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An (ι,κ)-fuzzy M closed and (ι,κ) - generalized fuzzy M closed sets in double fuzzy topological spaces

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Abstract

In double fuzzy topological spaces, (ι,κ)- fuzzy θ-closed, (ι,κ)-fuzzy M-closed sets and (ι,κ)-generalized fuzzy M-closed sets are introduced. Also we study some of their properties.

Keywords

(ι,κ)- fuzzy θ-closed, (ι,κ)-fuzzy M-closed sets, (ι,κ)-generalized fuzzy M-closed sets,(ι,κ)-generalized fuzzy M-interior, (ι,κ)-generalized fuzzy M-closure.

AMS Subject Classification

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Contents

1. Introduction and Preliminaries

" Intuitionistic fuzzy sets and Intuitionistic fuzzy topological space" were initiated by Atanassov [1] in 1993 and Coker [2] in 1997 respectively. In 2005, Garcia and Rodabaugh [7] coined the term "double" instead of "intuitionistic". In the past two decades many researchers [9–11, 16] doing more applications on double fuzzy topological spaces. From 2011, El-Maghrabi and Al - Johany [3–6] introduced and studied some properties on M-open sets and maps in topological spaces. Recently these kind of sets were studied in Ŝostak's Fuzzy topological spaces [14]. In this paper we introduce (ι,κ)- fuzzy θ-closed, (ι,κ)-fuzzy M-closed sets, (ι,κ) - generalized fuzzy M - closed sets and study some of their properties in double fuzzy topological spaces.

X denotes a non-empty set, $I=[0,1]$, $I_0=(0,1]$, $I_1=[0,1)$, $\underline{0}(X)=0$, $\underline{1}(X)=1$, $\iota\in I_0$ & $\kappa\in I_1$ and always $\iota+\kappa\leq 1$. I^X is a family of all fuzzy sets on X. In 2002 Samanta and Mondal[13] defined the double fuzzy topological space (briefly, dfts) denoted by (X,η,η^*), also defined (ι,κ)-fuzzy open (resp. (ι,κ)-fuzzy closed) (briefly (ι,κ)-fo (resp. (ι,κ)-fc)) set if $\eta(\gamma)\geq\iota$ and $\eta^*(\gamma)\leq\kappa$, (resp. $\underline{1}-\gamma$ is an (ι,κ)-fo set.) The double fuzzy interior and double fuzzy closure operators [8] are defined from $I^X\times I_0\times I_1\to I^X$ as follows

$$\begin{aligned} I_{\eta,\eta^*}(\gamma,\iota,\kappa) &= \bigvee\left\{\nu\in I^X|\nu\leq\gamma,\eta(\nu)\geq\iota,\eta^*(\nu)\leq\kappa\right\}, \\ C_{\eta,\eta^*}(\gamma,\iota,\kappa) &= \\ &\bigwedge\left\{\nu\in I^X|\nu\geq\gamma,\eta(\underline{1}-\nu)\geq\iota,\eta^*(\underline{1}-\nu)\leq\kappa\right\}, \end{aligned}$$

$\gamma\in I^X$, is called (ι,κ)-fuzzy regular open [12] (resp. (ι,κ)-fuzzy regular closed)(briefly (ι,κ)-fro (resp. (ι,κ)-frc)) set if $\gamma=I_{\eta,\eta^*}(C_{\eta,\eta^*}(\gamma,\iota,\kappa),\iota,\kappa)$ (resp. $\underline{1}-\gamma$ is (ι,κ)-fro set.) The operators $\delta C_{\eta,\eta^*}$ and $\delta I_{\eta,\eta^*}$: $I^X\times I_0\times I_1\to I^X$ [11] as follows

$$\delta I_{\eta,\eta^*}(\gamma,\iota,\kappa)=\bigvee\left\{\nu\in I^X|\nu\leq\gamma,\nu \text{ is an } (\iota,\kappa)\text{-}fro\right\},$$

$$\delta C_{\eta,\eta^*}(\gamma,\iota,\kappa)=\bigwedge\left\{\nu\in I^X|\nu\geq\gamma,\nu \text{ is an } (\iota,\kappa)\text{-}frc\right\}.$$

A fuzzy set $\gamma\in I^X$, is called (ι,κ)-fuzzy δ pre open (resp. (ι,κ)- fuzzy e open, (ι,κ)-fuzzy δ pre closed and (ι,κ)-fuzzy e closed) (briefly (ι,κ)-$f\delta po$ (resp. (ι,κ)-feo, (ι,κ)-$f\delta pc$ and (ι,κ)-fec)) [11]set if $\gamma\leq I_{\eta,\eta^*}(\delta C_{\eta,\eta^*}(\gamma,\iota,\kappa),\iota,\kappa)$ (resp. $\gamma\leq C_{\eta,\eta^*}(\delta I_{\eta,\eta^*}(\gamma,\iota,\kappa),\iota,\kappa)\vee I_{\eta,\eta^*}(\delta C_{\eta,\eta^*}(\gamma,\iota,\kappa),\iota,\kappa)$, $\underline{1}-\gamma$ is an (ι,κ)-$f\delta po$ and $\underline{1}-\gamma$ is an (ι,κ)-feo). The operators $\delta PC_{\eta,\eta^*}$, $\delta PI_{\eta,\eta^*}$, eC_{η,η^*} and eI_{η,η^*}:

