSs is a FSS \(\bigwedge_{i \in \Omega} (\Delta_i, P_i)|i \in \Omega\rangle = (\Delta, P)\), where \(P = \prod_{i \in \Omega} P_i\) and \(\Delta(p) = \bigwedge_{i \in \Omega} \Delta_i(p)\) for all \(p = (p_i)_{i \in \Omega} \in P\).

Definition 2.13. [20] Let \((\Delta, P)\) be a FSS over \(N\). Then \((\Delta, P)\) is called a fuzzy soft near-ring (FSN) for short if \(\Delta(p)\) is a fuzzy subnear-ring of \(N\) for all \(p \in P\).

Definition 2.14. [20] Let \(\eta\) and \(\tau\) be fuzzy subnear-rings of \(N\). Then \(\eta\) is called a fuzzy subnear-ring of \(\tau\) if \(\eta(u) \leq \tau(u)\) for all \(u \in N\).

Definition 2.15. [20] Let \(\eta\) be a fuzzy subset of \(N\) and \(\tau\) be a fuzzy subnear-ring of \(N\). Then \(\eta\) is called a fuzzy ideal of \(\tau\) if \(\eta(u) \leq \tau(u)\) for all \(u \in N\) and \(\eta\) is a fuzzy ideal of \(N\).

Definition 2.16. [20] Let \((\Delta_1, P_1)\) and \((\Theta_2, P_2)\) be FSNs over \(N\). Then \((\Delta_1, P_1)\) is called a fuzzy soft subnear-ring of \((\Theta_2, P_2)\) if \(P_1 \subseteq P_2\) and \(\Delta(p)\) is a fuzzy subnear-ring of \(\Theta(p)\) for all \(p \in P_1\).

Definition 2.17. [20] Let \((\eta, Q)\) be a FSS and \((\Delta, P)\) be a FSS over \(N\). Then \((\eta, Q)\) is called a fuzzy soft ideal (FSI, for short) of \((\Delta, P)\) if \(Q \subseteq P\) and \(\eta(p)\) is a fuzzy ideal of \(\Delta(p)\) for all \(p \in Q\).

3. \((\epsilon, \in \forall q_k)\)-fuzzy soft near-rings and \((\epsilon, \in \forall q_k)\)-fuzzy soft ideals

In this section, we introduce the notions of \((\epsilon, \in \forall q_k)\)-fuzzy soft near-ring and \((\epsilon, \in \forall q_k)\)-fuzzy soft ideal over a near-ring \(N\). Further we define the notion of \((\epsilon, \in \forall q_k)\)-fuzzy soft subnear-ring (resp. ideal) of an \((\epsilon, \in \forall q_k)\)-fuzzy soft near-ring and investigate some of their properties.

Definition 3.1. Let \((\Delta, P)\) be a FSS over \(N\). Then \((\Delta, P)\) is called an \((\epsilon, \in \forall q_k)\)-fuzzy soft near-ring (briefly, \((\epsilon, \in \forall q_k)\)-FSN) over \(N\) if for each \(p \in P\), \(\Delta(p) = \Delta_p\) is an \((\epsilon, \in \forall q_k)\)-FN of \(N\), that is,

\[ \begin{align*} (i) & \ u_r \in \Delta_p, \ v_s \in \Delta_p \Rightarrow (u+v)_{\min_{(r,s)}} \in \forall q_k \Delta_p, \\ (ii) & \ u_r \in \Delta_p \Rightarrow (-u)_r \in \forall q_k \Delta_p, \\ (iii) & \ u_r \in \Delta_p, \ v_s \in \Delta_p \Rightarrow (uv)_{\min_{(r,s)}} \in \forall q_k \Delta_p, \end{align*} \]

for all \(u, v \in N\) and \(r, s \in \{0,1\}\).

