An \((\iota, \kappa)\)-fuzzy \(M\) closed and \((\iota, \kappa)\)-generalized fuzzy \(M\) closed sets in double fuzzy topological spaces

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Abstract
In double fuzzy topological spaces, \((\iota, \kappa)\)-fuzzy \(\theta\)-closed, \((\iota, \kappa)\)-fuzzy \(M\)-closed sets and \((\iota, \kappa)\)-generalized fuzzy \(M\)-closed sets are introduced. Also we study some of their properties.

Keywords
\((\iota, \kappa)\)-fuzzy \(\theta\)-closed, \((\iota, \kappa)\)-fuzzy \(M\)-closed sets, \((\iota, \kappa)\)-generalized fuzzy \(M\)-closed sets, \((\iota, \kappa)\)-generalized fuzzy \(M\)-interior, \((\iota, \kappa)\)-generalized fuzzy \(M\)-closure.

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1. Introduction and Preliminaries

“Intuitionistic fuzzy sets and Intuitionistic fuzzy topological space” were initiated by Atanassov [1] in 1993 and Coker [2] in 1997 respectively. In 2005, Garcia and Rodabaugh [7] coined the term “double” instead of “intuitionistic”. In the past two decades many researchers [9–11, 16] did more applications on double fuzzy topological spaces. From 2011, El-Maghrabi and Al-Johany [3–6] introduced and studied some properties on \(M\)-open sets and maps in topological spaces. Recently these kind of sets were studied in Šostak’s Fuzzy topological spaces [14]. In this paper we introduce \((\iota, \kappa)\)-fuzzy \(\theta\)-closed, \((\iota, \kappa)\)-fuzzy \(M\)-closed sets, \((\iota, \kappa)\)-generalized fuzzy \(M\)-closed sets and study some of their properties in double fuzzy topological spaces.

\(X\) denotes a non-empty set, \(I = [0, 1], \ I_0 = (0, 1], \ I_1 = [0, 1), \ \emptyset(X) = 0, \ \emptyset(X) = 1, \ i \in I_0 \ & \ k \in I_1 \ \text{and always} \ t + k \leq 1, \ I^X \ \text{is a family of all fuzzy sets on} \ X. \ \text{In 200} \ \text{Samanta and Mondal[13] defined the double fuzzy topological space (briefly, dfts) denoted by} \ (X, \eta, \eta^*)\), also defined \((\iota, \kappa)\)-fuzzy open (resp. \((\iota, \kappa)\)-fuzzy closed) (briefly \((\iota, \kappa)\)-fo (resp. \((\iota, \kappa)\)-fc)) set if \(\eta(\gamma) \geq t \ \text{and} \ \eta^*(\gamma) \leq k, \ \text{(resp.} \ 1 - \gamma \ \text{is an} \ ((\iota, \kappa)\)-fo set.) The double fuzzy interior and double fuzzy closure operators [8] are defined from \(I^X \times I_0 \times I_1 \rightarrow I^X\) as follows

\[
I_{\eta, \eta^*}(\gamma, t, \kappa) = \bigvee \{v \in I^X | v \leq \gamma, \eta(v) \geq t, \eta^*(v) \leq k\},
\]

\[
C_{\eta, \eta^*}(\gamma, t, \kappa) = \bigwedge \{v \in I^X | v \geq \gamma, \eta(1 - v) \geq t, \eta^*(1 - v) \leq k\},
\]

\(\gamma \in I^X\), is called \((\iota, \kappa)\)-fuzzy regular open [12] (resp. \((\iota, \kappa)\)-fuzzy regular closed)(briefly \((\iota, \kappa)\)-fro (resp. \((\iota, \kappa)\)-frc)) set if \(\gamma = I_{\eta, \eta^*}(C_{\eta, \eta^*}(\gamma, t, \kappa), t, \kappa)\) (resp. \(1 - \gamma \ \text{is} \ ((\iota, \kappa)\)-fro set.) The operators \(\delta \theta \eta, \eta^*\) and \(\delta f \eta, \eta^*, \ \text{I}_0 \times I_0 \times I_1 \rightarrow I^X\) [11] as follows

\[
\delta I_{\eta, \eta^*}(\gamma, t, \kappa) = \bigvee \{v \in I^X | v \leq \gamma, v \ \text{is an} \ ((\iota, \kappa)\)-frf\},
\]

\[
\delta C_{\eta, \eta^*}(\gamma, t, \kappa) = \bigwedge \{v \in I^X | v \geq \gamma, v \ \text{is an} \ ((\iota, \kappa)\)-frf\}.
\]

