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Fuzzy semi pre-irresolute mappings in fuzzy topological spaces

M. Pritha*

Abstract

Y. C. Kim introduced r-fuzzy semi preopen (r-fuzzy semi preclosed) sets in $\hat{S}$ostak's fuzzy topological space. We investigate some properties of them. Moreover, we introduce and study fuzzy semi pre-irresolute mapping and fuzzy semi pre-connectedness in $\hat{S}$ostak's fuzzy topological space.

Keywords

r-fuzzy semi preopen (preclosed) sets, fuzzy semi pre-irresolute (semi pre-continuous) mapping, r-fuzzy semi pre-connected.

AMS Subject Classification

54A05, 54A10, 54A40.

Department of Mathematics, D.B. Jain College, Thoraipakkam, Chennai-600097, Tamil Nadu, India.

*Corresponding author: [1] prithasrinivasan@gmail.com;

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Contents

1. Introduction and preliminaries

$\hat{S}$ostak [7] introduced the fuzzy topology as an extension of Chang's fuzzy topology [1]. J. H. Park and B. Y. Lee [5] introduced fuzzy semi preopen and fuzzy semi preclosed sets in fuzzy topology.

In this paper, Y. C. Kim [4] introduced r-fuzzy semi preopen (r-fuzzy semi preclosed) sets in $\hat{S}$ostak's fuzzy topological space. Moreover, we establish some properties of fuzzy semi pre-continuous, fuzzy semi preopen (semi preclosed) maps and fuzzy semi pre-irresolute maps. Also, we investigate the relation between fuzzy semi pre-continuous maps and fuzzy semi pre irresolute maps. As a natural follow-up of the study of r-fuzzy semi preopen sets, we introduce and study r-fuzzy semi pre-connectedness of a fuzzy set.

Throughout this paper, let X be a nonempty set, $I=[0,\ 1]$ and $I_0=(0,\ 1]$. For $\alpha\in I$, $\overline{\alpha}(x)=\alpha$ for all $x\in X$. The family of all fuzzy sets on X denoted by I^X. For $\lambda,\mu\in I^X$, λ is called quasi-coincident with μ, denoted by $\lambda q\mu$, if there exists $x\in X$ such that $\lambda(x)+\mu(x)>1$. Otherwise we denote $\lambda\bar{q}\mu$.

Definition 1.1. [7] A function $\tau: I^X\to I$ is called a fuzzy topology on X if it satisfies the following conditions:
$(O1)$ $\tau(\overline{0})=\tau(\overline{1})=1$,
$(O2)$ $\tau(\bigvee_{i\in\Gamma}\mu_i)\geq\bigwedge_{i\in\Gamma}\tau(\mu_i)$, for any $\{\mu\}_{i\in\Gamma}\subset I^X$.
$(O3)$ $\tau(\mu_1\wedge\mu_2)\geq\tau(\mu_1)\wedge\tau(\mu_2)$, for any $\mu_1,\ \mu_2\in I^X$.
The pair $(X,\ \tau)$ is called a fuzzy topological space (for short, fts).

Remark 1.2. [4] Let (X,τ) be a fuzzy topological space. Then, for each $r\in I$, $\tau_r=\{\mu\in I^X:\tau(\mu)\geq r\}$ is a Change's fuzzy topology on X.

Theorem 1.3. [2], [3] Let $(X,\ \tau)$ be a fts. Then for each $r\in I_0,\ \lambda\in I^X$ we define an operator $C_\tau: I^X\times I_0\to I^X$ as follows

$C_\tau(\lambda,\ r)=\bigwedge\{\mu\in I^X:\lambda\leq\mu,\ \tau(\overline{1}-\mu)\geq r\}$.

For $\lambda,\ \mu\in I^X$ and $r,s\in I_0$, the operator C_τ satisfies the following conditions:

(1) $C_\tau(\overline{0},r)=\overline{0}$.

(2) $\lambda\leq C_\tau(\lambda,\ r)$

(3) $C_\tau(\lambda,\ r)\vee C_\tau(\mu,\ r)=C_\tau(\lambda\vee\mu,\ r)$.

(4) $C_\tau(\lambda,\ r)\leq C_\tau(\lambda,\ s)$ if $r\leq s$.

(5) $C_\tau(C_\tau(\lambda,\ r),\ r) = C_\tau(\lambda,\ r)$.

Theorem 1.4. [3] Let $(X,\ \tau)$ be a fts. Then for each $r \in I_0,\ \lambda \in I^X$ we define an operator $I_\tau : I^X \times I_0 \to I^X$ as follows
$I_\tau(\lambda,\ r) = \bigvee\{\mu \in I^X : \lambda \geq \mu,\ \tau(\mu) \geq r\}$.
For $\lambda,\ \mu \in I^X$ and $r, s \in I_0$, the operator I_τ satisfies the following conditions:

(1) $I_\tau(\overline{1} - \lambda, r) = \overline{1} - C_\tau(\lambda,\ r)$.

(2) $I_\tau(\overline{1}, r) = \overline{1}$.

(3) $\lambda \geq I_\tau(\lambda,\ r)$

(4) $I_\tau(\lambda,\ r) \wedge I_\tau(\mu,\ r) = I_\tau(\lambda \wedge \mu,\ r)$.

