A Subclass of close-to-star functions

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Abstract
In this paper, we generalise the concept of close-to-star functions by considering a subclass \( \mathcal{H}_R^*(\alpha, \beta) \) of normalized analytic functions in the unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) such that for some starlike function \( g(z) = z + b_2 z^2 + ... \),

\[
\frac{zf'(z)}{g(z)} < \frac{1 + (2\alpha - 1)\beta z}{1 + \beta z}, \quad 0 \leq \alpha < 1, \quad 0 < \beta \leq 1.
\]

We obtain results on sharp coefficient estimates, growth and distortion theorems.

Keywords
Analytic functions, coefficient estimates, starlike function.

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1. Introduction and Definitions

Let \( A \) be the normalized class of regular functions

\[
f(z) = z + az^2 + az^3 + ...
\]

which are univalent in the unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) and normalized by the conditions \( f(0) = f'(0) - 1 = 0 \).

If \( f \in A \) and satisfies \( \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad \forall \ z \in U \), then \( f(z) \) is said to be a starlike function of order \( \alpha \). We use \( S^*(\alpha) \) to denote the class of starlike function of order \( \alpha \).

On putting \( \alpha = 0 \), the class of functions is denoted by \( S^* \).

A function \( f(z) \) belonging to \( A \) is said to be convex if and only if \( \Re \left\{ \frac{zf''(z)}{f'(z)} + 1 \right\} > 0 \). We denote \( \mathcal{C} \) be the subclass of \( A \) consisting of all convex functions.

T.H. Macgregor [4], investigated the properties of the class \( R \) of functions \( f(z) \) which belong to \( A \) and satisfy \( \Re f'(z) > 0 \), for \( z \in U \). In [3], O.P. Juneja and M.L. Mogra considered the class \( R(\alpha, \beta) \) of functions \( f(z) \in A \), satisfying the condition

\[
\left| \frac{f'(z) - 1}{f'(z) + (1 - 2\alpha)} \right| < \beta, \quad (z \in U)
\]

for some \( \alpha, \beta (0 \leq \alpha < 1, \quad 0 < \beta \leq 1) \).

In 1983, Silvia [9] defined and studied the class \( \mathcal{C}_B[A, B] \) of functions \( f(z) \in A \) satisfying

\[
\Re \left\{ \frac{zf'(z)}{g'(z)} \right\} > \beta, \quad \text{where } g(z) \in S^*[A, B], \quad (z \in U).
\]

In [7], the author introduced new subclasses of the class \( \mathcal{C}_B[A, B] \) of functions \( f(z) \) defined by (1) to be an alpha-close-to-convex function if \( \frac{f(z)}{z} f'(z) \neq 0 \) in \( U \) and if for some nonnegative real number \( \alpha \) and for some starlike function \( \phi(z) = z + ... \),

\[
\Re \left\{ (1 - \alpha) \frac{zf'(z)}{\phi(z)} + \alpha \frac{(zf'(z))^2}{\phi'(z)} \right\} > 0, \quad (z \in U).
\]

In [8] the authors studied a subclass \( \mathcal{C}_B[A, B] \) of close to convex functions.
functions defined by the condition

$$\text{Re} \left( \frac{zf'(z)}{g(z)} \right) > 0, \text{ where } g \in \mathcal{C}.$$  

This work was further extended by Peng Zhigang [10], defining a subclass $\mathcal{C}^p(\alpha, \beta)$ by requiring that $f \in \mathcal{C}^p(\alpha, \beta)$ if and only if

$$\left| \frac{zf'(z)}{g(z)} - 1 \right| < \beta.$$  

In [11], the authors studied a subclass $\mathcal{C}_{\lambda}^p[A, B]$ of close to convex functions defined by the condition

$$\left| (zf'(z))' \right| \begin{vmatrix} g'(z) \end{vmatrix} - 1 < A - B \left( (zf'(z))' \right) \begin{vmatrix} g'(z) \end{vmatrix}, \text{ where } g \in \mathcal{C}$$

and $-1 \leq B < A \leq 1$.

Let $s$ and $t$ be analytic functions in $U$. Then $s$ is said to be subordinate to $t$, written $s(z) \prec t(z)$ if there is a Schwarz function $\omega(z)$ on $U$ such that $\omega(0) = 0$, $|\omega(z)| \leq |z|$ for every $|z| < 1$ and is such that

$$s(z) = t(\omega(z)).$$  

However it is well-known that, if $t \in \mathcal{S}$, then (2) is equivalent to $s(0) = t(0)$ and $s(U) = t(U)$.