$I^X \times I_0 \times I_1 \to I^X$ [11] defined as

$$\delta PI_{\eta,\eta^*}(\gamma,\iota,\kappa) = \bigvee \{\nu \in I^X | \nu \leq \gamma, \nu \text{ is an } (\iota,\kappa)\text{-}f\delta po\},$$

$$\delta PC_{\eta,\eta^*}(\gamma,\iota,\kappa) = \bigwedge \{\nu \in I^X | \nu \geq \gamma, \nu \text{ is an } (\iota,\kappa)\text{-}f\delta pc\},$$

$$eI_{\eta,\eta^*}(\gamma,\iota,\kappa) = \bigvee \{\nu \in I^X | \nu \leq \gamma, \nu \text{ is an } (\iota,\kappa)\text{-}feo\},$$

$$eC_{\eta,\eta^*}(\gamma,\iota,\kappa) = \bigwedge \{\nu \in I^X | \nu \geq \gamma, \nu \text{ is an } (\iota,\kappa)\text{-}fec\}.$$

All other notations are from fuzzy set theory, which is given by Zadeh [15] and his followers.

2. An (ι,κ)- fuzzy M closed sets

Definition 2.1. Let (X,η,η^*) be a dfts, $\forall \gamma,\nu \in I^X$, the operators (ι,κ)- fuzzy θ interior and (ι,κ)- fuzzy θ closure denoted by (ι,κ)-$\theta I_{\eta,\eta^*}$ and (ι,κ)-$\theta C_{\eta,\eta^*} : I^X \times I_0 \times I_1 \to I^X$ are defined as

$$\theta I_{\eta,\eta^*}(\gamma,\iota,\kappa) = \bigvee \{I_{\eta,\eta^*}(\nu,\iota,\kappa) | \nu \leq \gamma, \eta(\underline{1}-\nu) \geq \iota \,\&\, \eta^*(\underline{1}-\nu) \leq \kappa\}$$

and

$$\theta C_{\eta,\eta^*}(\gamma,\iota,\kappa) = \bigwedge \{C_{\eta,\eta^*}(\nu,\iota,\kappa) | \nu \geq \gamma, \eta(\nu) \geq \iota \,\&\, \eta^*(\nu) \leq \kappa\}.$$

Definition 2.2. In a dfts (X,η,η^*), $\gamma \in I^X$ is called an

1. (ι,κ) - fuzzy θ open (resp. (ι,κ) - fuzzy θ semi open) (briefly (ι,κ)-$f\theta o$ (resp. (ι,κ)-$f\theta so$)) set if $\gamma = \theta I_{\eta,\eta^*}(\gamma,\iota,\kappa)$. (resp. $\gamma \leq C_{\eta,\eta^*}(\theta I_{\eta,\eta^*}(\gamma,\iota,\kappa),\iota,\kappa)$.)
2. (ι,κ) - fuzzy θ closed (resp. (ι,κ) - fuzzy θ semi closed) (briefly (ι,κ)-$f\theta c$ (resp. (ι,κ)-$f\theta c$)) set if $\underline{1}-\gamma$ is an (ι,κ)-$f\theta o$ (resp. (ι,κ)-$f\theta so$) set.

Definition 2.3. In a dfts (X,η,η^*), $\gamma \in I^X$ is called an

1. (ι,κ)- fuzzy M closed (briefly (ι,κ)-fMc) set if $\gamma \geq I_{\eta,\eta^*}(\theta C_{\eta,\eta^*}(\gamma,\iota,\kappa),\iota,\kappa) \wedge C_{\eta,\eta^*}(\delta I_{\eta,\eta^*}(\gamma,\iota,\kappa),\iota,\kappa)$.
2. (ι,κ)- fuzzy M open (briefly (ι,κ)-fMo) set if $\underline{1}-\gamma$ is an (ι,κ)-fMc set.

Definition 2.4. Let (X,η,η^*) be a dfts, then the

1. union of all (ι,κ)-fMo (resp. (ι,κ)-$f\theta o$ and (ι,κ)-$f\theta so$) sets contained in γ is called the (ι,κ)-fuzzy M (resp. (ι,κ)-fuzzy θ and (ι,κ)-fuzzy θ semi) interior of γ and is denoted by $MI_{\eta,\eta^*}(\gamma,\iota,\kappa)$ (resp. $\theta I_{\eta,\eta^*}(\gamma,\iota,\kappa)$ and $\theta sI_{\eta,\eta^*}(\gamma,\iota,\kappa)$).
2. intersection of all (ι,κ)-fMc (resp. (ι,κ)-$f\theta c$ and (ι,κ)-$f\theta sc$) sets containing γ is called the (ι,κ)-fuzzy M (resp. (ι,κ)-fuzzy θ and (ι,κ)-fuzzy θ semi) closure of γ and is denoted by $MC_{\eta,\eta^*}(\gamma,\iota,\kappa)$ (resp. $\theta C_{\eta,\eta^*}(\gamma,\iota,\kappa)$ and $\theta sC_{\eta,\eta^*}(\gamma,\iota,\kappa)$).

Proposition 2.5. In a dfts (X,η,η^*), $\forall \gamma,\nu \in I^X$,

(a) $MI_{\eta,\eta^*}(\underline{0},\iota,\kappa) = \underline{0}$, and $MI_{\eta,\eta^*}(\underline{1},\iota,\kappa) = \underline{1}$.

