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A Subclass of close-to-star functions

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Abstract

In this paper, we generalise the concept of close-to-star functions by considering a subclass $\mathscr{K}_s^*(\alpha,\beta)$ of normalized analytic functions in the unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ such that for some starlike function $g(z)=z+b_2z^2+...,$

$$\frac{zf'(z)}{g(z)}\prec\frac{1+(2\alpha-1)\beta z}{1+\beta z},\quad 0\leq\alpha<1,\quad 0<\beta\leq 1.$$

We obtain results on sharp coefficient estimates, growth and distortion theorems. .

Keywords

Analytic functions, coefficient estimates,starlike function.

AMS Subject Classification: 30c45.

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1. Introduction and Definitions

Let $\mathscr{A}$ be the normalized class of regular functions

$$f(z)=z+a_2z^2+a_3z^3+... \tag{1.1}$$

which are univalent in the unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ and normalized by the conditions $f(0)=f'(0)-1=0$.

If $f\in\mathscr{A}$ and satisfies $\Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha,\quad\forall\ z\in\mathbb{U}$, then $f(z)$ is said to be a starlike functions of order α. We use $\mathscr{S}^*(\alpha)$ to denote the class of starlike function of order α. On putting $\alpha=0$, the class of functions is denoted by $\mathscr{S}^*$. A function $f(z)$ belonging to $\mathscr{A}$ is said to be convex if and only if $\Re\left\{\frac{zf''(z)}{f'(z)}+1\right\}>0$. We denote $\mathscr{C}$ be the subclass of $\mathscr{A}$ consisting of all convex functions.

T.H.Macgregor [4], investigated the properties of the class $\mathbf{R}$ of functions $f(z)$ which belong to $\mathscr{A}$ and satisfy $\Re f'(z)>0,\ for\ z\in\mathbb{U}$. In [3], O.P.Juneja and M.L.Mogra considered the class $\mathbf{R}(\alpha,\beta)$ of functions $f(z)\in\mathscr{A}$, satisfying the condition

$$\left|\frac{f'(z)-1}{f'(z)+(1-2\alpha)}\right|<\beta,\quad(z\in\mathbb{U})$$

for some $\alpha,\beta(0\leq\alpha<1,\quad 0<\beta\leq 1)$.

In 1983, Silvia [9] defined and studied the class $\mathscr{C}_\beta[A,B]$ of functions $f(z)\in\mathscr{A}$ satisfying

$$\Re\left\{\frac{z(f'(z))'}{g'(z)}\right\}>\beta,\qquad where\ \ g(z)\in\mathscr{S}^*[A,B],\quad(z\in\mathbb{U}).$$

In [7], the author introduced new subclasses of the class $\mathscr{C}_\alpha$ of functions $f(z)$ defined by (1) to be an alpha-close-to-convex function if $\frac{f(z)f'(z)}{z}\neq 0\quad in\ \ \mathbb{U}$ and if for some nonnegative real number α and for some starlike function $\phi(z)=z+...,$

$$\Re\left\{(1-\alpha)\frac{zf'(z)}{\phi(z)}+\alpha\frac{(zf'(z))'}{\phi'(z)}\right\}>0,\quad(z\in\mathbb{U}).$$

In [8] the authors studied a subclass $\mathscr{C}'$ of close to convex functions defined by the condition

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0, \quad where \ g \in \mathscr{C}.$$

This work was further extended by Peng Zhigang [10], defining a subclass $\mathscr{C}'(\alpha,\beta)$ by requiring that $f \in \mathscr{C}'(\alpha,\beta)$ if and only if

$$\left|\frac{\frac{zf'(z)}{g(z)} - 1}{\frac{zf'(z)}{g(z)} + (1-2\alpha)}\right| < \beta.$$