(5) $I_\tau(\lambda,\ r) \geq I_\tau(\lambda,\ s)$ if $r \leq s$.

(6) $I_\tau(I_\tau(\lambda,\ r),\ r) = I_\tau(\lambda,\ r)$.

(7) If $I_\tau(C_\tau(\lambda,\ r),\ r) = \lambda$, then $C_\tau(I_\tau(\overline{1} - \lambda, r), r) = \overline{1} - \lambda$.

Definition 1.5. [6] Let $(X,\ \tau)$ be a fts. For $\lambda \in I^X$ and $r \in I_0$.

(1) λ is called r-fuzzy preopen (r-fpo, for short) iff $\lambda \leq I_\tau(C_\tau(\lambda,\ r), r)$.

(2) λ is called r-fuzzy preclosed (r-fpc, for short) iff $\overline{1} - \lambda$ is r-fuzzy preopen set of X.

(3) The r-fuzzy pre-interiror of λ, denoted by $PI_\tau(\lambda, r)$, is defined by $PI_\tau(\lambda, r) = \bigvee\{\nu \in I^X : \nu \leq \lambda, \nu \text{ is } r\text{-fpo}\}$

(4) The r-fuzzy pre-closure of λ, denoted by $PC_\tau(\lambda, r)$, is defined by $PC_\tau(\lambda, r) = \bigwedge\{\nu \in I^X : \nu \geq \lambda, \nu \text{ is } r\text{-fpc}\}$.

Definition 1.6. [4] Let $(X,\ \tau)$ be a fts. For $\lambda \in I^X$ and $r \in I_0$,

(1) λ is called r-fuzzy semi preopen (for short, r-fspo) if $\lambda \leq C_\tau(I_\tau(C_\tau(\lambda,\ r),\ r),\ r)$;

(2) λ is called r-fuzzy semi preclosed (for short r-fspc) if $I_\tau(C_\tau(I_\tau(\lambda,\ r),\ r),\ r) \leq \lambda$.

Theorem 1.7. [4] Let $(X,\ \tau)$ be a fts. For $\lambda \in I^X$ and $r,\ s,\ r_i \in I_0$.

(1) λ is r-fspo iff $\overline{1} - \lambda$ is r-fspc.

(2) Any union of r-fspo sets is r-fspo.

(3) Any intersection of r-fspc sets is r-fspc.

(4) If λ is r-fpo, then λ is r-fspo.

Definition 1.8. [3], [7] Let $(X,\ \tau)$ and $(Y,\ \eta)$ be a fts's. Let $f : X \to Y$ be a mapping.

(1) f is called fuzzy continuous iff $\eta(\mu) \leq \tau(f^{-1}(\mu))$ for each $\mu \in I^Y$.

(2) f is called fuzzy open iff $\tau(\lambda) \leq \eta(f(\lambda))$ for each $\lambda \in I^X$.

(3) f is called fuzzy closed iff $\tau(\overline{1} - \lambda) \leq \eta(\overline{1} - f(\lambda))$ for each $\lambda \in I^X$.

2. r-fuzzy semi preopen and r-fuzzy semi preclosed sets

Theorem 2.1. Let $(X,\ \tau)$ be a fts. For each $r \in I_0,\ \lambda \in I^X$, we define an operator $SPC_\tau : I^X \times I_0 \to I^X$ as follows:
$SPC_\tau(\lambda,\ r) = \bigwedge\{\mu \in I^X : \lambda \leq \mu,\ \mu \text{ is } r\text{-fspc}\}$.
For $\lambda,\ \mu \in I^X$ and $r \in I_0$, it holds the following properties.

(1) $SPC_\tau(\overline{0}, r) = \overline{0}$.

(2) $\lambda \leq SPC_\tau(\lambda,\ r)$.

(3) $SPC_\tau(\lambda,\ r) \vee SPC_\tau(\mu,\ r) \leq SPC_\tau(\lambda \vee \mu,\ r)$.

(4) $SPC_\tau(SPC_\tau(\lambda,\ r),\ r) = SPC_\tau(\lambda,\ r)$.

(5) $SPC_\tau(\lambda,\ r) = \lambda$ iff λ is r-fspc.

(6) $C_\tau(SPC_\tau(\lambda,\ r),\ r) = SPC_\tau(C_\tau(\lambda,\ r),\ r) = C_\tau(\lambda, r)$.

(7) λ is r-fspc iff $SPC_\tau(\lambda,\ r) = \lambda$.

Proof. (1), (2), (5), (7) are easily proved from the definition of SPC_τ.
(3) Since $\lambda,\ \mu \leq \lambda \vee \mu$, we have

$$SPC_\tau(\lambda,\ r) \vee SPC_\tau(\mu,\ r) \leq SPC_\tau(\lambda \vee \mu,\ r).$$

(4) From (2), we only show $SPC_\tau(\lambda,\ r) \geq SPC_\tau(PC_\tau(\lambda,\ r), r)$. Suppose

$$SPC_\tau(\lambda,\ r) \not\geq SPC_\tau(SPC_\tau(\lambda,\ r), r).$$

There exist $x \in X$ and $t \in (0,\ 1)$ such that

$$SPC_\tau(\lambda,\ r)(x) < t < SPC_\tau(SPC_\tau(\lambda,\ r), r)(x). \tag{A}$$