A fuzzy set \(\gamma \in I^X\), is called \((\iota, \kappa)\)-fuzzy \(\delta\) pre open (resp. \((\iota, \kappa)\)-fuzzy \(e\) open, \((\iota, \kappa)\)-fuzzy \(\delta\) pre closed and \((\iota, \kappa)\)-fuzzy \(e\) closed) (briefly \((\iota, \kappa)\)-fpo (resp. \((\iota, \kappa)\)-feo, \((\iota, \kappa)\)-fpc and \((\iota, \kappa)\)-feo)) [11] set if \(\gamma \leq I_{\eta, \eta^*}(\delta \theta \eta, \eta^*, \gamma, t, \kappa)\) (resp. \(\gamma \leq C_{\eta, \eta^*}(\delta f \theta \eta, \eta^*, \gamma, t, \kappa)\) \(\text{and} \ 1 - \gamma \ \text{is} \ ((\iota, \kappa)\)-feo). The operators \(\delta PC_{\eta, \eta^*}, \delta P \theta \eta, \eta^*, \ e \theta \eta, \eta^*\) and \(\delta I_{\eta, \eta^*}\):
In $I^X \times I_0 \times I_1 \rightarrow I^X$ [11] defined as
\[ \delta Pl_{\eta} \gamma t, k = \sqrt{\{ v \in I^X | v \leq \gamma, v \text{ is an } (t, k)-f \delta po \} } \]
\[ \delta PC_{\eta} \gamma t, k = \sqrt{\{ v \in I^X | v \geq \gamma, v \text{ is an } (t, k)-f \delta pc \} } \]
el_{\eta} \gamma t, k = \sqrt{\{ v \in I^X | v \geq \gamma, v \text{ is an } (t, k)-f eo \} }
e_{\eta} \gamma t, k = \sqrt{\{ v \in I^X | v \geq \gamma, v \text{ is an } (t, k)-f ec \} }.
All other notations are from fuzzy set theory, which is given by Zadeh [15] and his followers.

2. An $(t, k)$-fuzzy $M$ closed sets

**Definition 2.1.** Let $(X, \eta, \eta^*)$ be a dfts, $\forall \gamma, v \in I^X$, the operators $(t, k)$-fuzzy $\theta$ interior and $(t, k)$-fuzzy $\theta$ closure denoted by $(t, k)-\theta I \eta, \eta^*$ and $(t, k)-\theta C \eta, \eta^*$ : $I^X \times I_0 \times I_1 \rightarrow I^X$ are defined as
\[ \theta I_{\eta, \eta} \gamma t, k = \sqrt{\{ v \in I_{\eta, \eta} \gamma t, k | v \leq \gamma, \eta (1 - v) \geq t & \eta^* (1 - v) \leq k \} } \]
and
\[ \theta C_{\eta, \eta} \gamma t, k = \sqrt{\{ v \in I_{\eta, \eta} \gamma t, k | v \geq \gamma, \eta (v) \geq t & \eta^* (v) \leq k \} } \]

**Definition 2.2.** In a dfts $(X, \eta, \eta^*)$, $\gamma \in I^X$ is called an
1. $(t, k)$-fuzzy $\theta$ open (resp. $(t, k)$-fuzzy $\theta$ semi open) (briefly $(t, k)$-f $\theta o$ (resp. $(t, k)$-f $\theta so$)) set if $\gamma = \theta I_{\eta, \eta} \gamma t, k.$ (resp. $\gamma \leq \theta I_{\eta, \eta} \gamma t, k.$)
2. $(t, k)$-fuzzy $\theta$ closed (resp. $(t, k)$-fuzzy $\theta$ semi closed) (briefly $(t, k)$-f $\theta c$ (resp. $(t, k)$-f $\theta co$)) set if $1 - \gamma = (t, k)-f \theta o$ (resp. $(t, k)-f \theta so$) set.

**Definition 2.3.** In a dfts $(X, \eta, \eta^*)$, $\gamma \in I^X$ is called an
1. $(t, k)$-fuzzy $M$ closed (briefly $(t, k)$-f $M c$) set if $\gamma \geq I_{\eta, \eta} \gamma t, k \cap C_{\eta, \eta} \gamma t, k$.
2. $(t, k)$-fuzzy $M$ open (briefly $(t, k)$-f $M o$) set if $1 - \gamma$ is an $(t, k)$-f $M c$ set.