In [11], the authors studied a subclass $\mathscr{C}^*[A,B]$ of close to convex functions defined by the condition

$$\left|\frac{(zf'(z))'}{g'(z)} - 1\right| < \left|A - B\frac{(zf'(z))'}{g'(z)}\right|, \quad where \ g \in \mathscr{C}$$

$$and \ -1 \leq B < A \leq 1.$$

Let s and t be analytic functions in $\mathbb{U}$. Then s is said to be subordinate to t, written $s(z) \prec t(z)$ if there is a Schwarz function $\omega(z)$ on $\mathbb{U}$ such that $\omega(0) = 0$,
$|\omega(z)| \leq |z|$ for every $|z| < 1$ and is such that

$$s(z) = t(w(z)). \tag{1.2}$$

However it is well-known that, if $t \in \mathscr{S}$, then (2) is equivalent to $s(0) = t(0)$ and $s(\mathbb{U}) = t(\mathbb{U})$.
In this paper. Motivated by the above concepts, we introduce and study
various properties of subclasses, $\mathscr{K}_s^*(\alpha,\beta)$ of close-to-star functions and
λ-close-to-star functions $\mathscr{C}_\lambda^*(\alpha,\beta)$.

Definition 1.1. *A normalized regular function $f(z)$ of the form (1) is said to be in the class $\mathscr{K}_s^*(\alpha,\beta)$ if and only if there is a starlike function*

$$g(z) = z + b_2z^2 + \ldots \in \mathscr{S}^*, \quad (z \in \mathbb{U}) \tag{1.3}$$

such that

$$\left|\frac{\frac{zf'(z)}{g(z)} - 1}{\frac{zf'(z)}{g(z)} + (1-2\alpha)}\right| < \beta, \quad 0 \leq \alpha < 1, \quad 0 < \beta \leq 1 \ (z \in \mathbb{U}). \tag{1.4}$$

Interms of subordination (2) can be put in the form $f \in \mathscr{K}_s^*(\alpha,\beta)$ if and only if

$$\frac{zf'(z)}{g(z)} \prec \frac{1 + (2\alpha-1)\beta z}{1 + \beta z}.$$

Note that $f \in \mathscr{K}_s^*(\alpha,\beta)$ if and only if $\frac{zf'(z)}{g(z)} \in \mathscr{P}((2\alpha - 1)\beta,\beta)$.

Definition 1.2. *A normalized regular function $f(z)$ of the form (1) is said to be in the class $\mathscr{C}_\lambda^*(\alpha,\beta)$ if and only if there is a starlike function $g(z)$ defined by (3) such that*

$$(1-\lambda)\frac{f'(z)}{g'(z)} + \lambda\frac{(zf'(z))'}{g'(z)} \prec \frac{1+(2\alpha-1)\beta z}{1+\beta z}, \quad (z \in \mathbb{U}), \tag{1.5}$$

where, $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$ and $0 < \beta \leq 1$.

2. Preliminaries

Lemma 2.1. *[1, 6] Let N and D be analytic in $\mathbb{U}$, D maps onto a many-sheeted starlike region, N(0) = 0 = D(0). Then*

$$\frac{N'(z)}{D'(z)} \prec \frac{1+Az}{1+Bz} \Rightarrow \frac{N(z)}{D(z)} \prec \frac{1+Az}{1+Bz}.$$

3. COEFFICIENT BOUNDS

Theorem 3.1. *If $h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n \in p\left((2\alpha-1)\beta,\beta\right)$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $z \in \mathbb{U}$, then $|h_n| \leq 2\beta(1-\alpha)$, for all $n \geq 1$. This inequality is sharp.*

Proof. As $h \in p((2\alpha-1)\beta,\beta)$, we have

$$1 + \sum_{n=1}^{\infty} h_n z^n \prec \frac{1+(2\alpha-1)\beta z}{1+\beta z}, \quad (0 \leq \alpha < 1, 0 < \beta \leq 1).$$