We know that a fuzzy subset \(\Delta_p\) of \(N\) satisfies two conditions \(i)\) and \(ii)\) if and only if it satisfies: \(iv)\) \(u_r, v_s \in \Delta_p \Rightarrow (u-v)_{\min_{(r,s)}} \in \forall q_k \Delta_p\) for all \(u, v \in N\) and \(r, s \in \{0,1\}\).

An \((\epsilon, \in \forall q_k)\)-fuzzy soft near-ring over \(N\) with \(k = 0\) is an \((\epsilon, \in \forall q)\)-fuzzy soft near-ring over \(N\) (see [20]).

Lemma 3.2. The conditions \(i)\) – \(iv)\) in Definition 3.1 are equivalent to

\[ \begin{align*} (i) & \ \Delta_p(u+v) \geq \min_{(r,s)} \{\Delta_p(u), \Delta_p(v), \frac{1-k}{r+s}\}, \\ (ii) & \ \Delta_p(-u) \geq \min_{(r,s)} \{\Delta_p(u), \frac{1-k}{r+s}\}, \\ (iii) & \ \Delta_p(uv) \geq \min_{(r,s)} \{\Delta_p(u), \Delta_p(v), \frac{1-k}{r+s}\}, \\ (iv) & \ \Delta_p(u-v) \geq \min_{(r,s)} \{\Delta_p(u), \Delta_p(v), \frac{1-k}{r+s}\}. \end{align*} \]

for all \(u, v \in N\) respectively.

Proof. From Lemma 3.2 and Lemma 3.3 of [9], the proof can be achieved. \(\square\)

Remark 3.3. The condition \(i)\) with the condition \(ii)\) in Lemma 3.2 is equivalent to the condition \(iv)\).

Theorem 3.4. Let \((\Delta, P)\) be a FSS over \(N\). Then \((\Delta, P)\) is called an \((\epsilon, \in \forall q_k)\)-FSN over \(N\) if and only if for all \(p \in P\) and \(u, v \in N\),

\[ \begin{align*} (i) & \ \Delta_p(u-v) \geq \min_{(r,s)} \{\Delta_p(u), \Delta_p(v), \frac{1-k}{r+s}\}, \\ (ii) & \ \Delta_p(uv) \geq \min_{(r,s)} \{\Delta_p(u), \Delta_p(v), \frac{1-k}{r+s}\}. \end{align*} \]

Proof. The proof follows from Lemma 3.2. \(\square\)

Example 3.5. Let \(N = \{0, u, v, w\}\) be a near-ring with \((N, +)\) and \((N, \cdot)\) as defined below (Scheme 20 : (7,8,1,2) see [22], p. 408).

\[
\begin{array}{cccc}
+ & 0 & u & v \\
0 & 0 & u & v \\
u & u & 0 & w \\
v & v & w & 0 \\
w & w & v & u
\end{array}
\]

Define a FSS \((\Delta, P)\) over \(N\), where \(P = \{a, b, c\}\), as follows:

\[ \begin{align*} \Delta_a = \{(0, 0.7), (u, 0.5), (v, 0.6), (w, 0.5)\}, \\ \Delta_b = \{(0, 0.6), (u, 0.5), (v, 0.2), (w, 0.2)\} \text{ and } \Delta_c = \{(0, 0.4), (u, 0.1), (v, 0.3), (w, 0.1)\}. \end{align*} \]

We can easily check that \((\Delta, P)\) is an \((\epsilon, \in \forall q_k)\)-FSN over \(N\) with \(k = 0.4\).
Definition 3.6. Let $(\eta, Q)$ be a FSS over $N$. Then $(\eta, Q)$ is called an $(\in, \in \forall k_q)$-fuzzy soft ideal (briefly, $(\in, \in \forall k_q)$-FSI) over $N$ if for each $p \in Q$, $\eta(p) = \eta$ is an $(\in, \in \forall k_q)$-FI of $N$, that is,

(i) $u_r \in \eta_r, v_s \in \eta \Rightarrow (u-v)_{\min(r,s)} \in \forall k_q \eta_p$.