**Definition 2.4.** Let $(X, \eta, \eta^*)$ be a dfts, then the
1. union of all $(t, k)$-f $M o$ (resp. $(t, k)$-f $\theta o$ and $(t, k)$-f $\theta so$) set contained in $\gamma$ is called the $(t, k)$-fuzzy $M$ (resp. $(t, k)$-fuzzy $\theta$ and $(t, k)$-fuzzy $\theta$ semi) interior of $\gamma$ and is denoted by $\theta I_{\eta, \eta} \gamma t, k.$ (resp. $\theta I_{\eta, \eta} \gamma t, k.$ and $\theta s I_{\eta, \eta} \gamma t, k.$).
2. intersection of all $(t, k)$-f $M c$ (resp. $(t, k)$-f $\theta c$ and $(t, k)$-f $\theta co$) sets containing $\gamma$ is called the $(t, k)$-fuzzy $M$ (resp. $(t, k)$-fuzzy $\theta$ and $(t, k)$-fuzzy $\theta$ semi) closure of $\gamma$ and is denoted by $\theta C_{\eta, \eta} \gamma t, k.$ (resp. $\theta C_{\eta, \eta} \gamma t, k.$ and $\theta s C_{\eta, \eta} \gamma t, k.$).

**Proposition 2.5.** In a dfts $(X, \eta, \eta^*)$, $\forall \gamma, v \in I^X$,
(a) $M I_{\eta, \eta} \gamma t, k = 0$ and $M I_{\eta, \eta} \gamma t, k = 1$.
(b) $M C_{\eta, \eta} \gamma t, k = 0$ and $M C_{\eta, \eta} \gamma t, k = 1$.
(c) $M I_{\eta, \eta} \gamma t, k = M C_{\eta, \eta} \gamma t, k = M C_{\eta, \eta} \gamma t, k.$
(d) $M C_{\eta, \eta} \gamma t, k = M C_{\eta, \eta} \gamma t, k.$
(e) If $\gamma < v$ then $M I_{\eta, \eta} \gamma t, k < M I_{\eta, \eta} \gamma t, k.$
(f) If $\gamma \leq v$ then $M C_{\eta, \eta} \gamma t, k \leq M C_{\eta, \eta} \gamma t, k.$
(g) $M I_{\eta, \eta} \gamma t, k \leq \gamma \leq M C_{\eta, \eta} \gamma t, k.$
(h) $M C_{\eta, \eta} \gamma t, k \geq M C_{\eta, \eta} \gamma t, k.$
(i) $M C_{\eta, \eta} \gamma t, k \cap M C_{\eta, \eta} \gamma t, k.$
(j) $M C_{\eta, \eta} \gamma t, k \cap M C_{\eta, \eta} \gamma t, k.$
(k) $M C_{\eta, \eta} \gamma t, k \gamma t, k.$
(l) $M C_{\eta, \eta} \gamma t, k \gamma t, k.$
(m) $M C_{\eta, \eta} \gamma t, k \gamma t, k.$
(n) $M C_{\eta, \eta} \gamma t, k \gamma t, k.$
(o) $M C_{\eta, \eta} \gamma t, k \gamma t, k.$
(p) $M C_{\eta, \eta} \gamma t, k \gamma t, k.$
(q) $\gamma \leq \gamma \gamma t, k \leq \gamma \gamma t, k.$
(r) $\gamma \gamma t, k \leq \gamma \gamma t, k \leq \gamma \gamma t, k.$

**Theorem 2.6.** In any dfts $(X, \eta, \eta^*)$, Every
1. $(t, k)$-f $\theta sc$ (resp. $(t, k)$-f $\theta pc$) set is an $(t, k)$-f $M c$ set.
2. $(t, k)$-f $\theta c$ set is an $(t, k)$-f $\theta sc$ set.
3. $(t, k)$-f $\theta c$ set is an $(t, k)$-f $c$ set.
4. $(t, k)$-f $c$ set is an $(t, k)$-f $\delta p c$ set.
5. $(t, k)$-f $M c$ set is an $(t, k)$-f $c$ set.

**Remark 2.7.** The converse of the above theorem, in general, need not be true. It can be verified from the following examples.