This implies there is a Schwarz function $\omega(z)$ such that

$$1 + \sum_{n=1}^{\infty} h_n z^n = \frac{1+(2\alpha-1)\beta\omega(z)}{1+\beta\omega(z)},$$

$$where \ \omega(0) = 0, |\omega(z)| \leq |z| \ forall \ |z| < 1. \tag{3.1}$$

Then (6) gives,

$$\left[((2\alpha-1)\beta - \beta) - \sum_{k=1}^{\infty} \beta h_k z^k\right]\left[\sum_{k=1}^{\infty} c_k z^k\right] = \sum_{k=1}^{\infty} h_k z^k,$$

$$where \ \omega(z) = \sum_{n=1}^{\infty} c_k z^k. \tag{3.2}$$

Applying Cauchy's product on the left hand side of (7) and comparing
coefficients, the nth coefficient h_n on the right hand side of (7) depends only on $h_1, h_2, \ldots, h_{n-1}$ on the left of (7).Therefore from (7), for all $n \geq 1$, it follows that

$$\left|2(\alpha-1)\beta - \sum_{k=1}^{n-1} \beta h_k z^k\right| |\omega(z)| = \sum_{k=1}^{\infty} h_k z^k + \sum_{k=n+1}^{\infty} b_k z^k$$

where b_k $(k = n+1, n+2, \ldots)$ are some complex numbers. Since $|\omega(z)| < 1$, we get,

$$\left|\sum_{k=1}^{\infty} h_k z^k + \sum_{k=n+1}^{\infty} b_k z^k\right| = \left|2(\alpha-1)\beta - \sum_{k=1}^{n-1} \beta h_k z^k\right| |\omega(z)|$$

$$\leq \left|2(\alpha-1)\beta - \sum_{k=1}^{n-1} \beta h_k z^k\right|. \tag{3.3}$$

Squaring and integrating around the circle $z = re^{i\theta}$, $0 \le \theta \le 2\pi$,

$$\frac{1}{2\pi}\int_0^{2\pi}\left|2(\alpha-1)\beta - \sum_{k=1}^{n-1}\beta h_k\, r^k e^{ik\theta}\right|^2 \, d\theta$$

$$\ge \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=1}^{n} h_k\, r^k e^{ik\theta} + \sum_{k=n+1}^{\infty} b_r\, r^k e^{ik\theta}\right|^2 \, d\theta.$$

Taking limit $r \to 1^-$ we get,

$$4\beta^2(\alpha-1)^2 + \sum_{k=1}^{n-1}\beta^2\,|h_k|^2 \ge \sum_{k=1}^{n-1}\beta\,|h_k|^2 + |h_n|^2,$$

which gives

$$|h_n| \le 2\beta(1-\alpha), \quad \forall\ n \ge 1.$$

Equality is attained for the functions $h(z) = \frac{1+(2\alpha-1)\beta z^n}{1+\beta z^n}$, $n = 2,3,\ldots.$ □

Theorem 3.2. *If an analytic function f in $\mathbb{U}$ defined by (1) satisfies the inequality*

$$\sum_{n=2}^{\infty} n(1+\beta)\,|a_n| + \sum_{n=2}^{\infty}(1+\beta(2\alpha-1))\,|b_n| \le 2(1-\alpha)\,\beta, \tag{3.4}$$

where for n=2,3,...,the coefficients b_n are given by (3) then $f \in \mathscr{K}_s^*(\alpha,\beta)$.

Proof. We set for f given by (1) and g defined by (3), then

$$M = \left|zf'(z) - g(z)\right| - \left|\beta(1-2\alpha)g(z) - \beta(zf'(z))\right|$$

$$= \left|\sum_{n=2}^{\infty} n\, a_n z^n - \sum_{n=2}^{\infty} b_n z^n\right|$$

$$- \left|2\beta(1-\alpha)(z) + \beta(1-2\alpha)\sum_{n=2}^{\infty} b_n z^n + \sum_{n=2}^{\infty}\beta n\, a_n z^n\right|$$

$$\le \sum_{n=2}^{\infty} n\,|a_n|\,|z|^n + \sum_{n=2}^{\infty}|b_n|\,|z|^n$$

$$- \left[2(1-\alpha)\beta\,|z| - \beta(1-2\alpha)\sum_{n=2}^{\infty}|b_n|\,|z|^n + \sum_{n=2}^{\infty}\beta n\,|a_n|\,|z|^n\right]$$