In this paper, Motivated by the above concepts, we introduce and study various properties of subclasses, $\mathcal{K}_{\lambda}^p(\alpha, \beta)$ of close-to-star functions and $\lambda$-close-to-star functions $\mathcal{K}_{\lambda}^p(\alpha, \beta)$.

**Definition 1.1.** A normalized regular function $f(z)$ of the form (1) is said to be in the class $\mathcal{K}_{\lambda}^p(\alpha, \beta)$ if and only if there is a starlike function

$$g(z) = z + b_2 z^2 + \ldots \in \mathcal{S}, \ (z \in \mathbb{U})$$  

such that

$$\left( \frac{zf'(z)}{g(z)} \right) - 1 \prec \beta, \ 0 \leq \alpha < 1, \ 0 < \beta \leq 1 \ (z \in \mathbb{U}).$$  

Interms of subordination (2) can be put in the form $f \in \mathcal{K}_{\lambda}^p(\alpha, \beta)$ if and only if

$$zf'(z) \prec \frac{1 + (2\alpha - 1)\beta z}{1 + \beta z},$$  

Note that $f \in \mathcal{K}_{\lambda}^p(\alpha, \beta)$ if and only if $zf'(z) \in \mathcal{S}(2\alpha - 1)\beta, \beta)$.

**Definition 1.2.** A normalized regular function $f(z)$ of the form (1) is said to be in the class $\mathcal{K}_{\lambda}^p(\alpha, \beta)$ if and only if there is a starlike function $g(z)$ defined by (3) such that

$$(1 - \lambda) \frac{f'(z)}{g'(z)} + \lambda \frac{(zf'(z))'}{g'(z)} \prec \frac{1 + (2\alpha - 1)\beta z}{1 + \beta z}, \ (z \in \mathbb{U}),$$  

where, $0 \leq \lambda \leq 1, \ 0 \leq \alpha < 1 \text{ and } 0 < \beta \leq 1.$

**2. Preliminaries**

**Lemma 2.1.** [1, 6] Let $N$ and $D$ be analytic in $U$, $D$ maps onto a many-sheeted starlike region, $N(0) = 0 = D(0).$ Then

$$\frac{N'(z)}{D'(z)} \prec \frac{1 + Az}{1 + Bz} \Rightarrow \frac{N(z)}{D(z)} \prec \frac{1 + Az}{1 + Bz}.$$  

**3. COEFFICIENT BOUNDS**

**Theorem 3.1.** If $h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n \in p((2\alpha - 1)\beta, \beta),$ $0 \leq \alpha < 1, \ 0 \leq \beta \leq 1,$ $z \in \mathbb{U},$ then $|h_n| \leq 2\beta(1 - \alpha), \text{ for all } n \geq 1.$ This inequality is sharp.

**Proof.** As $h \in p((2\alpha - 1)\beta, \beta),$ we have

$$1 + \sum_{n=1}^{\infty} h_n z^n \prec \frac{1 + (2\alpha - 1)\beta z}{1 + \beta z}, \ (0 \leq \alpha < 1, 0 < \beta \leq 1).$$

This implies there is a Schwarz function $\omega(z)$ such that

$$1 + \sum_{n=1}^{\infty} h_n z^n \equiv \frac{1 + (2\alpha - 1)\beta \omega(z)}{1 + \beta \omega(z)},$$

where $\omega(0) = 0, |\omega(z)| \leq |z| \text{ forall } |z| < 1.$  

Then (6) gives,

$$\left[(2\alpha - 1)\beta \omega(z) - \sum_{k=1}^{\infty} \beta b_k z^k \right] \left( \sum_{k=1}^{\infty} c_k z^k \right) = \sum_{k=1}^{\infty} h_k z^k,$$

where $\omega(z) = \sum_{k=1}^{\infty} c_k z^k.$  

Applying Cauchy’s product on the left hand side of (7) and comparing coefficients, the $n$th coefficient $h_n$ on the right hand side of (7) depends only on $h_1, h_2, \ldots, h_{n-1}$ on the left of (7). Therefore from (7), for all $n \geq 1$, it follows that

$$\left| (2\alpha - 1)\beta - \sum_{k=1}^{n-1} \beta b_k z^k \right| |\omega(z)| = \sum_{k=1}^{\infty} h_k z^k + \sum_{k=n+1}^{\infty} b_k z^k,$$