Since $SPC_\tau(\lambda,\ r)(x) < t$, by the Definition of SPC_τ, there exists r-fspc set λ_1 with $\lambda \leq \lambda_1$ such that

$$SPC_\tau(\lambda,\ r)(x) \leq \lambda_1(x) < t.$$

Since $\lambda \leq \lambda_1$, we have $SPC_\tau(\lambda,\ r) \leq \lambda_1$. Again, by the Definition of SPC_τ,

$$SPC_\tau(SPC_\tau(\lambda,\ r), r) \leq \lambda_1.$$

Hence $SPC_\tau(SPC_\tau(\lambda,\ r), r)(x) \leq \lambda_1(x) < t$. It is a contradiction for (A). Thus,

$$SPC_\tau(\lambda,\ r) \geq SPC_\tau(SPC_\tau(\lambda,\ r), r).$$

(6) From (2) and $C_\tau(\lambda,\ r)$ is r-fspc we have, $SPC_\tau(C_\tau(\lambda,\ r), r) = C_\tau(\lambda,\ r)$. We only show that

$$C_\tau(SPC_\tau(\lambda,\ r), r) = C_\tau(\lambda,\ r).$$

Since $\lambda \leq SPC_\tau(\lambda,\ r)$,

$$C_\tau(SPC_\tau(\lambda,\ r),r) \geq C_\tau(\lambda,\ r).$$

Suppose that

$$C_\tau(SPC_\tau(\lambda,\ r),r) \nleq C_\tau(\lambda,\ r).$$

There exist $x \in X$ and $r \in I_0$ such that

$$C_\tau(SPC_\tau(\lambda,\ r),r)(x) > C_\tau(\lambda,\ r)(x).$$

By the definition of C_τ, there exists $\rho \in I^X$ with $\lambda \leq \rho$ and $\tau(\overline{1}-\rho) \geq r$ such that

$$C_\tau(SPC_\tau(\lambda,\ r),r)(x) > \rho(x) \geq C_\tau(\lambda,\ r)(x).$$

On the other hand, since $\rho = C_\tau(\rho,\ r)$, $\lambda \leq \rho$ implies

$SPC_\tau(\lambda,\ r) \leq SPC_\tau(\rho,\ r) = SPC_\tau(C_\tau(\rho,\ r),r) = C_\tau(\rho,\ r) = \rho.$
Thus,

$$C_\tau(SPC_\tau(\lambda,\ r),r) \leq \rho.$$

It is a contradiction. Hence $C_\tau(SPC_\tau(\lambda,\ r),r) \leq C_\tau(\lambda,\ r)$.

Theorem 2.2. Let $(X,\ \tau)$ be a fts. Define an operator $SPI_\tau : I^X \times I_0 \to I^X$ as follows:
$SPI_\tau(\lambda,\ r) = \bigvee\{\mu \in I^X |\ \mu \leq \lambda, \mu \text{ is } r\text{-fspo }\}$.
Then:

(1) $SPI_\tau(\overline{1}-\lambda,r) = \overline{1} - SPC_\tau(\lambda,\ r)$.

(2) $I_\tau(\lambda,r) \leq PI_\tau(\lambda,r) \leq SPI_\tau(\lambda,r) \leq \lambda \leq SPC_\tau(\lambda,\ r) \leq PC_\tau(\lambda,\ r) \leq C_\tau(\lambda,\ r)$.

(3) λ is r-fspo iff $SPI_\tau(\lambda,r) = \lambda$.

Proof. (1) For each $\lambda \in I^X$ and $r \in I_0$, we have
$SPI_\tau(\overline{1}-\lambda,r) = \bigvee\{\mu \in I^X |\ \mu \leq \overline{1}-\lambda,\ \mu \text{ is } r\text{-fspo }\}$
$= \overline{1} - \bigwedge\{\overline{1}-\mu \in I^X | \overline{1}-\mu \geq \lambda, \overline{1}-\mu$ is r-fspc $\}$
$= \overline{1} - SPC_\tau(\lambda,r)$.
(2) and (3) are easily proved.

3. Fuzzy semi pre-irresolute mappings

Definition 3.1. Let $(X,\ \tau)$ and $(Y,\ \eta)$ be a fts's. Let $f : X \to Y$ be a mapping.

(1) f is fuzzy semi pre-irresolute (resp. fuzzy semi pre-continuous [4]) iff $f^{-1}(\mu)$ is r-fspo set of X for each r-fspo set $\mu \in I^Y$ (resp. $\eta(\mu) \geq r$).

(2) f is fuzzy semi pre-irresolute open (resp. fuzzy semi pre-open [4]) iff $f(\mu)$ is r-fspo set of Y for each r-fspo set $\mu \in I^X$ (resp. $\tau(\mu) \geq r$).

(3) f is fuzzy semi pre-irresolute closed (resp. fuzzy semi pre-closed [4]) iff $f(\mu)$ is r-fspc set of Y for each r-fspc set $\mu \in I^X$ (resp. $\tau(\overline{1}-\mu) \geq r$).

(4) f is fuzzy semi pre-irresolute homeomorphism iff f is bijective and both of f and f^{-1} are semi pre-irresolute.

Theorem 3.2. Let $(X,\ \tau_1)$ and $(Y,\ \tau_2)$ be a fts's. Let $f : X \to Y$ be a mapping. The following statements are equivalent.