$$= -2(1-\alpha)\beta\,|z| + \sum_{n=2}^{\infty}(n(\beta+1))\,|a_n|\,|z|^n$$

$$+ \sum_{n=2}^{\infty}(1+\beta(2\alpha-1))|b_n|\,|z|^n.$$

Hence for $z \in \mathbb{U}$ using (9) we have

$$+M \le -2(1-\alpha)\beta + \sum_{n=2}^{\infty}(n(1+\beta))\,|a_n|$$

$$\sum_{n=2}^{\infty}(1+\beta(2\alpha-1))\,|b_n| \le 0.$$

From the above calculation, we have

$$|zf'(z) - g(z)| < |\beta(1-2\alpha)g(z) - \beta(zf'(z))|,$$

Which is equivalent to the inequality (4).Hence $f \in \mathscr{K}_s^*(\alpha,\beta)$. □

Theorem 3.3. *Let $f(z)$ of the form (1) be in the class $\mathscr{K}_s^*(\alpha,\beta)$. Then for $z \in \mathbb{U}$*

$$|a_n| \le [1-2\beta(1-\alpha)] + (n+1)(1-\alpha)\beta.$$

Proof. If $f \in \mathscr{K}_s^*(\alpha,\beta)$, then there is a function $g \in \mathscr{S}^*$ with $g(z)$ is given by (3) such that

$$\frac{zf'(z)}{g(z)} \in \mathscr{P}((2\alpha-1)\beta,\beta). \tag{3.5}$$

Now

$$\frac{zf'(z)}{g(z)} = p(z) \prec \frac{1+(2\alpha-1)\beta z}{1+\beta z}.$$

Equating the coefficients of z^n on both sides, we get

$$na_n = p_{n-1} + b_2 p_{n-2} + \ldots + b_{n-1}p_1 + b_n. \tag{3.6}$$

Since $g(z) \in \mathscr{S}^*$, it follows that $|b_n| \le n$, for all $n \ge 2$ and from Theorem 1,
$|p_n| \le 2\beta(1-\alpha)$ for all $n \ge 1$. Therefore, (11) yields

$$\begin{aligned} |na_n| &\le |b_n| + 2\beta(1-\alpha)[1+b_2+b_3+\ldots+b_{n-1}] \\ &\le n + 2\beta(1-\alpha)[1+2+3+\ldots+(n-1)]. \end{aligned} \tag{3.7}$$

On simplifying, we get the required assertion of this theorem. Applying similar arguments, we get the following theorem. □

Theorem 3.4. *If the function $f(z) \in \mathscr{C}_\lambda^*(\alpha,\beta)$ is defined by (1), then*

$$|a_n| \le \frac{1}{1+\lambda(n-1)} \left[n(1-2\beta(1-\alpha)) + \frac{(1-\alpha)\beta(n+1)(2n+1)}{3}\right] \tag{3.8}$$

where, $0 \le \lambda \le 1$, $0 \le \alpha < 1$ *and* $0 < \beta \le 1$.

4. Growth and distortion theorems

In this section, we prove the distortion and growth results for the function classes $\mathscr{K}_s^*(\alpha,\beta)$ and $\mathscr{C}_\lambda^*(\alpha,\beta)$.

Theorem 4.1. *If* $f \in \mathscr{K}_s^*(\alpha,\beta)$, $\quad 0 \le \alpha < 1$ *and* $0 < \beta \le 1$ $|z| = r < 1$, *then*
if $\beta = 1$,

$$\frac{1-(1-2\alpha)r}{(1+r)^3} \le |f'(z)| \le \frac{1+(1-2\alpha)r}{(1-r)^3}$$

and

$$\frac{(1-\alpha)r(2+r)}{(1+r)^2} - \frac{(1-2\alpha)r}{(1+r)} \le |f(z)| \le \frac{(1-\alpha)r(2-r)}{(1-r)^2} - \frac{(1-2\alpha)r}{(1-r)}.$$

if $\beta \ne 1$,

$$\frac{1-(1-2\alpha)\beta r}{(1+\beta r)(1+r)^2} \le |f'(z)| \le \frac{1+(1-2\alpha)\beta r}{(1-\beta r)(1-r)^2}$$