(ii) $u_r \in \eta_r, v_n \in N \Rightarrow (v+u-v) \in \forall q_k \eta_r$.

(iii) $u_r \in \eta_r, v_n \in N \Rightarrow (uv), \in \forall k_q \eta_r$.

(iv) $u_r \in \eta_r, v_n \in N \Rightarrow (v(w+u)-vw), \in \forall k_q \eta_r$.

for all $u, v, w \in N$ and $r,s \in (0, 1]$.

An $(\in, \in \forall k_q)$-fuzzy soft ideal over $N$ with $k = 0$ is called an $(\in, \in \forall q)$-fuzzy soft ideal over $N$.

Lemma 3.7. The conditions (i) – (iv) in Definition 3.6 are equivalent to

(i) $\eta_p(u-v) \geq \min\{\eta_p(u), \eta_p(v), \frac{1-k}{2}\}$,

(ii) $\eta_p(v+u-v) \geq \min\{\eta_p(u), \frac{1-k}{2}\}$,

(iii) $\eta_p(uv) \geq \min\{\eta_p(u), \frac{1-k}{2}\}$,

(iv) $\eta_p(u(v+w)-uv) \geq \min\{\eta_p(w), \frac{1-k}{2}\}$.

for all $u, v, w \in N$, respectively.

Proof. From Lemma 3.11 of [9], the proof can be achieved.

Theorem 3.8. Let $(\eta, Q)$ be a FSS over $N$. Then $(\eta, Q)$ is an $(\in, \in \forall k_q)$-FSI over $N$ if and only if for all $p \in P$ and $u, v, w \in N$,

(i) $\eta_p(u-v) \geq \min\{\eta_p(u), \eta_p(v), \frac{1-k}{2}\}$,

(ii) $\eta_p(v+u-v) \geq \min\{\eta_p(u), \frac{1-k}{2}\}$,

(iii) $\eta_p(uv) \geq \min\{\eta_p(u), \frac{1-k}{2}\}$,

(iv) $\eta_p(u(v+w)-uv) \geq \min\{\eta_p(u), \frac{1-k}{2}\}$.

for all $u, v, w \in N$.

Proof. The proof follows from Lemma 3.7.

Example 3.9. Let $N = \{0, u, v, w\}$ be a near-ring as in Example 3.5. Define a FSS $(\eta, Q)$ over $N$, where $Q = \{a, b\}$, as shown below

$\eta_a = \{(0.7), (u, 0.5), (v, 0.2), (w, 0.2)\}$ and

$\eta_b = \{(0.6), (u, 0.3), (v, 0.3), (w, 0.5)\}$.

We can easily verify that $(\eta, Q)$ is an $(\in, \in \forall k_q)$-FSN and an $(\in, \in \forall k_q)$-FSI over $N$ with $k = 0.4$.

Remark 3.10. It is obvious that every $(\in, \in \forall k_q)$-FSI over $N$ is an $(\in, \in \forall k_q)$-FSN over $N$, but the converse is not true in general as explained below.

Example 3.11. Consider the $(\in, \in \forall k_q)$-FSN as defined in Example 3.5. Since $\Delta_k(u) = \Delta_k(u) = 0.1 \geq 0.3 = \min\{0.3, 0.3\} = \min\{\Delta_k(v), \frac{1-k}{2}\}$, $(\in, \in \forall k_q)$-FI over $N$.

Remark 3.12. For any $(\in, \in \forall k_q)$-FSN (resp. $(\in, \in \forall k_q)$-FSI) $(\in, \in \forall k_q)$-FI over $N$, we can conclude that

(i) if $\Delta_k(u) < \frac{1-k}{2}$ for all $p \in P$ and $u \in N$, then $(\in, \in \forall k_q)$-FSN over $N$.