**Example 2.8.** Consider the dfts $(X, \eta, \eta^*)$ with $X = \{a, b, c\}$ and
\[ \eta(\gamma) = \begin{cases} 1, & \text{if } \gamma \in \{0, 1\}, \\ \frac{1}{2}, & \text{if } \gamma = 0 \frac{1}{2}, \\ 0, & \text{otherwise}. \end{cases} \]
\[ \eta^*(\gamma) = \begin{cases} 0, & \text{if } \gamma \in \{0, 1\}, \\ \frac{1}{2}, & \text{if } \gamma = 0 \frac{1}{2}, \\ 1, & \text{otherwise}. \end{cases} \]
Then the fuzzy set $0.1$ is an $(\frac{1}{2}, 0.1)$-f $M c$ set but not an $(\frac{1}{2}, 0.1)$-f $\delta p c$ set.
Example 2.9. Consider the dfts \((X, \eta, \eta^*)\) with \(X = \{a, b, c\}\) and
\[
\eta(\gamma) = \begin{cases} 
1, & \text{if } \gamma \in \{0, 1\}, \\
\frac{4}{5}, & \text{if } \gamma = \gamma_1, \\
\frac{1}{5}, & \text{if } \gamma = \gamma_2, \\
0, & \text{Otherwise.}
\end{cases}
\eta^*(\gamma) = \begin{cases} 
0, & \text{if } \gamma \in \{0, 1\}, \\
\frac{4}{5}, & \text{if } \gamma = \gamma_1, \\
\frac{1}{5}, & \text{if } \gamma = \gamma_2, \\
1, & \text{Otherwise.}
\end{cases}
\]
where \(\gamma_1(a) = 0.3, \gamma_1(b) = 0.4, \gamma_1(c) = 0.5, \gamma_2(a) = 0.6, \gamma_2(b) = 0.9, \gamma_2(c) = 0.5, \gamma_3(a) = 0.3, \gamma_3(b) = 0\) and \(\gamma_3(c) = 0.5\). Then the fuzzy set \(\gamma\) is an \((\frac{3}{4}, \frac{3}{4})\)-fMo set but not an \((\frac{3}{4}, \frac{3}{4})\)-fMc set.

Example 2.10. Consider the dfts \((X, \eta, \eta^*)\) with \(X = \{a, b, c\}\) and
\[
\eta(\gamma) = \begin{cases} 
1, & \text{if } \gamma \in \{0, 1\}, \\
\frac{4}{5}, & \text{if } \gamma = \gamma_1, \\
\frac{1}{5}, & \text{if } \gamma = \gamma_2, \\
0, & \text{Otherwise.}
\end{cases}
\eta^*(\gamma) = \begin{cases} 
0, & \text{if } \gamma \in \{0, 1\}, \\
\frac{4}{5}, & \text{if } \gamma = \gamma_1, \\
\frac{1}{5}, & \text{if } \gamma = \gamma_2, \\
1, & \text{Otherwise.}
\end{cases}
\]
where \(\gamma_1(a) = 0.3, \gamma_1(b) = 0.4, \gamma_1(c) = 0.5, \gamma_2(a) = 0.6, \gamma_2(b) = 0.5, \gamma_2(c) = 0.5\). Then the fuzzy set \(\gamma\) is an \((\frac{3}{4}, \frac{3}{4})\)-fMc set but not an \((\frac{3}{4}, \frac{3}{4})\)-fThc set.

Example 2.11. Consider the dfts \((X, \eta, \eta^*)\) with \(X = \{a, b, c\}\) and
\[
\eta(\gamma) = \begin{cases} 
1, & \text{if } \gamma \in \{0, 1\}, \\
\frac{4}{5}, & \text{if } \gamma = \gamma_1, \\
\frac{1}{5}, & \text{if } \gamma = \gamma_2, \\
0, & \text{Otherwise.}
\end{cases}
\eta^*(\gamma) = \begin{cases} 
0, & \text{if } \gamma \in \{0, 1\}, \\
\frac{4}{5}, & \text{if } \gamma = \gamma_1, \\
\frac{1}{5}, & \text{if } \gamma = \gamma_2, \\
1, & \text{Otherwise.}
\end{cases}
\]
where \(\gamma_1(a) = 0.3, \gamma_1(b) = 0.5, \gamma_1(c) = 0.5\), then the fuzzy set \(1 - \gamma\) is an \((\frac{1}{5}, \frac{1}{5})\)-fc set but not an \((\frac{1}{5}, \frac{1}{5})\)-fMc set.

Example 2.12. Consider the dfts \((X, \eta, \eta^*)\) with \(X = \{a, b, c\}\) and
\[
\eta(\gamma) = \begin{cases} 
1, & \text{if } \gamma \in \{0, 1\}, \\
\frac{4}{5}, & \text{if } \gamma = \gamma_1, \\
\frac{1}{5}, & \text{if } \gamma = \gamma_2, \\
0, & \text{Otherwise.}
\end{cases}
\eta^*(\gamma) = \begin{cases} 
0, & \text{if } \gamma \in \{0, 1\}, \\
\frac{4}{5}, & \text{if } \gamma = \gamma_1, \\
\frac{1}{5}, & \text{if } \gamma = \gamma_2, \\
1, & \text{Otherwise.}
\end{cases}
\]
where \(\gamma_1(a) = 0.5, \gamma_1(b) = 0.3, \gamma_1(c) = 0.2, \gamma_2(a) = 0.5, \gamma_2(b) = 0.6\) and \(\gamma_2(c) = 0.6\), then the fuzzy set \(\gamma_2\) is an \((\frac{3}{6}, \frac{3}{6})\)-fc set but not an \((\frac{3}{6}, \frac{3}{6})\)-fMo set.