(b) $MC_{\eta,\eta^*}(\underline{0},\iota,\kappa) = \underline{0}$, and $MC_{\eta,\eta^*}(\underline{1},\iota,\kappa) = \underline{1}$.

(c) $\underline{1} - MI_{\eta,\eta^*}(\gamma,\iota,\kappa) = MC_{\eta,\eta^*}(\underline{1}-\gamma,\iota,\kappa)$.

(d) $\underline{1} - MC_{\eta,\eta^*}(\gamma,\iota,\kappa) = MI_{\eta,\eta^*}(\underline{1}-\gamma,\iota,\kappa)$.

(e) If $\gamma < \nu$ then $MI_{\eta,\eta^*}(\gamma,\iota,\kappa) < MI_{\eta,\eta^*}(\nu,\iota,\kappa)$.

(f) If $\gamma \leq \nu$ then $MC_{\eta,\eta^*}(\gamma,\iota,\kappa) \leq MC_{\eta,\eta^*}(\nu,\iota,\kappa)$.

(g) $MI_{\eta,\eta^*}(\gamma,\iota,\kappa). \leq \gamma \leq MC_{\eta,\eta^*}(\gamma,\iota,\kappa)$.

(h) $MC_{\eta,\eta^*}(\gamma \vee \nu,\iota,\kappa) \geq MC_{\eta,\eta^*}(\gamma,\iota,\kappa) \vee MC_{\eta,\eta^*}(\nu,\iota,\kappa)$.

(i) $MC_{\eta,\eta^*}(\gamma \wedge \nu,\iota,\kappa) \leq MC_{\eta,\eta^*}(\gamma,\iota,\kappa) \wedge MC_{\eta,\eta^*}(\nu,\iota,\kappa)$.

(j) $MC_{\eta,\eta^*}(MC_{\eta,\eta^*}(\gamma,\iota,\kappa),\iota,\kappa) = MC_{\eta,\eta^*}(\gamma,\iota,\kappa)$.

(k) If γ is (ι,κ)-fMc set then $MC_{\eta,\eta^*}(\gamma,\iota,\kappa) = \gamma$.

(l) If ν is (ι,κ)-fMo set then $\nu\, q\, \gamma$ iff $\nu\, q\, MC_{\eta,\eta^*}(\gamma,\iota,\kappa)$.

(m) $MI_{\eta,\eta^*}(\gamma \vee \nu,\iota,\kappa) \geq MI_{\eta,\eta^*}(\gamma,\iota,\kappa) \vee MI_{\eta,\eta^*}(\nu,\iota,\kappa)$.

(n) $MI_{\eta,\eta^*}(\gamma \wedge \nu,\iota,\kappa) \leq MI_{\eta,\eta^*}(\gamma,\iota,\kappa) \wedge MI_{\eta,\eta^*}(\nu,\iota,\kappa)$.

(o) $MI_{\eta,\eta^*}(MI_{\eta,\eta^*}(\gamma,\iota,\kappa),\iota,\kappa) = MI_{\eta,\eta^*}(\gamma,\iota,\kappa)$.

(p) If γ is (ι,κ)-fMo set then $MI_{\eta,\eta^*}(\gamma,\iota,\kappa) = \gamma$.

(q) $\gamma \leq C_{\eta,\eta^*}(\gamma,\iota,\kappa) \leq \delta C_{\eta,\eta^*}(\gamma,\iota,\kappa) \leq \theta C_{\eta,\eta^*}(\gamma,\iota,\kappa)$.

(r) $\theta I_{\eta,\eta^*}(\gamma,\iota,\kappa) \leq \delta I_{\eta,\eta^*}(\gamma,\iota,\kappa) \leq I_{\eta,\eta^*}(\gamma,\iota,\kappa) \leq \gamma$.

Theorem 2.6. In any dfts (X,η,η^*), Every

1. (ι,κ)- $f\theta sc$ (resp. (ι,κ)- $f\delta pc$) set is an (ι,κ)-fMc set.
2. (ι,κ)- $f\theta c$ set is an (ι,κ)-$f\theta sc$ set.
3. (ι,κ)- $f\theta c$ set is an (ι,κ)- fc set.
4. (ι,κ)- fc set is an (ι,κ)-$f\delta pc$ set.
5. (ι,κ)- fMc set is an (ι,κ)- fec set.

Remark 2.7. The converse of the above theorem, in general, need not be true. It can be verified from the following examples.

Example 2.8. Consider the dfts (X,η,η^*) with $X = \{a,b,c\}$ and

$$\eta(\gamma) = \begin{cases} 1, & \text{if } \gamma \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2}, & \text{if } \gamma = \underline{0.1} \\ 0, & \text{Otherwise.} \end{cases} \qquad \eta^*(\gamma) = \begin{cases} 0, & \text{if } \gamma \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2}, & \text{if } \gamma = \underline{0.1} \\ 1, & \text{Otherwise.} \end{cases}$$

Then the fuzzy set $\underline{0.1}$ is an $(\frac{1}{2},\frac{1}{2})$-$fMc$ set but not an $(\frac{1}{2},\frac{1}{2})$-$f\delta pc$ set.