(1) A map f is fuzzy semi pre-irresolute.

(2) For each r-fspc $\mu \in I^Y$, $f^{-1}(\mu)$ is r-fspc.

(3) $f(SPC_{\tau_1}(\lambda,r)) \leq SPC_{\tau_2}(f(\lambda),r)$, for each $\lambda \in I^X$ and $r \in I_0$.

(4) $SPC_{\tau_1}(f^{-1}(\mu),r) \leq f^{-1}(SPC_{\tau_2}(\mu,r))$, for each $\mu \in I^Y$ and $r \in I_0$.

(5) $f^{-1}(SPI_{\tau_2}(\mu,r)) \leq SPI_{\tau_1}(f^{-1}(\mu),r)$, for each $\mu \in I^Y$ and $r \in I_0$.

Proof. (1) $\Leftrightarrow$ (2) It is easily proved from Theorem 1.7, and $f^{-1}(\overline{1}-\mu) = \overline{1} - f^{-1}(\mu)$.

(2) $\Rightarrow$ (3) Suppose there exist $\lambda \in I^X$ and $r \in I_0$ such that

$$f(SPC_{\tau_1}(\lambda,r)) \nleq SPC_{\tau_2}(f(\lambda),r).$$

There exist $y \in Y$ and $t \in I_0$ such that

$$f(SPC_{\tau_1}(\lambda,r))(y) > t > SPC_{\tau_2}(f(\lambda),r)(y).$$

If $f^{-1}(\{y\}) = \phi$, it is a contradiction because $f(SPC_{\tau_1}(\lambda,r))(y) = 0$.
If $f^{-1}(\{y\}) \neq \phi$, there exists $x \in f^{-1}(\{y\})$ such that

$$f(SPC_{\tau_1}(\lambda,r))(y) \geq SPC_{\tau_1}(\lambda,r)(x) > t > SPC_{\tau_2}(f(\lambda),r)(f(x)). \tag{A}$$

Since $SPC_{\tau_2}(f(\lambda),r)(f(x)) < t$, there exists r-fspc $\mu \in I^Y$ with $f(\lambda) \leq \mu$ such that

$$SPC_{\tau_2}(f(\lambda),r)(f(x)) \leq \mu(f(x)) < t.$$

Moreover, $f(\lambda) \leq \mu$ implies $\lambda \leq f^{-1}(\mu)$. From (2), $f^{-1}(\mu)$ is r-fspc. Thus, $SPC_{\tau_1}(\lambda,r)(x) \leq f^{-1}(\mu)(x) = \mu(f(x)) < t$. It is a contradiction for (A).

(3) $\Rightarrow$ (4) For all $\mu \in I^Y$, $r \in I_0$, put $\lambda = f^{-1}(\mu)$. From (3), we have

$$f(SPC_{\tau_1}(f^{-1}(\mu),r)) \leq SPC_{\tau_2}(f(f^{-1}(\mu)),r) \leq SPC_{\tau_2}(\mu,r).$$

It implies

$$SPC_{\tau_1}(f^{-1}(\mu),r)) \leq f^{-1}(f(SPC_{\tau_1}(f^{-1}(\mu),r))) \leq f^{-1}(SPC_{\tau_2}(\mu,r)).$$

(4) $\Rightarrow$ (5) It is easily proved from Theorem 3(1)

(5) $\Rightarrow$ (1) Let μ be r-fspo set of Y. From Theorem 3(3), $\mu = SPI_{\tau_2}(\mu, r)$. By (5),

$$f^{-1}(\mu) \leq SPI_{\tau_1}(f^{-1}(\mu), r).$$

On the other hand, by Theorem 3(2),

$$f^{-1}(\mu) \geq SPI_{\tau_1}(f^{-1}(\mu), r)$$

Thus, $f^{-1}(\mu) = SPI_{\tau_1}(f^{-1}(\mu), r)$, that is, $f^{-1}(\mu)$ is r-fspo.

The following theorem is similarly proved as Theorem 3.2

Theorem 3.3. Let $(X,\ \tau_1)$ and $(Y,\ \tau_2)$ be a fts's. Let $f : X \to Y$ be a mapping. The following statements are equivalent.

(1) A map f is fuzzy semi pre-continuous.

(2) $f(SPC_{\tau_1}(\lambda, r)) \leq C_{\tau_2}(f(\lambda), r)$, for each $\lambda \in I^X$ and $r \in I_0$.

(3) $SPC_{\tau_1}(f^{-1}(\mu), r) \leq f^{-1}(C_{\tau_2}(\mu, r))$, for each $\mu \in I^Y$ and $r \in I_0$.

(4) $f^{-1}(I_{\tau_2}(\mu, r)) \leq SPI_{\tau_1}(f^{-1}(\mu), r)$, for each $\mu \in I^Y$ and $r \in I_0$.

Theorem 3.4. Let $(X,\ \tau)$ and $(Y,\ \eta)$ be a fts's. Let $f : X \to Y$ be a bijective mapping. The following statements are equivalent.

(1) A map f is fuzzy semi pre-irresolute.

(2) $SPI_\eta(f(\lambda), r) \leq f(SPI_\tau(\lambda, r))$, for each $\lambda \in I^X$ and $r \in I_0$.