(ii) if $k = 0$, then $(\in, \in \forall k_q)$-FSI is an $(\in, \in \forall k_q)$-fuzzy soft near-ring (resp. ideal) over $N$.

Remark 3.13. Every FSN (resp. FSI) and $(\in, \in \forall k_q)$-FSN (resp. $(\in, \in \forall k_q)$-FSI) is an $(\in, \in \forall k_q)$-FSN (resp. $(\in, \in \forall k_q)$-FSI) over $N$, but in general the converses are not true as demonstrated below.

Example 3.14. Consider the near-ring $N$ as in Example 3.5. Define a FSS $(\Psi, R)$ over $N$, where $R = \{a, b\}$, is given by

$\Psi_0 = \{(0, 0.4), (u, 0.7), (v, 0.4), (w, 0.6)\}$ and

$\Psi_0 = \{(0, 0.6), (u, 0.6), (v, 0.3), (w, 0.3)\}$.

Then $(\Psi, R)$ is an $(\in, \in \forall k_q)$-FSN (resp. $(\in, \in \forall k_q)$-FSI) over $N$ with $k = 0.2$. But

(i) $(\Psi, R)$ is not a FSN over $N$, since $0.4 = \Psi_0(0) = \Psi_1(u-u) \geq \min\{\Psi_0(u), \Psi_0(u)\} = 0.7$. \(\Psi_0\) is not a fuzzy subnear-ring of $N$.

(ii) $(\Psi, R)$ is not an $(\in, \in \forall k_q)$-FSN (resp. $(\in, \in \forall k_q)$-FSI) over $N$, since $u \in \Psi_0$, and $(u-u)_{\min} \in \forall k_q \Psi_0$, \(\Psi_0\) is not an $(\in, \in \forall q)$-fuzzy subnear-ring (resp. ideal) of $N$.

Theorem 3.15. The restricted intersection of two $(\in, \in \forall k_q)$-FSNs (resp. $(\in, \in \forall k_q)$-FSIs) over $N$ is an $(\in, \in \forall k_q)$-FSN (resp. $(\in, \in \forall k_q)$-FSI) over $N$ when it is non-null.

Proof. Let $(\Delta, P_1)$ and $(\Theta, P_2)$ be two $(\in, \in \forall k_q)$-FSNs of $N$ and let $(\Delta, P_1) \cap \Theta(P_2) = (\Psi, \Psi)$, where $P_1 = \bigcap \{P_1 \cap P_2 \mid \Delta(0) \cap \Theta(0) = \Psi(0)\}$ for all $p \in \supp(\Psi, \Psi)$. Since $\Delta(p) \cap \Theta(p)$ are $(\in, \in \forall k_q)$-FNS of $N$, by Theorem 3.16 of [9], $(\Delta(p) \cap \Theta(p) = \Psi(p)$ is $(\in, \in \forall k_q)$-FSN of $N$ for all $p \in \supp(\Psi, \Psi)$. Hence $(\Psi, \Psi)$ is an $(\in, \in \forall k_q)$-FSN over $N$. Similarly, we can prove for $(\in, \in \forall q)$-FSIs.

Theorem 3.16. The extended intersection of two $(\in, \in \forall k_q)$-FSNs (resp. $(\in, \in \forall k_q)$-FSIs) over $N$ is an $(\in, \in \forall k_q)$-FSN (resp. $(\in, \in \forall k_q)$-FSI) over $N$ when it is non-null.