Example 2.9. Consider the dfts (X,η,η^*) with $X=\{a,b,c\}$ and

$$\eta(\gamma)=\begin{cases}1, & \text{if } \gamma\in\{\underline{0},\underline{1}\},\\ \frac{2}{3}, & \text{if } \gamma=\gamma_1,\\ \frac{1}{3}, & \text{if } \gamma=\gamma_2,\\ 0, & \text{Otherwise.}\end{cases}\qquad \eta^*(\gamma)=\begin{cases}0, & \text{if } \gamma\in\{\underline{0},\underline{1}\},\\ \frac{1}{3}, & \text{if } \gamma=\gamma_1,\\ \frac{2}{3}, & \text{if } \gamma=\gamma_2,\\ 1, & \text{Otherwise.}\end{cases}$$

where $\gamma_1(a)=0.3$, $\gamma_1(b)=0.4$, $\gamma_1(c)=0.5$, $\gamma_2(a)=0.6$, $\gamma_2(b)=0.9$, $\gamma_2(c)=0.5$, $\gamma_3(a)=0.3$, $\gamma_3(b)=0$ and $\gamma_3(c)=0.5$. Then the fuzzy set γ_3 is an $(\frac{1}{3},\frac{2}{3})$-$fMc$ set but not an $(\frac{1}{3},\frac{2}{3})$- $f\theta sc$ set.

Example 2.10. Consider the dfts (X,η,η^*) with $X=\{a,b,c\}$ and

$$\eta(\gamma)=\begin{cases}1, & \text{if } \gamma\in\{\underline{0},\underline{1}\},\\ \frac{3}{4}, & \text{if } \gamma=\gamma_1,\\ \frac{1}{4}, & \text{if } \gamma=\gamma_2,\\ 0, & \text{Otherwise.}\end{cases}\qquad \eta^*(\gamma)=\begin{cases}0, & \text{if } \gamma\in\{\underline{0},\underline{1}\},\\ \frac{1}{4}, & \text{if } \gamma=\gamma_1,\\ \frac{3}{4}, & \text{if } \gamma=\gamma_2,\\ 1, & \text{Otherwise,}\end{cases}$$

where $\gamma_1(a)=0.3$, $\gamma_1(b)=0.4$, $\gamma_1(c)=0.5$, $\gamma_2(a)=0.6$, $\gamma_2(b)=0.5$, and $\gamma_2(c)=0.5$. Then the fuzzy set γ_1 is an $(\frac{1}{4},\frac{3}{4})$- $f\theta sc$ set but not an $(\frac{1}{4},\frac{3}{4})$- $f\theta c$ set.

Example 2.11. Consider the dfts (X,η,η^*) with $X=\{a,b,c\}$ and

$$\eta(\gamma)=\begin{cases}1, & \text{if } \gamma\in\{\underline{0},\underline{1}\},\\ \frac{2}{5}, & \text{if } \gamma=\underline{0.5},\\ \frac{1}{5}, & \text{if } \gamma=\gamma_1,\\ 0, & \text{Otherwise.}\end{cases}\qquad \eta^*(\gamma)=\begin{cases}0, & \text{if } \gamma\in\{\underline{0},\underline{1}\},\\ \frac{3}{5}, & \text{if } \gamma=\underline{0.5},\\ \frac{4}{5}, & \text{if } \gamma=\gamma_1,\\ 1, & \text{Otherwise,}\end{cases}$$

where $\gamma_1(a)=0.3$, $\gamma_1(b)=0.5$, and $\gamma_1(c)=0.5$, then the fuzzy set $\underline{1}-\gamma_1$ is an $(\frac{1}{5},\frac{4}{5})$- fc set but not an $(\frac{1}{5},\frac{4}{5})$- $f\theta c$ set.

Example 2.12. Consider the dfts (X,η,η^*) with $X=\{a,b,c\}$ and

$$\eta(\gamma)=\begin{cases}1, & \text{if } \gamma\in\{\underline{0},\underline{1}\},\\ \frac{1}{6}, & \text{if } \gamma=\gamma_1,\\ 0, & \text{Otherwise.}\end{cases}\qquad \eta^*(\gamma)=\begin{cases}0, & \text{if } \gamma\in\{\underline{0},\underline{1}\},\\ \frac{5}{6}, & \text{if } \gamma=\gamma_1,\\ 1, & \text{Otherwise,}\end{cases}$$

where $\gamma_1(a)=0.5$, $\gamma_1(b)=0.3$, $\gamma_1(c)=0.2$, $\gamma_2(a)=0.5$, $\gamma_2(b)=0.6$ and $\gamma_2(c)=0.6$, then the fuzzy set γ_2 is an $(\frac{1}{6},\frac{5}{6})$-$fec$ set but not an $(\frac{1}{6},\frac{5}{6})$- fMc set.

From the above theorem, examples and by [11], the following implications are hold.

$$\begin{tikzcd}
 & (\iota,\kappa)\text{-}f\theta c \arrow[dl] \arrow[dr] & \\
(\iota,\kappa)\text{-}f\theta sc \arrow[dd] & & (\iota,\kappa)\text{-}fc \arrow[dl] \arrow[dd] \\
 & (\iota,\kappa)\text{-}f\delta pc \arrow[dl] & \\
(\iota,\kappa)\text{-}\mathbf{fMc} \arrow[dr] & & (\iota,\kappa)\text{-}f\delta sc \arrow[dl] \\
 & (\iota,\kappa)\text{-}fec &
\end{tikzcd}$$

Note: $A\to B$ denotes A implies B, but not conversely.

Theorem 2.13. Let (X,η,η^*) be a dfts,

(i) $\bigvee\limits_{i\in \mathrm{N}}\gamma_i$ is an (ι,κ)-fMo set if $\forall\ i\in\mathrm{N}$, γ_i be an (ι,κ)-fMo set.