where $b_k \ (k = n + 1, n + 2, \ldots)$ are some complex numbers. Since $|\omega(z)| < 1,$ we get,

$$\sum_{k=1}^{\infty} h_k z^k + \sum_{k=n+1}^{\infty} b_k z^k \leq \left| 2(\alpha - 1)\beta - \sum_{k=1}^{n-1} \beta b_k z^k \right| |\omega(z)| \leq \left| 2(\alpha - 1)\beta - \sum_{k=1}^{n-1} \beta b_k z^k \right|.$$  

| Image: | Description: | 291/295 |
Squaring and integrating around the circle \( z = r e^{i\theta}, \ 0 \leq \theta \leq 2\pi \),

\[
\frac{1}{2\pi} \int_0^{2\pi} \left| 2(\alpha - 1)\beta - \sum_{k=1}^{n-1} \beta h_k r^k e^{i\theta} \right|^2 \, d\theta \geq \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=1}^{n} h_k r^k e^{i\theta} + \sum_{k=n+1}^{\infty} b_r r^k e^{i\theta} \right|^2 \, d\theta.
\]

Taking limit \( r \to 1^- \) we get,

\[4\beta^2(\alpha - 1)^2 + \sum_{k=1}^{n-1} \beta^2 |h_k|^2 \geq \sum_{k=1}^{n-1} \beta |h_k|^2 + |h_n|^2,\]

which gives

\[|h_n| \leq 2\beta(1 - \alpha), \ \forall \ n \geq 1.\]

Equality is attained for the functions \( h(z) = \frac{1 + (2\alpha - 1)\beta z^n}{1 + \beta z^n}, \ n = 2, 3, \ldots. \)

**Theorem 3.2.** If an analytic function \( f \) in \( \mathbb{U} \) defined by (1) satisfies the inequality

\[
\sum_{n=2}^{\infty} (1 + \beta)(2\alpha - 1)|b_n| \leq 2(1 - \alpha)\beta,
\]

(3.4)

where for \( n=2,3,... \), the coefficients \( b_n \) are given by (3) then \( f \in \mathcal{K}_s^* (\alpha, \beta) \).

**Proof.** We set for \( f \) given by (1) and \( g \) defined by (3), then

\[
M = |zf'(z) - g(z)| - |\beta(1 - 2\alpha)g(z) - \beta(zf'(z))|
= \sum_{n=2}^{\infty} n a_n z^n - \sum_{n=2}^{\infty} b_n z^n
- 2\beta(1 - \alpha)z + \beta(1 - 2\alpha) \sum_{n=2}^{\infty} b_n z^n + \sum_{n=2}^{\infty} \beta n a_n z^n
\]

\[\leq \sum_{n=2}^{\infty} n |a_n| |z|^n + \sum_{n=2}^{\infty} |b_n| |z|^n
- 2(1 - \alpha)\beta |z| - \beta(1 - 2\alpha) \sum_{n=2}^{\infty} b_n |z|^n + \sum_{n=2}^{\infty} \beta n |a_n| |z|^n
= -2(1 - \alpha)\beta |z| + \sum_{n=2}^{\infty} (n(\beta + 1)) |a_n| |z|^n
+ \sum_{n=2}^{\infty} (1 + \beta(2\alpha - 1)) |b_n| |z|^n.
\]

Hence for \( z \in \mathbb{U} \) using (9) we have

\[+M \leq -2(1 - \alpha)\beta + \sum_{n=2}^{\infty} (n(1 + \beta)) |a_n|\]

\[\sum_{n=2}^{\infty} (1 + \beta(2\alpha - 1)) |b_n| \leq 0.
\]

From the above calculation, we have

\[|zf'(z) - g(z)| < |\beta(1 - 2\alpha)g(z) - \beta(zf'(z))|,
\]

which is equivalent to the inequality (4). Hence \( f \in \mathcal{K}_s^*(\alpha, \beta) \).