From the above theorem, examples and by [11], the following implications are hold.

Theorem 2.13. Let \((X, \eta, \eta^*)\) be a dfts,

(i) \(\bigvee_{i \in \mathbb{N}} \gamma_i\) is an \((i, \kappa)\)-fMo set if \(\forall i \in \mathbb{N}, \gamma_i\) be an \((i, \kappa)\)-fMo set.

(ii) \(\bigwedge_{i \in \mathbb{N}} \gamma_i\) is an \((i, \kappa)\)-fMc set if \(\forall i \in \mathbb{N}, \gamma_i\) be an \((i, \kappa)\)-fMc set.

Proof: (i) Let \(\gamma_i\) be an \((i, \kappa)\)-fMo set, \(\forall i \in \mathbb{N}\) then
\[
\gamma_i \subseteq C_{\eta, \eta^*}(\Theta_{\eta, \eta^*}(\gamma_{i, \kappa}, \kappa), t, \kappa) \quad \forall i \in \mathbb{N},
\]
\[
\Rightarrow \bigvee_{i \in \mathbb{N}} \gamma_i \subseteq \bigvee_{i \in \mathbb{N}} (C_{\eta, \eta^*}(\Theta_{\eta, \eta^*}(\gamma_{i, \kappa}, \kappa), t, \kappa)) \quad \forall i \in \mathbb{N},
\]
\[
\subseteq C_{\eta, \eta^*}(\Theta_{\eta, \eta^*}(\bigvee_{i \in \mathbb{N}} \gamma_{i, \kappa}, \kappa), t, \kappa).
\]
Thus \(\bigvee_{i \in \mathbb{N}} \gamma_i\) is an \((i, \kappa)\)-fMo set.

(ii) Similar to the proof of (i).

Theorem 2.14. In dfts \((X, \eta, \eta^*)\), let \(\gamma, v \in f^X\)

(i) \(\gamma \land v\) is an \((i, \kappa)\)-fMo set if \(\gamma\) is an \((i, \kappa)\)-fMo set and \(\eta(v) \geq t, \eta^*(v) \leq \kappa\).

(ii) \(\gamma \lor v\) is an \((i, \kappa)\)-fMc set if \(\gamma\) is an \((i, \kappa)\)-fMc set and \(\eta(1 - v) \geq t, \eta^*(1 - v) \leq \kappa\).

Proof: (i) Let \(\gamma\) is an \((i, \kappa)\)-fMo set, and a crisp set \(v \in f^X\)
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with $\eta(v) \geq t$, $\eta^*(v) \leq \kappa$, then

$$\gamma \land v \leq (C_{\eta, \eta^*}(\Theta_{I, \eta}^-(\gamma, t, \kappa), I, \kappa)) \land v$$

$$= (C_{\eta, \eta^*}(\Theta_{I, \eta}^-(\gamma, t, \kappa), I, \kappa) \land v)$$

$$\leq (C_{\eta, \eta^*}(\Theta_{I, \eta}^-(\gamma \land v, t, \kappa), I, \kappa) \land v)$$

$$\leq (C_{\eta, \eta^*}(\Theta_{I, \eta}^-(\gamma \land v, t, \kappa), I, \kappa)) \land v$$

Hence $\gamma \land v$ is an $(t, \kappa)$-fMo set.

(ii) Similar to the proof of (i).

\section*{Theorem 2.15}

If $\gamma \in I^X$ is both $(t, \kappa)$-fMo and $(t, \kappa)$-fc set in $(X, \eta, \eta^*)$, then $\gamma$ is an $(t, \kappa)$-fMo set.

\textbf{Proof:} Let $\gamma$ be an $(t, \kappa)$-fMo set then

$$\gamma \leq C_{\eta, \eta^*}(\Theta_{I, \eta}^-(\nu, t, \kappa), I, \kappa)$$

$$= C_{\eta, \eta^*}(\Theta_{I, \eta}^-(\nu, t, \kappa), I, \kappa) \land v$$

$$\leq (C_{\eta, \eta^*}(\Theta_{I, \eta}^-(\nu \land v, t, \kappa), I, \kappa)) \land v$$

$$\leq (C_{\eta, \eta^*}(\Theta_{I, \eta}^-(\nu \land v, t, \kappa), I, \kappa)) \land v$$

Hence $\gamma$ is an $(t, \kappa)$-fMo set.

\section*{Theorem 2.16}

If $\gamma \in I^X$ is both $(t, \kappa)$-fMo and $(t, \kappa)$-fc set in $(X, \eta, \eta^*)$, then $\gamma$ is an $(t, \kappa)$-fMo set.