(ii) $\bigwedge\limits_{i\in \mathrm{N}}\gamma_i$ is an (ι,κ)-fMc set if $\forall\ i\in\mathrm{N}$, γ_i be an (ι,κ)-fMc set.

Proof: (i) Let γ_i be an (ι,κ)-fMo set, $\forall i\in\mathrm{N}$ then

$$\begin{aligned}
\gamma_i &\subseteq C_{\eta,\eta^*}(\theta I_{\eta,\eta^*}(\gamma_i,\iota,\kappa),\iota,\kappa)\\
&\quad \vee I_{\eta,\eta^*}(\delta C_{\eta,\eta^*}(\gamma_i,\iota,\kappa),\iota,\kappa)\quad \forall i\in\mathrm{N},\\
\Rightarrow \bigvee_{i\in\mathrm{N}}\gamma_i &\subseteq \bigvee_{i\in I}(C_{\eta,\eta^*}(\theta I_{\eta,\eta^*}(\gamma_i,\iota,\kappa),\iota,\kappa)\\
&\qquad\qquad \vee I_{\eta,\eta^*}(\delta C_{\eta,\eta^*}(\gamma_i,\iota,\kappa),\iota,\kappa))\\
&\subseteq C_{\eta,\eta^*}(\theta I_{\eta,\eta^*}(\bigvee_{i\in I}\gamma_i,\iota,\kappa),\iota,\kappa)\\
&\qquad\qquad \bigvee I_{\eta,\eta^*}(\delta C_{\eta,\eta^*}(\bigvee_{i\in I}\gamma_i,\iota,\kappa),\iota,\kappa).
\end{aligned}$$

Thus $\bigvee\limits_{i\in I}\gamma_i$ is an (ι,κ)-fMo set.
(ii) Similar to the proof of (i). Ξ

Theorem 2.14. In dfts (X,η,η^*), let $\gamma,\nu\in I^X$

(i) $\gamma\wedge\nu$ is an (ι,κ)-fMo set if γ is an (ι,κ)-fMo set and $\eta(\nu)\geq\iota$, $\eta^*(\nu)\leq\kappa$.

(ii) $\gamma\vee\nu$ is an (ι,κ)-fMc set if γ is an (ι,κ)-fMc set and $\eta(\underline{1}-\nu)\geq\iota$, $\eta^*(\underline{1}-\nu)\leq\kappa$.

Proof: (i) Let γ is an (ι,κ)-fMo set, and a crisp set $\nu\in I^X$

with $\eta(\nu)\geq\iota,\ \eta^*(\nu)\leq\kappa$, then

$$\begin{aligned}\gamma\wedge\nu &\leq (C_{\eta,\eta^*}(\theta I_{\eta,\eta^*}(\gamma,\iota,\kappa),\iota,\kappa)\\ &\quad\vee I_{\eta,\eta^*}(\delta C_{\eta,\eta^*}(\gamma,\iota,\kappa),\iota,\kappa))\wedge\nu\\ &= (C_{\eta,\eta^*}(\theta I_{\eta,\eta^*}(\gamma,\iota,\kappa),\iota,\kappa)\wedge\nu)\\ &\quad\vee(I_{\eta,\eta^*}(\delta C_{\eta,\eta^*}(\gamma,\iota,\kappa),\iota,\kappa)\wedge\nu)\\ &\leq (C_{\eta,\eta^*}(\theta I_{\eta,\eta^*}(\gamma,\iota,\kappa)\wedge\nu,\iota,\kappa))\vee\\ &\quad (I_{\eta,\eta^*}(\delta C_{\eta,\eta^*}(\gamma,\iota,\kappa)\wedge\nu,\iota,\kappa))\\ &\leq (C_{\eta,\eta^*}(\theta I_{\eta,\eta^*}(\gamma\wedge\nu,\iota,\kappa),\iota,\kappa))\vee\\ &\quad (I_{\eta,\eta^*}(\delta C_{\eta,\eta^*}(\gamma\wedge\nu,\iota,\kappa),\iota,\kappa))\end{aligned}$$

Hence $\gamma\wedge\nu$ is an (ι,κ)-fMo set.
(ii) Similar to the proof of (i). Ξ

Theorem 2.15. If $\gamma\in I^X$ is both (ι,κ)-fMo and (ι,κ)-fc set in (X,η,η^*) then γ is an (ι,κ)-$f\theta so$.

Proof: Let γ be an (ι,κ)-fMo then

$$\begin{aligned}\gamma &\leq C_{\eta,\eta^*}(\theta I_{\eta,\eta^*}(\gamma,\iota,\kappa),\iota,\kappa)\\ &\quad\vee I_{\eta,\eta^*}(\delta C_{\eta,\eta^*}(\gamma,\iota,\kappa),\iota,\kappa)\\ &= C_{\eta,\eta^*}(\theta I_{\eta,\eta^*}(\gamma,\iota,\kappa),\iota,\kappa)\vee I_{\eta,\eta^*}(\gamma,\iota,\kappa)\\ &\leq C_{\eta,\eta^*}(\theta I_{\eta,\eta^*}(\gamma,\iota,\kappa),\iota,\kappa).\end{aligned}$$

Hence γ is an (ι,κ)-$f\theta so$. Ξ

Theorem 2.16. If $\gamma\in I^X$ is both (ι,κ)-fMc and (ι,κ)-fo set in (X,η,η^*) then γ is an (ι,κ)-$f\theta sc$.