and

$$\frac{-2(1-\alpha)\beta}{(1-\beta)^2}\log\left(\frac{1+r}{1+\beta r}\right) + \frac{1+(1-2\alpha)\beta}{(1-\beta)}\frac{r}{1+r} \le |f(z)| \le \frac{2(1-\alpha)\beta}{(1-\beta)^2}\log\left(\frac{1-r}{1-\beta r}\right) + \frac{1+(1-2\alpha)\beta}{(1-\beta)}\frac{r}{1-r}.$$

Proof. As $f \in \mathscr{K}_s^*(\alpha,\beta)$, there exists a function $g(z) \in \mathscr{S}^*$, such that

$$zf'(z) = p(z)g(z),$$

where $p(z) \prec \frac{1+(2\alpha-1)\beta z}{1+\beta z}$.
It is clear that

$$\frac{1-(1-2\alpha)\beta r}{1+\beta r} \le |p(z)| \le \frac{1+(1-2\alpha)\beta r}{1-\beta r}.$$

Again $g(z) \in \mathscr{S}^*$, then

$$\frac{r}{(1+r)^2} \le |g(z)| \le \frac{r}{(1-r)^2}.$$

Therefore,

$$\frac{1-(1-2\alpha)\beta r}{(1+\beta r)(1+r)^2} \le |f'(z)| \le \frac{1+(1-2\alpha)\beta r}{(1-\beta r)(1-r)^2}. \quad (4.1)$$

For $z = re^{i\theta}$, with $0 < r < 1$, we have

$$f(z) = \int_0^r f'(te^{i\theta})e^{i\theta}dt. \quad (4.2)$$

From (13) and (14) we get

$$f(z) = \int_0^r f'(te^{i\theta})e^{i\theta}dt \le \int_0^r \frac{1+(1-2\alpha)\beta t}{(1-\beta t)(1-t)^2}dt. \quad (4.3)$$

If $\beta = 1$, then

$$|f(z)| \le \frac{(1-\alpha)r(2-r)}{(1-r)^2} - \frac{(1-2\alpha)r}{(1-r)}.$$

If $\beta \ne 1$, then

$$|f(z)| \le \frac{2(1-\alpha)\beta}{(1-\beta)^2}\log\left(\frac{1-r}{1-\beta r}\right) + \frac{1+(1-2\alpha)\beta}{(1-\beta)}\frac{r}{1-r}.$$

To prove the lower bound of $|f(z)|$ we proceed as follows. Let σ be the radius of the open disc contained in the map of $\mathbb{D}$ by $f(z)$. Let z_0 be the point of $|z| = r$ for which $|f(z)|$ attains its minimum value. This minimum increases with r and is less than σ. Hence, the linear segment Γ connecting the origin with the point $f(z_0)$ will be covered entirely by the values of $f(z)$ in $\mathbb{D}$. Let γ be the arc in $\mathbb{D}$ which is mapped by $\omega = f(z)$ onto this linear segment. Then

$$|f(z_0)| = \int_\Gamma |d\omega| = \int_\Gamma |f'(z)\,dz| \ge \int_\Gamma |f'(z)|\,|dz| \ge \int_0^r \frac{1-(1-2\alpha)\beta t}{(1+\beta t)(1+t)^2}dt. \quad (4.4)$$

If $\beta = 1$, then

$$|f(z)| \ge \frac{(1-\alpha)r(2+r)}{(1+r)^2} - \frac{(1-2\alpha)r}{(1+r)}.$$

If $\beta \ne 1$, then

$$|f(z)| \ge \frac{-2(1-\alpha)\beta}{(1-\beta)^2}\log\left(\frac{1+r}{1+\beta r}\right) + \frac{1+(1-2\alpha)\beta}{(1-\beta)}\frac{r}{1+r}.$$