Proof. Let $(\Delta, P_1)$ and $(\Theta, P_2)$ be two $(\in, \in \forall k_q)$-FSNs over $N$ and let $(\Delta, P_1) \cap \Theta(P_2) = (\Psi, \Psi)$ be the extended intersection of $(\Delta, P_1)$ and $(\Theta, P_2)$. Then $P_1 = \bigcap \{P_1 \cap P_2 \mid \Delta(0) \cap \Theta(0) = \Psi(0) \}$ is an $(\in, \in \forall k_q)$-FSN of $N$, if $p \in P_1 \cap P_2$, then $\Psi(p) = \Delta(p)$ is an $(\in, \in \forall k_q)$-FSN of $N$, if $p \in P_1 \cap P_2$, then $\Psi(p) = \Delta(p) \cap \Theta(p)$ is an $(\in, \in \forall k_q)$-FSN of $N$, since the intersection of two $(\in, \in \forall k_q)$-FSNs of $N$ is an $(\in, \in \forall k_q)$-FSN of $N$. Therefore $(\Psi, \Psi)$ is an $(\in, \in \forall k_q)$-FSN of $N$ for all $p \in \supp(\Psi, \Psi)$. Hence $(\Psi, \Psi)$ is an $(\in, \in \forall q)$-FSN over $N$. Similarly, we can show for $(\in, \in \forall q)$-FSIs.

Theorem 3.17. If $(\Delta, P_1)$ and $(\Theta, P_2)$ are two $(\in, \in \forall k_q)$-FSNs (resp. $(\in, \in \forall k_q)$-FSIs) over $N$ such that $P_1 \cap P_2 = \emptyset$, then their union is an $(\in, \in \forall k_q)$-FSN (resp. $(\in, \in \forall k_q)$-FSI) over $N$. 253/256
Proof. Using Definition 2.7, we can write \((\Delta, P_1) \cup (\Theta, P_2) = (\Psi, P), \) where \(P = P_1 \cup P_2\). Since \(P_1 \cap P_2 = \emptyset\), we have either \(\rho \in P_1 \setminus P_2\) or \(\rho \in P_2 \setminus P_1\) for all \(\rho \in \text{Supp}(\Psi, P)\). For any \(\rho \in \text{Supp}(\Psi, P)\), if \(\rho \in P_1 \setminus P_2\), then \(\Psi(\rho) = \Delta(\rho)\) is an \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FN of \(N\), and if \(\rho \in P_2 \setminus P_1\), then \(\Psi(\rho) = \Theta(\rho)\) is an \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FN of \(N\). Thus \(\Psi(\rho)\) is an \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FN of \(N\) for all \(\rho \in \text{Supp}(\Psi, P)\). Hence \((\Psi, P)\) is an \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FSN over \(N\). Similarly, we can prove for \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FSIs.

**Theorem 3.18.** Let \((\Delta, P_1)\) and \((\Theta, P_2)\) be \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FSNs (resp. \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FSIs) over \(N\). Then \((\Delta, P_1) \cup (\Theta, P_2)\) is an \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FSN (resp. \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FSI) over \(N\) when it is non-null.

**Proof.** By Definition 2.8, we can write \((\Delta, P_1) \cup (\Theta, P_2) = (\Psi, P)\), where \(P = P_1 \cup P_2\) and \(\Psi(\rho, \sigma) = \Delta(\rho) \cup \Theta(\sigma)\) for all \((\rho, \sigma) \in P_1 \times P_2\). Since \(\Delta(\rho)\) and \(\Theta(\sigma)\) are \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FNs of \(N\) for all \(\rho \in P_1\) and \(\sigma \in P_2\), \(\Delta(\rho) \cup \Theta(\sigma)\) is also an \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FN of \(N\) for all \((\rho, \sigma) \in P_1 \times P_2\). Therefore \((\Psi, P)\) is an \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FSN over \(N\). Similarly, we can show for \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FSIs.

In the subsequent example, we illustrate that \((\Delta, P_1) \cup (\Theta, P_2)\) is not an \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FSN over \(N\), in general.