Proof. (1) $\Rightarrow$ (2): Let f be fuzzy semi pre-irresolute mapping and $\lambda \in I^X,\ r \in I_0$. Then $f^{-1}(SPI_\eta(f(\lambda), r))$ is r-fspo set in X. By Theorem 3.2 and the fact that f is one-to-one we have

$$\begin{aligned} f^{-1}(SPI_\eta(f(\lambda), r)) &\leq SPI_\tau(f^{-1}(f(\lambda), r)) \\ &= SPI_\tau(\lambda, r). \end{aligned}$$

Again since f is onto we have

$$\begin{aligned} SPI_\eta(f(\lambda), r)) &= ff^{-1}(SPI_\eta(f(\lambda), r)) \\ &\leq f(SPI_\tau(\lambda, r)). \end{aligned}$$

(2) $\Rightarrow$ (1): Let μ be r-fspo set of Y. Then by Theorem 3(3), $\mu = SPI_\eta(\mu, r)$. By (2)

$$\begin{aligned} f(SPI_\tau(f^{-1}(\mu), r)) &\geq SPI_\eta(ff^{-1}(\mu), r) \\ &= SPI_\eta(\mu, r)) \\ &= \mu. \end{aligned}$$

It implies

$$\begin{aligned} SPI_\tau(f^{-1}(\mu), r)) &= f^{-1}f(SPI_\tau(f^{-1}(\mu), r) \\ &\geq f^{-1}(\mu). \end{aligned}$$

Hence, $f^{-1}(\mu) = SPI_\tau(f^{-1}(\mu), r)$, that is, $f^{-1}(\mu)$ is r-fspo set of X.

Theorem 3.5. Let (X, τ) and (Y, η) be fts's. Let $f : X \to Y$ be a mapping. The following statements are equivalent:

(1) f is called fuzzy semi-pre-irresolute open.

(2) $f(SPI_\tau(\lambda, r)) \leq SPI_\eta(f(\lambda), r)$, for each $\lambda \in I^X$ and $r \in I_0$.

(3) $SPI_\tau(f^{-1}(\mu), r) \leq f^{-1}(SPI_\eta(\mu, r))$, for each $\mu \in I^Y$ and $r \in I_0$.

(4) For any $\mu \in I^Y$ and any r-fpsc $\lambda \in I^X$ with $f^{-1}(\mu) \leq \lambda$, there exits a r-fspc $\rho \in I^Y$ with $\mu \leq \rho$ such that $f^{-1}(\rho) \leq \lambda$.

Proof. (1)$\Rightarrow$ (2): For each $\lambda \in I^X$, since $SPI_\tau(\lambda, r) \leq \lambda$ from Theorem (2), we have $f(SPI_\tau(\lambda, r)) \leq f(\lambda)$. From (1), $f(SPI_\tau(\lambda, r))$ is r-fspo. Hence $f(SPI_\tau(\lambda, r)) \leq SPI_\eta(f(\lambda), r)$.

(2)$\Rightarrow$ (3): For all $\mu \in I^Y$, $r \in I_0$, put $\lambda = f^{-1}(\mu)$ from (2). Then $f(SPI_\tau(f^{-1}(\mu), r) \leq SPI_\eta(f(f^{-1}(\mu)), r) \leq SPI_\eta(\mu, r)$.
It implies $SPI_\tau(f^{-1}(\mu), r) \leq f^{-1}(SPI_\eta(\mu, r))$.
(3) $\Rightarrow$ (4) : Let λ be r-fspc set of X such that $f^{-1}(\mu) \leq \lambda$. Since $\overline{1} - \lambda \leq f^{-1}(\overline{1} - \mu)$ and $SPI_\tau(\overline{1} - \lambda,\ r) = \overline{1} - \lambda$,

$$SPI_\tau(\overline{1} - \lambda,\ r) = \overline{1} - \lambda \leq SPI_\tau(f^{-1}(\overline{1} - \mu),\ r).$$

From(3),

$$\overline{1} - \lambda \leq SPI_\tau(f^{-1}(\overline{1} - \mu),\ r) \leq f^{-1}(SPI_\eta(\overline{1} - \mu,\ r)).$$

It implies

$$\begin{aligned} \lambda &\geq \overline{1} - f^{-1}(SPI_\eta(\overline{1} - \mu,\ r)) \\ &= f^{-1}(\overline{1} - SPI_\eta(\overline{1} - \mu,\ r)) \\ &= f^{-1}(SPC_\eta(\mu,\ r)). \end{aligned}$$

Hence there exists a r-fspc $SPC_\eta(\mu,\ r) \in I^Y$ with $\mu \leq SPC_\eta(\mu,\ r)$ such that $f^{-1}(SPC_\eta(\mu,\ r)) \leq r\lambda$.

(4) $\Rightarrow$ (1) : Let ω be r-fspo set X. Put $\mu = \overline{1} - f(\omega)$ and $\lambda = \overline{1} - \omega$ such that λ is r-fspc. We obtain

$$f^{-1}(\mu) = f^{-1}(\overline{1} - f(\omega)) = \overline{1} - f^{-1}(f(\omega)) \leq \overline{1} - \omega = \lambda.$$

From(4), there exists r-fspc set ρ with $\mu \leq \rho$ such that $f^{-1}(\rho) \leq \lambda = \overline{1} - \omega$. It implies $\omega \leq \overline{1} - f^{-1}(\rho) = f^{-1}(\overline{1} - \rho)$. Thus, $f(\omega) \leq f(f^{-1}(\overline{1} - \rho)) \leq \overline{1} - \rho$. On the other hand, since $\mu \leq \rho$,

$$f(\omega) = \overline{1} - \mu \geq \overline{1} - \rho.$$

Hence $f(\omega) = \overline{1} - \rho$, that is, $f(\omega)$ is r-fspo.