□

Theorem 4.2. *Let* $0 \le \lambda \le 1$, $0 \le \alpha < 1$, $0 < \beta \le 1$ *and* $f \in \mathscr{C}_\lambda^*(\alpha,\beta)$.
(1) If $\lambda = 0$, *then for* $|z| = r < 1$, *we have*

$$\int_0^r \frac{1-(1-2\alpha)\beta t}{(1+\beta t)}\frac{(1-t)}{(1+t)^3}\,dt \le |f(z)| \le \int_0^r \frac{1+(1-2\alpha)\beta t}{(1-\beta t)}\frac{(1+t)}{(1-t)^3}\,dt. \quad (4.5)$$

which gives

$$\mathscr{L}_1 \le |f(z)| \le \mathscr{L}_2,$$

where

$$\mathscr{L}_1 = \frac{2\beta(1-\alpha)(1+\beta)}{(1-\beta)^3 r}\log\left(\frac{1+r}{1+\beta r}\right) + \frac{(1-2\alpha)\beta^2 + 2\beta(3\alpha-2)-1}{(1-\beta)^2}\frac{1}{1+r} + \frac{[1+(1-2\alpha)\beta]}{(1-\beta)}\frac{r+2}{(1+r)^2},$$

and

$$\mathscr{L}_2 = \frac{-2\beta(1-\alpha)(1+\beta)}{(1-\beta)^3 r}\log\left(\frac{1-r}{1-\beta r}\right) + \frac{(1-2\alpha)\beta^2 + 2\beta(3\alpha-2)-1}{(1-\beta)^2}\frac{1}{1-r}$$

$$- \frac{2[1+(1-2\alpha)\beta]}{(1-\beta)} \frac{r-2}{(1-r)^2}.$$

(2)If $0<\lambda\le 1$, then for $|z|=r<1$, we have

$$\mathscr{M}_1 \le |f(z)| \le \mathscr{M}_2,$$

where

$$\mathscr{M}_1 = \frac{1}{\lambda} r^{1-\frac{1}{\lambda}} \int_0^r \int_0^s \frac{1-(1-2\alpha)\beta t}{(1+\beta t)} \frac{1-t}{(1+t)^3}\, dt\, s^{\frac{1}{\lambda}-2}\, ds.$$
$$\mathscr{M}_2 = \frac{1}{\lambda} r^{1-\frac{1}{\lambda}} \int_0^r \int_0^s \frac{1+(1-2\alpha)\beta t}{(1-\beta t)} \frac{1+t}{(1-t)^3}\, dt\, s^{\frac{1}{\lambda}-2}\, ds. \tag{4.6}$$

Proof. Suppose that $f\in\mathscr{C}_\lambda^*(\alpha,\beta)$, there exists a function $g(z)\in\mathscr{S}^*$,
such that$(1-\lambda)f'(z) + \lambda(zf'(z))' = g'(z)h(z)$, where

$$h(z) = \frac{1+(2\alpha-1)\beta z}{1+\beta z}.$$

It is clear to see that then

$$\frac{1-r}{(1+r)^3} \le |g'(z)| \le \frac{1+r}{(1-r)^3}.$$

Setting $F'(z)=(1-\alpha)f'(z) + \alpha(zf'(z))'$, we have

$$\frac{1-(1-2\alpha)\beta r}{(1+\beta r)} \frac{1-r}{(1+r)^3} \le |F'(z)|$$
$$; \le \frac{1+(1-2\alpha)\beta r}{(1-\beta r)} \frac{1+r}{(1-r)^3}. \tag{4.7}$$

Therefore,

$$\int_0^r \frac{1-(1-2\alpha)\beta t}{(1+\beta t)} \frac{1-t}{(1+t)^3}\, dt \le |F(z)| \le \int_0^r \frac{1+(1-2\alpha)\beta t}{(1-\beta t)} \frac{1+t}{(1-t)^3}\, dt. \tag{4.8}$$

$$\int_0^r \frac{1-(1-2\alpha)\beta t}{(1+\beta t)} \frac{1-t}{(1+t)^3}\, dt \le |(1-\lambda)f(z) + \lambda z f'(z)| \le \int_0^r \frac{1+(1-2\alpha)\beta t}{(1-\beta t)} \frac{1+t}{(1-t)^3}\, dt. \tag{4.9}$$