**Example 3.19.** Consider the near-ring \(N\) as given in Example 3.5. Define FSSs \((\Delta, P_1)\) and \((\Theta, P_2)\) over \(N\), where

\[
P_1 = \{a, b, c\}
\]

\[
P_2 = \{b, c\}.
\]

Define a FSS \((\Psi, P)\) as \((\Delta, P_1) \cup (\Theta, P_2)\) over \(N\) as follows:

\[
\Psi_{\{a\}} = \{(0, 0.5), (0.3, 0.6), (0.3, 0.2)\},
\]

\[
\Psi_{\{b\}} = \{(0.4, 0.1, 0.3, 0.1)\},
\]

\[
\Psi_{\{c\}} = \{(0.4, 0.1, 0.3, 0.1)\}.
\]

Then \((\Delta, P_1)\) and \((\Theta, P_2)\) are \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FSNs over \(N\) with \(k = 0.2\).

Let \(P = P_1 \times P_2 = \{(a, b), (a, c), (b, b), (b, c), (c, b), (c, c)\}\).

Define a FSS \((\Psi, P)\) as \((\Delta, P_1) \cup (\Theta, P_2)\) over \(N\) as follows:

\[
\Psi_{\{a\}} = \{(0.0.7, (0.5, 0.7), (0.8, 0.8)\},
\]

\[
\Psi_{\{b\}} = \{(0.0.8, (0.5, 0.7), (0.8, 0.8)\},
\]

\[
\Psi_{\{c\}} = \{(0.0.5, (0.3, 0.7), (0.3, 0.3)\}.
\]

Then \((\Psi, P)\) is an \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FSN and also an \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FSI over \(N\) with \(k = 0.2\).

Define a FSS \((\Gamma, P)\) as \((\Delta, P_1) \cup (\Theta, P_2)\) as follows:

\[
\Gamma_{\{a\}} = \{(0.0.7, (0.5, 0.7), (0.8, 0.8)\},
\]

\[
\Gamma_{\{b\}} = \{(0.0.8, (0.5, 0.7), (0.8, 0.8)\},
\]

\[
\Gamma_{\{c\}} = \{(0.0.5, (0.3, 0.7), (0.3, 0.3)\}.
\]

Then \((\Gamma, P)\) is not an \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FSN over \(N\) with \(k = 0.2\), because \(\Gamma_{\{c\}}(u - v) = \Gamma_{\{c\}}(w) = 0.1 \geq 0.4\).


In the following theorem, we give an additional condition for \((\Delta, P_1) \cup (\Theta, P_2)\) to be an \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FSN.

**Theorem 3.20.** Let \((\Delta, P_1)\) and \((\Theta, P_2)\) be \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FSNs over \(N\) and let \(\Delta(\rho) \subseteq \Theta(\sigma)\) or \(\Theta(\sigma) \subseteq \Delta(\rho)\) for all \((\rho, \sigma) \in P_1 \times P_2\). Then \((\Delta, P_1) \cup (\Theta, P_2)\) is an \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FSN over \(N\).

**Proof.** Using Definition 2.8, we can write \((\Delta, P_1) \cup (\Theta, P_2) = (\Psi, P), \) where \(P = P_1 \times P_2\) and \(\Psi(\rho, \sigma) = \Delta(\rho) \cup \Theta(\sigma)\) for all \((\rho, \sigma) \in P_1 \times P_2\). For any \((\rho, \sigma) \in P_1 \times P_2\), if \(\Delta(\rho) \subseteq \Theta(\sigma)\), then \(\Psi(\rho, \sigma) = \Delta(\rho) \cup \Theta(\sigma) = \Theta(\sigma)\) is an \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FN of \(N\) and if \(\Theta(\sigma) \subseteq \Delta(\rho)\), then \(\Psi(\rho, \sigma) = \Delta(\rho) \cup \Theta(\sigma) = \Delta(\rho)\) is an \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FN of \(N\). Hence \((\Psi, P)\) is an \((\epsilon, \in \mathfrak{v}_{\mathfrak{q}_k})\)-FSN over \(N\).
Proof. Straightforward.