\textbf{Proof:} Follows from theorem 2.15.

\section*{Theorem 2.17}

In a dfts $(X, \eta, \eta^*)$, $\forall \gamma \in I^X$,

1. If $\eta(\gamma) \geq t$ and $\eta^*(\gamma) \leq \kappa$ then $\gamma$ is an $(t, \kappa)$-fMo set.

2. $I_{\eta, \eta^*}(\gamma, t, \kappa)$ is an $(t, \kappa)$-fMo set.

3. $C_{\eta, \eta^*}(\gamma, t, \kappa)$ is an $(t, \kappa)$-fMo set.

\section*{3. An $(t, \kappa)$-generalized fuzzy $M$ closed sets}

\textbf{Definition 3.1.} In a $(X, \eta, \eta^*)$ be a dfts, $\gamma, v \in I^X$, a fuzzy set $\gamma$ is called an $(t, \kappa)$-generalized fuzzy $M$ closed (resp. $(t, \kappa)$-generalized fuzzy $M$ open) (briefly $(t, \kappa)$-gMo (resp. $(t, \kappa)$-gMo)) set if $MC_{\eta, \eta^*}(\gamma, t, \kappa) \leq v$ whenever $\gamma \leq v$ and $\eta(\gamma) \geq t$ and $\eta^*(\gamma) \leq \kappa$ (resp. $\gamma \leq \kappa$ is an $(t, \kappa)$-gMo set).

\textbf{Theorem 3.2.} In a dfts $(X, \eta, \eta^*)$, $\gamma \in I^X$ is $(t, \kappa)$-gMo set iff $v \leq MI_{\eta, \eta^*}(\gamma, t, \kappa)$ whenever $\gamma \leq v$, $\eta(1 - v) \geq t$ and $\eta^*(1 - v) \leq \kappa$.

\textbf{Definition 3.3.} In a dfts $(X, \eta, \eta^*)$, $\forall \gamma \in I^X$, an $(t, \kappa)$-generalized fuzzy $M$ closure operator denoted as $(t, \kappa)$-GMC$_{\eta, \eta^*}$: $I^X \times I_0 \times I_1 \rightarrow I^X$ defined as

$$GMC_{\eta, \eta^*}(\gamma, t, \kappa) = \bigwedge \{v \in I^X | \gamma \leq v \land v \text{ is } (t, \kappa)-gMo\}.

\textbf{Theorem 3.4.} In a dfts $(X, \eta, \eta^*)$, $\forall \gamma, v \in I^X$, then the operator $(t, \kappa)$-GMC$_{\eta, \eta^*}$ satisfies the following statements

1. $GMC_{\eta, \eta^*}(0, t, \kappa) = 0$ and $GMC_{\eta, \eta^*}(1, t, \kappa) = 1$.

2. $\gamma \leq GMC_{\eta, \eta^*}(\gamma, t, \kappa)$.

3. $GMC_{\eta, \eta^*}(\gamma \lor v, t, \kappa) \geq GMC_{\eta, \eta^*}(\gamma, \gamma) \lor GMC_{\eta, \eta^*}(v, \kappa)$.

4. $GMC_{\eta, \eta^*}(\gamma, t, \kappa) = GMC_{\eta, \eta^*}(\gamma, \kappa)$.

5. If $\nu$ is $(t, \kappa)$-gMo set then $GMC_{\eta, \eta^*}(\nu, \kappa) = \nu$.

6. $GMC_{\eta, \eta^*}(\gamma, \gamma, \kappa) \leq MC_{\eta, \eta^*}(\gamma, \kappa) \leq \kappa$.

\textbf{Theorem 3.5.} In a dfts $(X, \eta, \eta^*)$, $\forall \gamma \in I^X$, an $(t, \kappa)$-generalized fuzzy $M$ interior operator denoted as $(t, \kappa)$-GMI$_{\eta, \eta^*}$: $I^X \times I_0 \times I_1 \rightarrow I^X$ given by

$$GMI_{\eta, \eta^*}(\gamma, t, \kappa) = \bigcap \{v \in I^X | \gamma \geq v \land v \text{ is } (t, \kappa)-gMo\},$$

and

$$GMI_{\eta, \eta^*}(\gamma, t, \kappa) = 1 - GMC_{\eta, \eta^*}(\gamma, t, \kappa).$$

\textbf{Proposition 3.6.} In a dfts $(X, \eta, \eta^*)$, let an $(t, \kappa)$-gMo set $\gamma \in I^X$ and

1. if $\nu$ is $(t, \kappa)$-gMo set then $\gamma$ is an $(t, \kappa)$-gMo set.