**Theorem 3.3.** Let \( f(z) \) of the form (1) be in the class \( \mathcal{K}_s^*(\alpha, \beta) \).

Then for \( z \in \mathbb{U} \)

\[|a_n| \leq [1 - 2\beta(1 - \alpha)] + (n + 1 - n)\beta.
\]

**Proof.** If \( f \in \mathcal{K}_s^*(\alpha, \beta) \), then there is a function \( g \in \mathcal{S}^* \) with \( g(z) \) is given by (3) such that

\[
\frac{zf'(z)}{g(z)} \in \mathcal{P}((2\alpha - 1)\beta, \beta).
\]

(3.5)

Now

\[
\frac{zf'(z)}{g(z)} = p(z) < \frac{1 + (2\alpha - 1)\beta}{1 + \beta z}.
\]

Equating the coefficients of \( z^n \) on both sides, we get

\[na_n = p_{n-1} + b_2 p_{n-2} + \cdots + b_{n-1} p_1 + b_n. \]

(3.6)

Since \( g(z) \in \mathcal{S}^* \), it follows that \( |b_n| \leq n \), for all \( n \geq 2 \) and from Theorem 1,

\[|p_n| \leq 2\beta(1 - \alpha) \text{ for all } n \geq 1. \]

Therefore, (11) yields

\[|na_n| \leq |b_n| + 2\beta(1 - \alpha) \left[ 1 + b_2 + b_3 + \cdots + b_{n-1} \right] \]
\[\leq n + 2\beta(1 - \alpha) \left[ 1 + 2 + 3 + \cdots + (n - 1) \right]. \]

(3.7)

On simplifying, we get the required assertion of this theorem. Applying similar arguments, we get the following theorem.

**Theorem 3.4.** If the function \( f(z) \in \mathcal{C}_s^*(\alpha, \beta) \) is defined by (1), then

\[
|a_n| \leq \frac{1}{1 + \lambda(n - 1)} \left[ (n1 - 2\beta(1 - \lambda)) + \frac{(1 - \lambda)\beta(n + 1)(2n + 1)}{3} \right] \]

(3.8)

where, \( 0 \leq \lambda \leq 1, \ 0 \leq \alpha < 1 \) and \( 0 < \beta \leq 1 \).

**4. Growth and distortion theorems**

In this section, we prove the distortion and growth results for the function classes \( \mathcal{K}_s^*(\alpha, \beta) \) and \( \mathcal{C}_s^*(\alpha, \beta) \).
Theorem 4.1. If $f \in \mathcal{K}_r^\alpha(\alpha, \beta)$, $0 \leq \alpha < 1$ and $0 < \beta \leq 1$, then if $\beta = 1$, then
\[
|f(z)| \leq \frac{(1-\alpha)r(2-r)}{(1-r)^2} - \frac{(1-2\alpha)r}{(1-r)}.
\]
and
\[
(1-\alpha)r(2+r) - (1-2\alpha)r
\]
\[
\leq |f(z)| \leq \frac{(1-\alpha)r(2-r)}{(1-r)^2} - \frac{(1-2\alpha)r}{(1-r)}.
\]
If $\beta \neq 1$, then
\[
\frac{1-(1-2\alpha)\beta r}{(1+\beta r)(1+r)^2} \leq |f'(z)| \leq \frac{1+(1-2\alpha)\beta r}{(1-\beta)(1-r)^2}
\]
and
\[
\frac{-2(1-\alpha)\beta}{(1-\beta)^2} \log \left( \frac{1+r}{1+\beta r} \right) + \frac{1+(1-2\alpha)\beta r}{(1-\beta)}\frac{r}{1+r}
\]
\[
\leq |f(z)| \leq \frac{2(1-\alpha)\beta}{(1-\beta)^2} \log \left( \frac{1-r}{1+\beta r} \right) + \frac{1+(1-2\alpha)\beta r}{(1-\beta)}\frac{r}{1+r}.
\]
Proof. As $f \in \mathcal{K}_r^\alpha(\alpha, \beta)$, there exists a function $g(z) \in \mathcal{K}_r^\alpha$, such that
\[
z^\beta f'(z) = p(z)g(z),
\]
where $p(z) = \frac{1+2\alpha-1\beta}{1+\beta z}$.
It is clear that
\[
\frac{1-(1-2\alpha)\beta r}{1+\beta r} \leq |p(z)| \leq \frac{1+(1-2\alpha)\beta r}{1-\beta r}.
\]
Again $g(z) \in \mathcal{K}_r^\alpha$, then
\[
\frac{r}{(1+r)^2} \leq |g(z)| \leq \frac{r}{(1-r)^2}.
\]
Therefore,
\[
\frac{1-(1-2\alpha)\beta r}{(1+\beta r)(1+r)^2} \leq |f'(z)| \leq \frac{1+(1-2\alpha)\beta r}{(1-\beta)(1-r)^2}.
\]
For $z = re^{\theta}$, with $0 < r < 1$, we have
\[
f(z) = \int_0^r f'(te^{\theta})e^{\theta} dt.
\]
From (13) and (14) we get
\[
f(z) = \int_0^r f'(te^{\theta})e^{\theta} dt \leq \int_0^r \frac{1+(1-2\alpha)\beta t}{(1-\beta)(1-t)^3} dt.
\]
If $\beta = 1$, then
\[
|f(z)| \leq \frac{(1-\alpha)r(2-r)}{(1-r)^2} - \frac{(1-2\alpha)r}{(1-r)}.
\]
If $\beta \neq 1$, then
\[
|f(z)| \leq \frac{2(1-\alpha)\beta}{(1-\beta)^2} \log \left( \frac{1-r}{1+\beta r} \right) + \frac{1+(1-2\alpha)\beta r}{(1-\beta)}\frac{r}{1+r}.
\]
To prove the lower bound of $|f(z)|$ we proceed as follows. Let $\sigma$ be the radius of the open disc contained in the map of $\mathbb{D}$ by $f(z)$. Let $z_0$ be the point of $|z| = r$ for which $|f(z)|$ attains its minimum value. This minimum increases with $r$ and is less than $\sigma$. Hence, the linear segment $\Gamma$ connecting the origin with the point $f(z_0)$ will be covered entirely by the values of $f(z)$ in $\mathbb{D}$. Let $\gamma$ be the arc in $\mathbb{D}$ which is mapped by $\omega = f(z)$ onto this linear segment. Then
\[
|f(z_0)| = \int_{\Gamma} |d\omega| = \int_{\Gamma} |f'(z)| dz
\]
\[
\geq \int_{\Gamma} |f'(z)| |dz| \geq \int_0^r \frac{1-(1-2\alpha)\beta t}{(1+\beta t)(1+t)^2} dt.
\]
If $\beta = 1$, then
\[
|f(z)| \geq \frac{(1-\alpha)r(2+r)}{(1+r)^2} - \frac{(1-2\alpha)r}{1+r}.
\]
If $\beta \neq 1$, then
\[
|f(z)| \geq \frac{2(1-\alpha)\beta}{(1-\beta)^2} \log \left( \frac{1+r}{1+\beta r} \right) + \frac{1+(1-2\alpha)\beta r}{(1-\beta)}\frac{r}{1+r}.
\]
\]
\]