Proof: Follows from theorem 2.15. Ξ

Theorem 2.17. In a dfts (X,η,η^*), $\forall\gamma\in I^X$,

1. If $\eta(\gamma)\geq\iota$ and $\eta^*(\gamma)\leq\kappa$ then γ is an (ι,κ)-fMo set.
2. $I_{\eta,\eta^*}(\gamma,\iota,\kappa)$ is an (ι,κ)-fMo set.
3. $C_{\eta,\eta^*}(\gamma,\iota,\kappa)$ is an (ι,κ)-fMc set.

3. An (ι,κ)-generalized fuzzy M closed sets

Definition 3.1. In a (X,η,η^*) be a dfts, $\gamma,\nu\in I^X$, a fuzzy set γ is called an (ι,κ)- generalized fuzzy M closed (resp. (ι,κ)-generalized fuzzy M open) (briefly (ι,κ)-$gfMc$ (resp. (ι,κ)-$gfMo$)) set if $MC_{\eta,\eta^*}(\gamma,\iota,\kappa)\leq\nu$ whenever $\gamma\leq\nu$, $\eta(\nu)\geq\iota$ and $\eta^*\leq\kappa$ (resp. $\underline{1}-\gamma$ is an (ι,κ)-$gfMc$ set).

Theorem 3.2. In a dfts (X,η,η^*), $\gamma\in I^X$ is (ι,κ)-$gfMo$ set iff $\nu\leq MI_{\eta,\eta^*}(\gamma,\iota,\kappa)$ whenever $\nu\leq\gamma$, $\eta(\underline{1}-\nu)\geq\iota$ and $\eta^*(\underline{1}-\nu)\leq\kappa$.

Definition 3.3. In a dfts (X,η,η^*), $\forall\gamma\in I^X$, an (ι,κ)- generalized fuzzy M closure operator denoted as (ι,κ)-GMC_{η,η^*} : $I^X\times I_0\times I_1\to I^X$ defined as

$$GMC_{\eta,\eta^*}(\gamma,\iota,\kappa)=\bigwedge\left\{\nu\in I^X|\gamma\leq\nu \text{ and } \nu \text{ is } (\iota,\kappa)\text{-}gfMc\right\}.$$

Theorem 3.4. In a dfts (X,η,η^*), $\forall\gamma,\nu\in I^X$, then the operator (ι,κ)-GMC_{η,η^*} satisfies the following statements

1. $GMC_{\eta,\eta^*}(\underline{0},\iota,\kappa)=\underline{0}$, and $GMC_{\eta,\eta^*}(\underline{1},\iota,\kappa)=\underline{1}$.
2. $\gamma\leq GMC_{\eta,\eta^*}(\gamma,\iota,\kappa)$.
3. $GMC_{\eta,\eta^*}(\gamma\vee\nu,\iota,\kappa)$ $\geq GMC_{\eta,\eta^*}(\gamma,\iota,\kappa)\vee GMC_{\eta,\eta^*}(\nu,\iota,\kappa)$.
4. $GMC_{\eta,\eta^*}(GMC_{\eta,\eta^*}(\gamma,\iota,\kappa),\iota,\kappa)=GMC_{\eta,\eta^*}(\gamma,\iota,\kappa)$.
5. If γ is (ι,κ)-$gfMc$ set then $GMC_{\eta,\eta^*}(\gamma,\iota,\kappa)=\gamma$.
6. $GMC_{\eta,\eta^*}(\gamma,\iota,\kappa)\leq MC_{\eta,\eta^*}(\gamma,\iota,\kappa)\leq C_{\eta,\eta^*}(\gamma,\iota,\kappa)$.

Theorem 3.5. In a dfts (X,η,η^*), $\forall\gamma\in I^X$, an (ι,κ)- generalized fuzzy M interior operator denoted as (ι,κ)-GMI_{η,η^*} : $I^X\times I_0\times I_1\to I^X$ given by

$$GMI_{\eta,\eta^*}(\gamma,\iota,\kappa)=\bigvee\left\{\nu\in I^X|\gamma\geq\nu \text{ and } \nu \text{ is } (\iota,\kappa)\text{-}gfMo\right\},$$

and

$$GMI_{\eta,\eta^*}(\underline{1}-\gamma,\iota,\kappa)=\underline{1}-GMC_{\eta,\eta^*}(\gamma,\iota,\kappa).$$

Proposition 3.6. In a dfts (X,η,η^*), let an (ι,κ)-$gfMc$ set $\gamma\in I^X$ and

1. if γ is an (ι,κ)-fMo set then γ is an (ι,κ)-fMc set.
2. if γ is an (ι,κ)-fo set then $\gamma\wedge\nu$ is an (ι,κ)-fMc set whenever $\nu\leq MC_{\eta,\eta^*}(\gamma,\iota,\kappa)$.