2. if $\gamma$ is an $(t, \kappa)$-gMo set then $\gamma \land \nu$ is an $(t, \kappa)$-gMo set whenever $\nu \leq MC_{\eta, \eta^*}(\gamma, t, \kappa)$.

\textbf{Proof:} (1) Let $\gamma$ be an $(t, \kappa)$-gMo set and an $(t, \kappa)$-gMo set such that $\gamma \leq \gamma$.

$$\Rightarrow MC_{\eta, \eta^*}(\gamma, t, \kappa) \leq \gamma.$$ Since $\gamma \leq MC_{\eta, \eta^*}(\gamma, t, \kappa)$.

$$\Rightarrow \gamma = MC_{\eta, \eta^*}(\gamma, t, \kappa).$$ Therefore $\gamma$ is an $(t, \kappa)$-gMo set.

(2) Let $\gamma$ be an $(t, \kappa)$-gMo set and an $(t, \kappa)$-gMo set, then

$$MC_{\eta, \eta^*}(\gamma, t, \kappa) \leq \gamma \Rightarrow \gamma \text{ is an (t, k)-gMo set}$$

$$\Rightarrow \gamma \land \nu \text{ is an (t, k)-gMo set}$$

$$\Rightarrow \gamma \land \nu \text{ is an (t, k)-gMo set}.$$
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Hence, $v \leq M_{\eta, \eta} (\gamma, t, \kappa)$.

Conversely, let $v \leq M_{\eta, \eta} (\gamma, t, \kappa)$ whenever $v \leq \gamma$ and $v$ is an $(t, \kappa)$-g$fMc$ set, now, $1 - M_{\eta, \eta} (\gamma, t, \kappa) \leq 1 - v$.

Thus $MC_{\eta, \eta} (1 - \gamma, t, \kappa) \leq 1 - v$. Therefore $1 - \gamma$ is an $(t, \kappa)$-g$fMc$ set. Hence $\gamma$ is an $(t, \kappa)$-g$fMo$ set.

Proposition 3.9. In a dfts $(X, \eta, \eta^*)$, $\gamma \wedge v$ is an $(t, \kappa)$-g$fMc$ set if $\gamma$ and $v$ are $(t, \kappa)$-g$fMc$ sets.

Proof: Assume $\gamma$ and $v$ are $(t, \kappa)$-g$fMc$ sets, $\gamma \wedge v \leq v$, $\forall (t, \kappa)$-g$fMc$ set $v$, $MC_{\eta, \eta} (\gamma, t, \kappa) \leq v$, and since $\gamma$ is an $(t, \kappa)$-g$fMc$ set $MC_{\eta, \eta} (\gamma, t, \kappa) \leq u \forall (t, \kappa)$-g$fMc$ set $u \in I^X$ and $\gamma \leq u$. Also $v$ is an $(t, \kappa)$-g$fMc$ set $MC_{\eta, \eta} (v, t, \kappa) \leq v \forall (t, \kappa)$-g$fMc$ set $v \in I^X$, and $\gamma \leq v$. Therefore

$$MC_{\eta, \eta} (\gamma, t, \kappa) \wedge MC_{\eta, \eta} (v, t, \kappa) \leq u$$

whenever $\gamma \wedge v \leq v$. Hence $\gamma \wedge v$ is an $(t, \kappa)$-g$fMc$ sets in $I^X$.

Proposition 3.10. In a dfts $(X, \eta, \eta^*)$, $\gamma \wedge v$ is an $(t, \kappa)$-g$fMc$ set if $\gamma$ is an $(t, \kappa)$-g$fMc$ set and $\eta (v) \geq t, \eta^* (v) \leq \kappa$.

Proof: Since every $(t, \kappa)$-g$fMc$ set is an $(t, \kappa)$-g$fMc$ set and from the proposition 3.9 we have the proof.

Proposition 3.11. In a dfts $(X, \eta, \eta^*)$, $\gamma$ is an $(t, \kappa)$-g$fMc$ set and

1. if $\gamma$ is an $(t, \kappa)$-g$fMo$ set then $\gamma$ is an $(t, \kappa)$-g$fMc$ set.

2. if $\gamma \leq v \leq MC_{\eta, \eta} (\gamma, t, \kappa)$ then $\gamma$ is an $(t, \kappa)$-g$fMc$ set.

Proof: (1) Suppose that $\gamma$ is both $(t, \kappa)$-g$fMo$ and $(t, \kappa)$-g$fMc$ set in $I^X$. Since $\gamma$ is an $(t, \kappa)$-g$fMo$ set, $MC_{\eta, \eta} (\gamma, t, \kappa) \leq \gamma$, since $\gamma$ is an $(t, \kappa)$-g$fMc$ set $\gamma \leq MC_{\eta, \eta} (\gamma, t, \kappa)$. Therefore $\gamma = MC_{\eta, \eta} (\gamma, t, \kappa)$.