Theorem 4.2. Let $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$ and $f \in \mathcal{K}_r^\alpha(\alpha, \beta)$.
(1) If $\lambda = 0$, then for $|z| = r < 1$, we have
\[
\int_0^r \frac{1-(1-2\alpha)\beta t}{(1+\beta t)(1+t)^3} dt \leq |f(z)| \leq \int_0^r \frac{1+(1-2\alpha)\beta t}{(1-\beta)(1-t)^3} dt.
\]
which gives
\[
\mathcal{L}_1 \leq |f(z)| \leq \mathcal{L}_2,
\]
where
\[
\mathcal{L}_1 = \frac{2\beta(1-\alpha)(1+\beta)}{(1-\beta)^3} \log \left( \frac{1+r}{1+\beta r} \right) + \frac{(1-2\alpha)\beta^2 + 2\beta(3\alpha-2) - 1}{(1-\beta)^2} \frac{1}{1+r} + \frac{[1+(1-2\alpha)\beta]}{(1-\beta)^2}(r + 2) \frac{1}{1+r}.
\]
and
\[
\mathcal{L}_2 = \frac{-2\beta(1-\alpha)(1+\beta)}{(1-\beta)^3} \log \left( \frac{1-r}{1+\beta r} \right) + \frac{(1-2\alpha)\beta^2 + 2\beta(3\alpha-2) - 1}{(1-\beta)^2} \frac{1}{1-r}.
\]
If we get (17).

Setting $F$ Suppose that

$$
\frac{(1-\alpha)\beta r}{1+\beta r} \leq \int_0^r \frac{1-\alpha}{1+\beta t} \frac{1-t}{1+t} \, dt \leq \frac{1}{1+r} (1-r)^3.
$$

(4.6)

Proof. Suppose that $f \in \mathcal{K}^*_\alpha(\alpha, \beta)$, there exists a function $g(z) \in \mathcal{P}$, such that $(1-\lambda)f(z) + \lambda(zf'(z))' = g'(z)h(z)$, where

$$
h(z) = \frac{1+(2\alpha-1)\beta z}{1+\beta z}.
$$

It is clear to see that then

$$
\frac{1-r}{1+r} \leq |g'(z)| \leq \frac{1+r}{1-r}.
$$

Setting $F'(z) = (1-\alpha)f'(z) + \alpha(zf'(z))'$, we have

$$
\frac{1-(2\alpha-1)\beta r}{1+\beta r} \leq |F'(z)| \leq \frac{1-(2\alpha-1)\beta r}{1-\beta r} \frac{1+r}{1+\beta r} (1-r)^3.
$$