Consider the following two cases:
(1) When $\lambda=0$. From (20) and using simple computation, we get (16).
(2) When $0<\lambda\le 1$, using a simple computation from (20), we get (17). □

Theorem 4.3. *Let $f(z)\in\mathscr{K}_s^*(\alpha,\beta), 0\le\alpha<1, 0<\beta\le 1$. Then*

$$Re\left\{\frac{zf'(z)}{g(z)}\right\} \ge \mathscr{M} \tag{4.10}$$

where

$$\mathscr{M} = \begin{cases} \frac{1-r}{1+r} - \frac{2(1-\alpha)\beta r}{(1+\beta r)[1+(2\alpha-1)\beta r]}, & \text{if } r_1 \le r_2,\\ \frac{1-r}{1+r} + \gamma - \frac{\alpha}{1-\alpha}, & \text{if } r_1 \ge r_2. \end{cases}$$

and

$$7\gamma = \frac{\sqrt{(1+\beta)[1+(2\alpha-1)\beta](1-\beta r^2)[1-(2\alpha-1)\beta r^2]}}{(1-\alpha)\beta(1-r^2)}. \frac{-[1+(1-2\alpha)\beta^2 r^2]}{(1-\alpha)\beta(1-r^2)} \tag{4.11}$$

where r_1 and r_2 are the unique roots of the equations

$$(2\alpha-1)\beta^3 r^3 - \beta[(2\alpha-1)\beta-1]r^2 - [2(2\alpha-1)\beta-1]r - 1 = 0 \tag{4.12}$$

and

$$[1-(1-2\alpha)\beta]\beta r^4 - 2[1-(1-2\alpha)\beta]\beta r^3 - \{(1-\beta)[1-(1-2\alpha)\beta] + 2\beta(\alpha-1)\}r^2 + 2[1-(1-2\alpha)\beta]r - [1-(1-2\alpha)\beta] = 0 \tag{4.13}$$

respectively.

Proof. Since $f(z)\in\mathscr{K}_s^*(\alpha,\beta)$, we can write

$$\frac{zf'(z)}{g(z)} = p(z) \prec \frac{1+(2\alpha-1)\beta z}{1+\beta z} \quad \textit{where } g(z)\in\mathscr{S}^*.$$

Hence $F(z) = zf'(z)$ and therefore

$$\frac{zF'(z)}{F(z)} = \frac{zg'(z)}{g(z)} + \frac{zp'(z)}{p(z)}. \tag{4.14}$$

For starlike function $g(z)$ of order α $(0\le\alpha<1)$, we have

$$Re\left\{\frac{zg'(z)}{g(z)}\right\} \ge \frac{1-r}{1+r}. \tag{4.15}$$

As $p(z) \prec \frac{1+(2\alpha-1)\beta z}{1+\beta z}$, we have [2],

$$Re\left\{\frac{zp'(z)}{p(z)}\right\} \ge \begin{cases} \frac{-2(1-\alpha)\beta r}{(1+\beta r)[1+(2\alpha-1)\beta r]}, & \text{if } r_1\le r_2;\\ \gamma - \frac{\alpha}{1-\alpha}, & \text{if } r_1 \ge r_2. \end{cases} \tag{4.16}$$

where

$$7\gamma = \frac{\sqrt{(1+\beta)[1+(2\alpha-1)\beta](1-\beta r^2)[1-(2\alpha-1)\beta r^2]}}{(1-\alpha)\beta(1-r^2)} \frac{-[1+(1-2\alpha)\beta^2 r^2]}{(1-\alpha)\beta(1-r^2)} \tag{4.17}$$

and r_1, r_2 are the unique roots of the equation

$$(2\alpha-1)\beta^2 r^4 - 2(2\alpha-1)\beta^2 r^3 - [1+4\alpha\beta+(2\alpha-1)\beta^2]r^2 - 2r + 1 = 0 \tag{4.18}$$

in the interval (0,1].From (24) (25) and (26), we get the required result.

□

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