Proof: (1) Let γ be an (ι,κ)-$gfMc$ set and an (ι,κ)-fMo set such that $\gamma\leq\gamma$.
$\Rightarrow$ $MC_{\eta,\eta^*}(\gamma,\iota,\kappa)\leq\gamma$. Since $\gamma\leq MC_{\eta,\eta^*}(\gamma,\iota,\kappa)$,
$\Rightarrow$ $\gamma=MC_{\eta,\eta^*}(\gamma,\iota,\kappa)$. Therefore γ is an (ι,κ)-fMc set.
(2) Let γ be an (ι,κ)-$gfMc$ set and an (ι,κ)-fo set, then

$$\begin{aligned}MC_{\eta,\eta^*}(\gamma,\iota,\kappa)\leq\gamma &\Rightarrow \gamma \text{ is an } (\iota,\kappa)\text{-}fMc \text{ set}\\ &\Rightarrow \gamma\wedge\nu \text{ is an } (\iota,\kappa)\text{-}fMc \text{ set}\\ &\Rightarrow \gamma\wedge\nu \text{ is an } (\iota,\kappa)\text{-}gfMc \text{ set.}\end{aligned}$$

Ξ

Definition 3.7. In a dfts (X,η,η^*), a fuzzy set γ is called as (ι,κ)- generalized* fuzzy M closed (resp. (ι,κ)- generalized* fuzzy M open) (briefly (ι,κ)-g^*fMc (resp. (ι,κ)-g^*fMo)) set if $MC_{\eta,\eta^*}(\gamma,\iota,\kappa)\leq\nu$ whenever $\gamma\leq\nu$ and ν is an (ι,κ)-gfo set in I^X (resp. $\underline{1}-\gamma$ is (ι,κ)-gfM^*c set).

Theorem 3.8. In a dfts (X,η,η^*), a fuzzy set γ is (ι,κ)-g^*fMo set if and only if $\nu\leq MI_{\eta,\eta^*}(\gamma,\iota,\kappa)$ whenever ν is an (ι,κ)-gfc set.

Proof: Let γ be an (ι,κ)-g^*fMo set in I^X, and let ν is an (ι,κ)-gfc set $\ni\nu\leq\gamma$, by definition (3.1), $\underline{1}-\gamma$ is an (ι,κ)-gfo set in I^X & $\underline{1}-\gamma\leq\underline{1}-\nu$. But $\underline{1}-\gamma$ is an (ι,κ)-g^*fMc set, then $MC_{\eta,\eta^*}(\underline{1}-\gamma,\iota,\kappa)\leq\underline{1}-\nu$. But

$$MC_{\eta,\eta^*}(\underline{1}-\gamma,\iota,\kappa)=\underline{1}-MI_{\eta,\eta^*}(\gamma,\iota,\kappa)\leq\underline{1}-\nu.$$

Hence, $\nu \leq MI_{\eta,\eta^*}(\gamma,\iota,\kappa)$.

Conversely, let $\nu \leq MI_{\eta,\eta^*}(\gamma,\iota,\kappa)$ whenever $\nu \leq \gamma$ and ν is an (ι,κ)-gfc set, now, $\underline{1} - MI_{\eta,\eta^*}(\gamma,\iota,\kappa) \leq \underline{1} - \nu$, Thus $MC_{\eta,\eta^*}(\underline{1} - \gamma,\iota,\kappa) \leq \underline{1} - \nu$. Therefore $\underline{1} - \gamma$ is an (ι,κ)-$gfMc$ set. Hence γ is an (ι,κ)-$gfMo$ set. Ξ

Proposition 3.9. In a dfts (X,η,η^*), $\gamma \wedge \nu$ is an (ι,κ)-g^*fMc set if γ & ν are (ι,κ)-g^*fMc sets.

Proof: Assume γ and ν are (ι,κ)-g^*fMc sets, $\ni$ $\gamma \wedge \nu \leq \upsilon$, $\forall$ (ι,κ)-gfo set υ, $MC_{\eta,\eta^*}(\gamma,\iota,\kappa) \leq \upsilon$, and since γ is an (ι,κ)-g^*fMc set $MC_{\eta,\eta^*}(\gamma,\iota,\kappa) \leq \upsilon$ $\forall$ (ι,κ)-gfo set $\upsilon \in I^X$ and $\gamma \leq \upsilon$. Also ν is an (ι,κ)-g^*fMc set $MC_{\eta,\eta^*}(\nu,\iota,\kappa) \leq \upsilon$ $\forall$ (ι,κ)-gfo set $\upsilon \in I^X$, & $\nu \leq \upsilon$. Therefore

$$MC_{\eta,\eta^*}(\gamma,\iota,\kappa) \wedge MC_{\eta,\eta^*}(\nu,\iota,\kappa) \leq \upsilon$$

whenever $\gamma \wedge \nu \leq \upsilon$. Hence $\gamma \wedge \nu$ is an (ι,κ)-g^*fMc sets in I^X. Ξ

Proposition 3.10. In a dfts (X,η,η^*), $\gamma \wedge \nu$ is an (ι,κ)-g^*fMc set if γ is (ι,κ)-g^*fMc set and $\eta(\nu) \geq \iota, \eta^*(\nu) \leq \kappa$.

Proof: Since every (ι,κ)-fc set is an (ι,κ)-g^*fMc set and from the proposition 3.9 we have the proof. Ξ

Proposition 3.11. In a dfts (X,η,η^*), γ is an (ι,κ)-g^*fMc set and

1. if γ is an (ι,κ)-gfo set then γ is an (ι,κ)-fMc set.
2. if $\gamma \leq \nu \leq MC_{\eta,\eta^*}(\gamma,\iota,\kappa)$ then ν is an (ι,κ)-g^*fMc set.