(2) Suppose that $\gamma$ is an $(t, \kappa)$-g$fMc$ set and $v$ is an $(t, \kappa)$-g$fMo$ set in $I^X$, $\gamma \leq v \leq v$ and let $\gamma \leq MC_{\eta, \eta} (\gamma, t, \kappa)$. This implies that

$$MC_{\eta, \eta} (v, t, \kappa) \leq MC_{\eta, \eta} (MC_{\eta, \eta} (\gamma, t, \kappa), t, \kappa) = MC_{\eta, \eta} (\gamma, t, \kappa).$$

Since $\gamma$ is an $(t, \kappa)$-g$fMc$ set, $\gamma$ is an $(t, \kappa)$-g$fMo$ set and $\gamma \leq v$ we can say that $MC_{\eta, \eta} (\gamma, t, \kappa) v$, this implies that

$$MC_{\eta, \eta} (v, t, \kappa) \leq u.$$

Therefore $\gamma$ is an $(t, \kappa)$-g$fMc$ set.

Theorem 3.12. Let $(X, \eta_1, \eta_1^*)$ and $(Y, \eta_2, \eta_2^*)$ be dfts’s. If $\gamma \leq 1_Y \leq 1_X$, $\forall \gamma$ is $(t, \kappa)$-g$fMc$ set in $I^X$, then $\gamma$ is an $(t, \kappa)$-g$fMc$ set relative to $Y$.

Proof: Suppose that $\gamma \leq 1_Y \leq 1_X$ and $\gamma = (t, \kappa)$-g$fMc$ set. And let $\gamma \leq 1_Y \wedge v \forall v \in I^X$ is an $(t, \kappa)$-g$fMo$ set.

Since $\gamma \in I^X$ is $(t, \kappa)$-g$fMc$ set $\gamma \leq v \Rightarrow MC_{\eta, \eta} (\gamma, t, \kappa) \leq v$. So that $1_Y \wedge MC_{\eta, \eta} (\gamma, t, \kappa) \leq 1_Y \wedge v$. Therefore $\gamma$ is an $(t, \kappa)$-g$fMc$ set relative to $Y$.

Theorem 3.13. In a dfts $(X, \eta, \eta^*)$, $\forall v \leq \gamma$, if $v$ is $(t, \kappa)$-g$fMc$ set relative to $\gamma \Rightarrow \gamma$ is both $(t, \kappa)$-g$fMo$ and $(t, \kappa)$-g$fMc$ set of $I^X$ then $v$ is an $(t, \kappa)$-g$fMc$ set relative to $X$.

Proof: Let $v$ is an $(r, s)$-g$fMc$ set and $\eta (u) \geq t$ and $\eta^* (u) \leq \kappa \& v \leq u$. But we have, $v \leq \gamma \leq 1$ and so $v \leq \gamma \leq u$, hence $v \leq \gamma \wedge u$. Therefore, $v$ is an $(t, \kappa)$-g$fMc$ set relative to $\gamma$, i.e. $\gamma \wedge MC_{\eta, \eta} (v, t, \kappa) \leq \gamma \wedge u \Rightarrow \gamma \wedge MC_{\eta, \eta} (v, t, \kappa) \leq u$.

Thus $(\gamma \wedge MC_{\eta, \eta} (v, t, \kappa)) \wedge (1 - MC_{\eta, \eta} (v, t, \kappa)) \leq u \wedge (1 - MC_{\eta, \eta} (v, t, \kappa)).$

This implies that

$$\gamma \wedge (1 - MC_{\eta, \eta} (v, t, \kappa)) \leq u \wedge (1 - MC_{\eta, \eta} (v, t, \kappa)).$$

Because $\gamma$ is an $(t, \kappa)$-g$fMc$ set, $MC_{\eta, \eta} (\gamma, t, \kappa) \leq u \wedge (1 - \gamma)$. And so $v \leq \gamma \Rightarrow MC_{\eta, \eta} (v, t, \kappa) \leq MC_{\eta, \eta} (\gamma, t, \kappa)$.

Therefore

$$MC_{\eta, \eta} (v, t, \kappa) \leq MC_{\eta, \eta} (\gamma, t, \kappa) \leq u \wedge (1 - MC_{\eta, \eta} (v, t, \kappa)).$$

Thus $MC_{\eta, \eta} (\gamma, t, \kappa) \leq u$, $\forall u \in (1 - MC_{\eta, \eta} (v, t, \kappa))$. i.e. $v$ is an $(t, \kappa)$-g$fMc$ relative to $X$.

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References


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