Definition 3.25. Let $\eta$ be a fuzzy subset of $N$ and $\tau$ be an $(\varepsilon, \in \chi q_k)$-FNS of $N$. Then $\eta$ is called an $(\varepsilon, \in \chi q_k)$-FI of $\tau$ if $\eta(a) \leq \tau(a)$ for all $a \in N$ and $\eta$ is an $(\varepsilon, \in \chi q_k)$-FI of $N$.

Definition 3.26. Let $(\Delta, P)$ be an $(\varepsilon, \in \chi q_k)$-FSN over $N$. Then a FSS $(\eta, Q)$ over $N$ is called an $(\varepsilon, \in \chi q_k)$-fuzzy soft ideal of $(\Delta, P)$ expressed by $(\eta, Q) \sqsubseteq (\Delta, P)$ if $Q \subseteq P$ and $\eta(p) = \eta(p) \sqsubseteq (\Delta, P)$, for all $p \in \text{Supp}(\eta, Q)$.

Example 3.27. Let $(\Delta, P)$ be an $(\varepsilon, \in \chi q_k)$-FSN as in Example 3.5. Define a FSS $(\eta, Q)$ over $N$, where $Q = \{a, b\}$, as follows:

$\eta_a = \{(0, 0.4), (u, 0.5), (x, 0.6), (w, 0.4)\}$ and

$\eta_b = \{(0, 0.3), (u, 0.2), (x, 0.1), (w, 0.1)\}$

We can easily verify that $(\eta, Q)$ is an $(\varepsilon, \in \chi q_k)$-FSI of $(\Delta, P)$ over $N$ with $k = 0.3$.

Theorem 3.28. The restricted intersection of two $(\varepsilon, \in \chi q_k)$-FSIs of an $(\varepsilon, \in \chi q_k)$-FSN $(\Delta, P)$ over $N$ is an $(\varepsilon, \in \chi q_k)$-FSI of $(\Delta, P)$ when it is non-null.

Proof. Let $\left((\xi, Q_1) \sqsubseteq (\Delta, P)\right)$ and $\left((\tau, Q_2) \sqsubseteq (\Delta, P)\right)$. By Definition 2.10, we can write $\left((\xi, Q_1) \sqcap (\tau, Q_2) = (\eta, Q)\right)$, where $Q = Q_1 \cap Q_2 \neq \emptyset$ and $\eta(p) = \eta(p) \sqcap (\Delta, P)$ for all $p \in \text{Supp}(\eta, Q)$. Since $Q_1 \subseteq P$ and $Q_2 \subseteq P$, we have $Q_1 \cap Q_2 = Q \subseteq P$. Suppose that $(\eta, Q)$ is non-null. Since $(\xi, Q_1) \sqsubseteq (\Delta, P)$ and $(\tau, Q_2) \sqsubseteq (\Delta, P)$, we have $\eta(p) \sqcap (\Delta, P)$ and $\tau(p) \sqcap (\Delta, P)$ for all $p \in \text{Supp}(\eta, Q)$. Hence, $(\xi, Q_1) \sqcap (\tau, Q_2) = (\eta, Q)$. Thus $\eta(p) \sqcap (\Delta, P)$ for all $p \in \text{Supp}(\eta, Q)$. Therefore, $(\eta, Q)$ is an $(\varepsilon, \in \chi q_k)$-FSI of $(\Delta, P)$ over $N$.

Theorem 3.29. The union of two $(\varepsilon, \in \chi q_k)$-FSIs of an $(\varepsilon, \in \chi q_k)$-FSN $(\Delta, P)$ over $N$ is an $(\varepsilon, \in \chi q_k)$-FSI of $(\Delta, P)$.