The following theorem is similarly proved from Theorem 3.5

Theorem 3.6. Let $(X,\ \tau)$ and $(Y,\ \eta)$ be a fts's. Let $f: X \to Y$ be a mapping. The following statements are equivalent:

(1) f is called fuzzy semi pre-irresolute closed.

(2) $f(SPC_\tau(\lambda, r)) \leq SPC_\eta(f(\lambda), r)$, for each $\lambda \in I^X$ and $r \in I_0$.

Theorem 3.7. Let $(X,\ \tau)$ and $(Y,\ \eta)$ be a fts's. Let $f: X \to Y$ be a bijective mapping.

(1) f is a fuzzy semi pre-irresolute closed map iff, for each $\mu \in I^Y$ and $r \in I_0$, $f^{-1}(SPC_\eta(\mu,\ r)) \leq SPC_\tau(f^{-1}(\mu),\ r)$.

(1) f is a fuzzy semi pre-irresolute closed map iff f is a fuzzy semi pre-irresolute open map.

Proof. (1)($\Rightarrow$) Let f be fuzzy semi pre-irresolute closed. From Theorem 3.6(2), for each $\lambda \in I^X$ and $r \in I_0$.

$$f(SPC_\tau(\lambda, r)) \geq SPC_\eta(f(\lambda), r).$$

For all $\mu \in I^Y$, $r \in I_0$. Put $\lambda = f^{-1}(\mu)$ from (2) since f is onto, $f(f^{-1}(\mu)) = \mu$. Thus,

$$\begin{aligned} f(SPC_\tau(f^{-1}(\mu), r)) &\geq SPC_\eta(f(f^{-1}(\mu)), r). \\ &= SPC_\eta(\mu, r). \end{aligned}$$

It implies

$$\begin{aligned} SPC_\tau(f^{-1}(\mu), r)) &= f^{-1}(f(SPC_\tau(f^{-1}(\mu), r))) \\ &\geq f^{-1}(SPC_\eta(\mu, r)). \end{aligned}$$

($\Leftarrow$) Put $\mu = f(\lambda)$. Since f is injective,

$$\begin{aligned} f^{-1}(SPC_\eta(f(\lambda), r)) &\leq SPC_\tau(f^{-1}(f(\lambda)), r) \\ &= SPC_\tau(\lambda,\ r). \end{aligned}$$

Since f is onto,

$$SPC_\eta(f(\lambda), r) \leq f(SPC_\tau(\lambda,\ r)).$$

(2) It is easily proved from:

$$\begin{aligned} & f^{-1}(SPC_\eta(\mu,\ r) \leq SPC_\tau(f^{-1}(\mu), r) \\ \Leftrightarrow\ & f^{-1}(\overline{1} - SPI_\eta(\overline{1} - \mu, r)) \leq \overline{1} - SPI_\tau(\overline{1} - f^{-1}(\mu), r) \\ \Leftrightarrow\ & f^{-1}(SPI_\eta(\overline{1} - \mu, r)) \geq SPI_\tau(f^{-1}(\overline{1} - \mu), r). \end{aligned}$$

From the above Theorems we obtain the following theorem.

Theorem 3.8. Let $f: (X,\ \tau) \to (Y,\ \eta)$ be a bijective mapping from a fts $(X,\ \tau)$ into a fts $(Y,\ \eta)$. For each $\lambda \in I^X$, $\mu \in I^Y$ and $r \in I_0$. The following statements are equivalent:

(1) f is a fuzzy semi pre-irresolute homeomorphism.

(2) f is a fuzzy semi pre-irresolute and fuzzy semi pre-irresolute open.

(3) f is a fuzzy semi pre-irresolute and fuzzy semi pre-irresolute closed.

(4) $f(SPI_\tau(\lambda, r)) = SPI_\eta(f(\lambda), r)$.

(5) $f(SPC_\tau(\lambda, r)) = SPC_\eta(f(\lambda), r)$.

(6) $SPI_\tau(f^{-1}(\mu), r) = f^{-1}(SPI_\eta(\mu, r))$.

(7) $SPC_\tau(f^{-1}(\mu), r) = f^{-1}(SPC_\eta(\mu, r))$.

Remark 3.9. For the mapping $f: X \to Y$, the following statements are valid:

(1) f is a fuzzy semi pre-irresolute $\Rightarrow$ f is a fuzzy semi pre-continuous.

(2) f is a fuzzy continuous $\Rightarrow$ f is a fuzzy semi pre-continuous.

The converses of Remark 3.9(1,2) need not be true from the following example.