Proof: (1) Suppose that γ is both (ι,κ)-gfo and (ι,κ)-g^*fMc set in I^X. Since γ is an (ι,κ)-gfo set, $MC_{\eta,\eta^*}(\gamma,\iota,\kappa) \leq \gamma$, since γ is an (ι,κ)-g^*fMc set $\gamma \leq MC_{\eta,\eta^*}(\gamma,\iota,\kappa)$. Therefore $\gamma = MC_{\eta,\eta^*}(\gamma,\iota,\kappa)$.

(2) Suppose that γ is an (ι,κ)-g^*fMc set and υ is an (ι,κ)-gfo set in I^X $\ni$ $\gamma \leq \nu \leq \upsilon$ and let $\nu \leq MC_{\eta,\eta^*}(\gamma,\iota,\kappa)$, This implies that

$$\begin{aligned} MC_{\eta,\eta^*}(\nu,\iota,\kappa) &\leq MC_{\eta,\eta^*}(MC_{\eta,\eta^*}(\gamma,\iota,\kappa),\iota,\kappa) \\ &= MC_{\eta,\eta^*}(\gamma,\iota,\kappa). \end{aligned}$$

Since γ is an an (ι,κ)-g^*fMc, υ is an (ι,κ)-gfo and $\gamma \leq \upsilon$ we can say that $MC_{\eta,\eta^*}(\gamma,\iota,\kappa)\upsilon$, this implies that

$$MC_{\eta,\eta^*}(\nu,\iota,\kappa) \leq \upsilon.$$

Hence ν is an (ι,κ)-g^*fMc set. Ξ

Theorem 3.12. Let (X,η_1,η_1^*) and (Y,η_2,η_2^*) be dfts's. If $\gamma \leq \underline{1}_Y \leq \underline{1}_X$, $\ni$ γ is (ι,κ)-g^*fMc set in I^X, then γ is an (ι,κ)-g^*fMc set relative to Y.

Proof: Suppose that (X,η_1,η_1^*) and (Y,η_2,η_2^*) be dfts's $\ni$ $\gamma \leq \underline{1}_Y \leq \underline{1}_X$, & $\gamma \in I^X$ is (ι,κ)-g^*fMc set. And let $\gamma \leq \underline{1}_Y \wedge \nu$ $\ni$ $\nu \in I^X$ is an (ι,κ)-gfo set.
Since $\gamma \in I^X$ is (ι,κ)-g^*fMc set $\gamma \leq \nu \Rightarrow MC_{\eta,\eta^*}(\gamma,\iota,\kappa) \leq \nu$. So that $\underline{1}_Y \wedge MC_{\eta,\eta^*}(\gamma,\iota,\kappa) \leq \underline{1}_Y \wedge \nu$. Therefore γ is an (ι,κ)-g^*fMc set relative to Y. Ξ

Theorem 3.13. In a dfts (X,η,η^*), $\forall$ $\nu \leq \gamma$, if ν is (ι,κ)-g^*fMc set relative to γ $\ni$ γ is both (ι,κ)-gfo and (ι,κ)-g^*fMc set of I^X then ν is an (ι,κ)-g^*fMc set relative to X.

Proof: Let ν is an (r,s)-g^*fMc set and $\eta(\upsilon) \geq \iota$ and $\eta^*(\upsilon) \leq \kappa$ $\ni$ & $\nu \leq \upsilon$. But we have, $\nu \leq \gamma \leq \underline{1}$, and so $\nu \leq \gamma\, \nu \leq \upsilon$, hence $\nu \leq \gamma \wedge \upsilon$. Therefore, ν is an (ι,κ)-g^*fMc relative to γ, i.e. $\gamma \wedge MC_{\eta,\eta^*}(\nu,\iota,\kappa) \leq \gamma \wedge \upsilon \Rightarrow \gamma \wedge MC_{\eta,\eta^*}(\nu,\iota,\kappa) \leq \upsilon$.

Thus $(\gamma \wedge MC_{\eta,\eta^*}(\nu,\iota,\kappa)) \vee (\underline{1} - MC_{\eta,\eta^*}(\nu,\iota,\kappa)) \leq \upsilon \vee (\underline{1} - MC_{\eta,\eta^*}(\nu,\iota,\kappa))$.

This implies that

$$\gamma \vee (\underline{1} - MC_{\eta,\eta^*}(\nu,\iota,\kappa)) \leq \upsilon \vee (\underline{1} - MC_{\eta,\eta^*}(\nu,\iota,\kappa)).$$

Because γ is an (ι,κ)-g^*fMc set, $MC_{\eta,\eta^*}(\gamma,\iota,\kappa) \leq \upsilon \vee (\underline{1} - \nu)$. And so $\nu \leq \gamma \Rightarrow MC_{\eta,\eta^*}(\nu,\iota,\kappa) \leq MC_{\eta,\eta^*}(\gamma,\iota,\kappa)$.

Therefore

$$\begin{aligned} MC_{\eta,\eta^*}(\nu,\iota,\kappa) &\leq MC_{\eta,\eta^*}(\gamma,\iota,\kappa) \\ &\leq \upsilon \vee (\underline{1} - MC_{\eta,\eta^*}(\nu,\iota,\kappa)). \end{aligned}$$

Thus $MC_{\eta,\eta^*}(\gamma,\iota,\kappa) \leq \upsilon$, but $MC_{\eta,\eta^*}(\gamma,\iota,\kappa)$ is not contained in $(\underline{1} - MC_{\eta,\eta^*}(\nu,\iota,\kappa))$. i.e. ν is an (ι,κ)-g^*fMc relative to X. Ξ

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