(4.7)

Therefore,

$$
\int_0^r \frac{1-(2\alpha-1)\beta t}{1+\beta t} \frac{1-t}{1+t} \, dt \leq |F(z)| \leq \int_0^r \frac{1+(2\alpha-1)\beta t}{1+\beta t} \frac{1+t}{1-t} \, dt.
$$

(4.8)

Consider the following two cases:

1. When $\lambda = 0$. From (20) and using simple computation, we get (16).
2. When $0 < \lambda \leq 1$, using a simple computation from (20), we get (17).

\[\Box\]

Theorem 4.3. Let $f(z) \in \mathcal{K}^*_\alpha(\alpha, \beta), 0 \leq \alpha < 1, 0 < \beta \leq 1$. Then

$$
\text{Re} \left\{ \frac{zf'(z)}{g(z)} \right\} \geq \mathcal{M}
$$

where

$$
\mathcal{M} = \begin{cases}
\frac{1-r}{1+r} - \frac{2(1-\alpha)\beta r}{1+\beta r} & \text{if } r_1 \leq r_2, \\
\frac{1-r}{1+r} + \frac{\gamma}{\alpha} & \text{if } r_1 \geq r_2.
\end{cases}
$$

\[\text{and}\]

$$
7\gamma = \sqrt{\frac{(1+\beta)(1+(2\alpha-1)\beta)(1-\beta r^2)}{(1-\alpha)(1+r^2)}}
$$

(4.11)

where $r_1$ and $r_2$ are the unique roots of the equations

$$
\begin{align*}
(2\alpha-1)\beta^2 r^3 - \beta - (2\alpha-1)\beta - 1 &= 0, \\
(2\alpha-1)\beta^2 r^3 - \beta - (2\alpha-1)\beta - 1 &= 0
\end{align*}
$$

(4.12)

and

$$
\begin{align*}
[1-(2\alpha-1)\beta \beta^4 r^3] - [1-(2\alpha-1)\beta \beta^4 r^3] &- (1-\beta)(1-(2\alpha-1)\beta] \\
+ 2\beta(\alpha-1)\beta^2 r^3 - 2(1-(2\alpha-1)\beta) &- 1 = 0
\end{align*}
$$

(4.13)

respectively.

Proof. Since $f(z) \in \mathcal{K}^*_\alpha(\alpha, \beta)$, we can write

$$
\frac{zf'(z)}{g(z)} = \frac{p(z)}{1+\beta z} \leq \frac{1+(2\alpha-1)\beta z}{1+\beta z} \text{ where } g(z) \in \mathcal{P}.
$$

Hence $F(z) = zf'(z)$ and therefore

$$
\frac{zf'(z)}{g(z)} = \frac{g'(z)}{g(z)} + \frac{zp'(z)}{p(z)}.
$$

(4.14)

For starlike function $g(z)$ of order $\alpha (0 \leq \alpha < 1)$, we have

$$
\text{Re} \left\{ \frac{g'(z)}{g(z)} \right\} \geq \frac{1-r}{1+r}.
$$

(4.15)

As $p(z) < \frac{1+(2\alpha-1)\beta z}{1+\beta z}$, we have [2],

$$
\text{Re} \left\{ \frac{zp'(z)}{p(z)} \right\} \geq \begin{cases}
\frac{-2(1-\alpha)\beta r}{(1+\beta r)[1+(2\alpha-1)\beta r]} & \text{if } r_1 \leq r_2, \\
\frac{\gamma}{\alpha} & \text{if } r_1 \geq r_2.
\end{cases}
$$

(4.16)

where

$$
7\gamma = \sqrt{\frac{(1+\beta)(1+(2\alpha-1)\beta)(1-\beta r^2)}{(1-\alpha)(1+r^2)}}
$$

(4.17)

and $r_1$, $r_2$ are the unique roots of the equation

$$
(2\alpha-1)\beta^2 r^3 - 2(2\alpha-1)\beta^2 r^3
$$

(4.18)

in the interval $(0,1)$. From (24) (25) and (26), we get the required result. \[\Box\]
References