Example 3.10. Let λ_1, λ_2, λ_3 and λ_4 be fuzzy subsets of $X = \{a, b, c\}$ defined as follows:

$$\begin{aligned} &\lambda_1(a) = 0.2,\ \lambda_1(b) = 0.4,\ \lambda_1(c) = 0.5, \\ &\lambda_2(a) = 0.2,\ \lambda_2(b) = 0.2,\ \lambda_2(c) = 0.2, \\ &\lambda_3(a) = 0.5,\ \lambda_3(b) = 0.5,\ \lambda_3(c) = 0.5, \\ &\lambda_4(a) = 0.6,\ \lambda_4(b) = 0.6,\ \lambda_4(c) = 0.6. \end{aligned}$$

Then $\tau,\ \eta: I^X \to I$ defined as

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0}, \overline{1} \\ \frac{1}{2}, & \text{if } \lambda = \lambda_1,\ \lambda_2,\ \lambda_3 \\ 0, & \text{otherwise,} \end{cases} \qquad \eta(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0}, \overline{1} \\ \frac{1}{2}, & \text{if } \lambda = \lambda_1 \\ 0, & \text{otherwise} \end{cases}$$

are fuzzy topologies on X. Then, the identity mapping $id_X: (X, \tau) \to (X, \eta)$ is fuzzy continuous and fuzzy semi pre continuous but not fuzzy semi pre irresolute, because λ_4 is $\frac{1}{2}$-fspo but $f^{-1}(\lambda_4)$ is not $\frac{1}{2}$-fspo from

$\lambda_4 \leq C_\tau(I_\tau(C_\tau(\lambda_5, \frac{1}{2}), \frac{1}{2}), \frac{1}{2})$

$\lambda_5 \leq \overline{1}$

$\lambda_4 \leq C_\tau(I_\tau(C_\tau(\lambda_4, \frac{1}{2}), \frac{1}{2}), \frac{1}{2})$

$= C_\tau(I_\tau(\lambda_2^c, \frac{1}{2}), \frac{1}{2})$

$= I_\tau(\lambda_3, \frac{1}{2})$

$\not\leq \lambda_3$.

4. Fuzzy semi pre-connectedness

Definition 4.1. Let (X, τ) be a fts and $\lambda,\ \mu \in I^X$, $r \in I_0$. The two fuzzy sets λ and μ are said to be r-fuzzy semi pre-separated iff $\lambda \overline{q} SPC_\tau(\mu, r)$ and $\mu \overline{q} SPC_\tau(\lambda, r)$.

Definition 4.2. A fuzzy set which cannot be expressed as the union of two r-fuzzy semi pre-separated sets is said to be r-fuzzy semi pre-connected set.

Theorem 4.3. Let (X,τ) be a fts and $\lambda,\mu \in I^X$, $r \in I_0$.

(i) If λ, μ are r-fuzzy semi-pre-separated and ν, η are non-null fuzzy sets such that $\nu \leq \lambda$, $\eta \leq \mu$, then ν, η are also r-fuzzy semi pre separated.

(ii) If $\lambda\overline{q}\mu$ and either both are r-fspo or both r-fspc, then λ and μ are r-fuzzy semi pre separated.

(iii) If λ, μ are either both r-fspo or both r-fspc, then $\lambda \wedge (\overline{1}-\mu)$ and $\mu \wedge (\overline{1}-\lambda)$ are r-fuzzy semi pre-separated.

Proof. (1) and (2) are obvious.

(3) Let λ and μ be both r-fspo. Since $\lambda \wedge (\overline{1}-\mu) \leq \overline{1}-\mu$, $SPC_\tau(\lambda \wedge (\overline{1}-\mu), r) \leq \overline{1}-\mu$ and hence $SPC_\tau(\lambda \wedge (\overline{1}-\mu), r)\overline{q}\mu$. Then

$$SPC_\tau(\lambda \wedge (\overline{1}-\mu), r)\overline{q}(\mu \wedge (\overline{1}-\lambda)).$$

Again, since $\mu \wedge (\overline{1}-\lambda) \leq \overline{1}-\lambda$, $SPC_\tau(\mu \wedge (\overline{1}-\lambda), r) \leq \overline{1}-\lambda$ and hence $SPC_\tau(\mu \wedge (\overline{1}-\lambda), r)\overline{q}\lambda$. Then

$$SPC_\tau(\mu \wedge (\overline{1}-\lambda), r)\overline{q}(\lambda \wedge (\overline{1}-\mu)).$$

Thus $\lambda \wedge (\overline{1}-\mu)$ and $\mu \wedge (\overline{1}-\lambda)$ are r-fuzzy semi-pre-separated.

Similarly we can prove when λ and μ are r-fspc.

Theorem 4.4. Let (X,τ) be a fts and $r \in I_0$. The two non-null fuzzy sets λ and μ are r-fuzzy semi pre-separated iff there exists two r-fspo sets ν, ω such that $\lambda \leq \nu$, $\mu \leq \omega$, $\lambda\overline{q}\omega$ and $\mu\overline{q}\nu$.

Proof. For two r-fuzzy semi pre-separated sets λ and μ, $\mu \leq \overline{1} - SPC_\tau(\lambda, r) = \omega$ (say) and $\lambda \leq \overline{1} - SPC_\tau(\mu, r) = \nu$ (say), where ω and ν are clearly r-fspo, then $\omega\overline{q}SPC_\tau(\lambda, r)$ and $\nu\overline{q}SPC_\tau(\mu, r)$. Thus, $\lambda\overline{q}\omega$ and $\mu\overline{q}\nu$.