Proof. Let $\left((\xi, Q_1) \sqsubseteq (\Delta, P)\right)$ and $\left((\tau, Q_2) \sqsubseteq (\Delta, P)\right)$. By Definition 2.9, we can write $\left((\xi, Q_1) \cup (\tau, Q_2) = (\eta, Q)\right)$, where $Q = Q_1 \cup Q_2$ and for all $p \in \text{Supp}(\eta, Q)$,

$\eta(p) = \begin{cases} \xi(p) & \text{if } p \in Q_1 \setminus Q_2, \\ \tau(p) & \text{if } p \in Q_2 \setminus Q_1, \\ \xi(p) \cup \tau(p) & \text{if } p \in Q_1 \cap Q_2. \end{cases}$

Obviously $Q_1 \cup Q_2 = Q \subseteq P$. Since $Q_1 \cap Q_2 = \emptyset$, either $\rho \in Q_1 \setminus Q_2$ or $\rho \in Q_2 \setminus Q_1$, for all $p \in \text{Supp}(\eta, Q)$. If $p \in Q_1 \setminus Q_2$, then $\eta(p) = \xi(p) \sqsubseteq (\Delta, P)$ and if $p \in Q_2 \setminus Q_1$, then $\eta(p) = \tau(p) \sqsubseteq (\Delta, P)$. Thus $\eta(p) \sqsubseteq (\Delta, P)$ for all $p \in \text{Supp}(\eta, Q)$. Therefore, $(\eta, Q)$ is an $(\varepsilon, \in \chi q_k)$-FSI of $(\Delta, P)$ over $N$.

Theorem 3.30. Let $\{(\eta_i, Q_i)\}_{i \in \Omega}$ be a family of $(\varepsilon, \in \chi q_k)$-FSIs of an $(\varepsilon, \in \chi q_k)$-FSN $(\Delta, P)$ over $N$. Then

(i) $\cap \{(\eta_i, Q_i)\}_{i \in \Omega}$ is an $(\varepsilon, \in \chi q_k)$-FSI of $(\Delta, P)$ if $\cap \{(\eta_i, Q_i)\}_{i \in \Omega} \neq \Phi$.

(ii) $\cap \{(\eta_i, Q_i)\}_{i \in \Omega}$ is an $(\varepsilon, \in \chi q_k)$-FSI of $(\Delta, P)$ if $\cap \{(\eta_i, Q_i)\}_{i \in \Omega} \neq \Phi$.

(iii) if $Q_i \cap Q_j = \emptyset$ for all $i, j \in \Omega, i \neq j$, then $\cup \{(\eta_i, Q_i)\}_{i \in \Omega}$ is an $(\varepsilon, \in \chi q_k)$-FSI of $(\Delta, P)$.

Proof. The proofs of (i) – (iii) are similar to the proofs of Theorem 3.21 and Theorem 3.29 and so we omit the proofs.

4. Conclusion

In this article, as a generalization of $(\varepsilon, \in \chi q_k)$-fuzzy soft near-ring and $(\varepsilon, \in \chi q_k)$-fuzzy soft ideal, we have introduced the notions of $(\varepsilon, \in \chi q_k)$-fuzzy soft near-ring and $(\varepsilon, \in \chi q_k)$-fuzzy soft ideal over a near-ring respectively. We have also introduced the concept of $(\varepsilon, \in \chi q_k)$-fuzzy soft subnear-ring (resp. ideal) of an $(\varepsilon, \in \chi q_k)$-fuzzy soft near-ring and studied some of their properties with illustrated examples. We have applied the operations of fuzzy soft sets to $(\varepsilon, \in \chi q_k)$-fuzzy soft near-rings and $(\varepsilon, \in \chi q_k)$-fuzzy soft ideals.

Acknowledgment

The authors are highly grateful to anonymous reviewers for their valuable comments and suggestions.

References

\[ (\varepsilon, \varepsilon \lor q_k) \text{-fuzzy soft near-rings and } (\varepsilon, \varepsilon \lor q_k) \text{-fuzzy soft ideals over near-rings} \]