(Conversely) Let ν and ω be r-fspo sets such that $\lambda \leq \nu$, $\mu \leq \omega$, $\lambda\overline{q}\omega$ and $\mu\overline{q}\nu$. Then $\lambda \leq \overline{1}-\omega$, $\mu \leq \overline{1}-\nu$. Hence $SPC_\tau(\lambda, r) \leq \overline{1}-\omega$, $SPC_\tau(\mu, r) \leq \overline{1}-\nu$, which in turn imply that $SPC_\tau(\lambda, r)\overline{q}\mu$ and $SPC_\tau(\mu, r)\overline{q}\lambda$. Thus λ and μ are r fuzzy semi-pre-separated.

Theorem 4.5. Let (X,τ) be a fts, $r \in I_0$ and λ be a non-null r-fuzzy semi pre-connected set. If $\lambda \leq \mu \leq SPC_\tau(\lambda, r)$ then μ is also r-fuzzy semi pre-connected.

Proof. Suppose that μ is not r-fuzzy semi pre-connected. Then there exists r-fuzzy semi pre-separated sets ω_1 and ω_2 in X such that $\mu = \omega_1 \vee \omega_2$. Let $\nu = \lambda \wedge \omega_1$ and $\omega = \lambda \wedge \omega_2$. Then $\lambda = \nu \vee \omega$. Since $\nu \leq \omega_1$ and $\omega \leq \omega_2$, by Theorem 4.3(1), ν and ω are r-fuzzy semi pre-separated, contradicting the r-fuzzy semi pre-connectedness of λ. Thus μ is r-fuzzy semi pre-connected.

Theorem 4.6. Let $f: X \to Y$ be a fuzzy semi pre-irresolute mapping, $\lambda \in I^X$ and $r \in I_0$. If λ is r-fuzzy semi pre-connected set in X, then so is $f(\lambda)$ in Y.

Proof. Suppose that $f(\lambda)$ is not r-fuzzy semi pre-connected in Y. Then there exists r-fuzzy semi pre-separated sets μ and ν in Y such that $f(\lambda) = \mu \vee \nu$. Since μ and ν are r-fuzzy semi pre-separated, by Theorem 4.4 there exists two r-fspo sets ω_1 and ω_2 such that $\mu \leq \omega_1$, $\nu \leq \omega_2$, $\mu\overline{q}\omega_2$ and $\nu\overline{q}\omega_1$. Now, f being semi pre-irresolute, $f^{-1}(\omega_1)$ and $f^{-1}(\omega_2)$ are r-fspo sets in X and

$$\lambda \leq f^{-1}f(\lambda) = f^{-1}(\mu \vee \nu) = f^{-1}(\mu) \vee f^{-1}(\nu).$$

For $\mu\overline{q}\omega_2$ and $\nu\overline{q}\omega_1$, we have $\mu \leq \overline{1}-\omega_2$ and $\nu \leq \overline{1}-\omega_1$ i.e., $f^{-1}(\mu) \leq \overline{1} - f(\omega_2)$ and $f^{-1}(\nu) \leq \overline{1} - f(\omega_1)$. Hence $f^{-1}(\mu)\overline{q}f^{-1}(\omega_2)$ and $f^{-1}(\nu)\overline{q}f^{-1}(\omega_1)$. By Theorem 4.4, $f^{-1}(\mu)$ and $f^{-1}(\nu)$ are r-fuzzy semi pre-separated in X. Since $\lambda = (\lambda \wedge f^{-1}(\mu)) \vee (\lambda \wedge f^{-1}(\nu))$ and $\lambda \wedge f^{-1}(\mu)$ and $\lambda \wedge f^{-1}(\nu)$ are r-fuzzy semi pre-separated in X from Theorem 4.3 (1), λ is not r-fuzzy semi pre-connected set. It is a contradiction.

Corollary 4.7. Let $f: X \to Y$ be a fuzzy semi pre-irresolute mapping and $r \in I_0$. If X is r-fuzzy semi pre-connected, then so is $f(X)$.

References

[1] C.L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.*, 24 (1968), 182–190.

[2] K.C. Chattopadyay and S.K. Samanta, Fuzzy topology: fuzzy closure, fuzzy compactness and fuzzy connectedness, *Fuzzy Sets and Systems*, 54 (1993), 207–212.

[3] U. Höhle and S.E. Rodabaugh, Logic, Topology and Measure Theorey, *The Handbooks of Fuzzy Sets Series*, Kluwer Academic Publishers, Boston (1999).

[4] Y.C. Kim, A.A. Ramadan and S.E. Abbas, Weaker forms of continuity in Ŝostak's fuzzy topology, *Indian J. Pure and Appl. Math.*, (2003), 311–333.

[5] J.H. Park and B.Y. Lee, Fuzzy semi preopen sets and fuzzy semi pre continuous mappings, *Fuzzy Sets and Systems*, 67(3) (1994), 359–364.

[6] Seok Jong Lee and Eun Pyo Lee, Fuzzy r-preopen sets and fuzzy r-pre-continuous maps, *Bull. Korean Math. Soc.*, 36(1) (1999), 91–108.

[7] A.P. Ŝostak, On a fuzzy topological structure, *Rend. Circ. Matem. Palermo Ser. II*, 11 (1985), 89–103.