Fuzzy semi pre-irresolute mappings in fuzzy topological spaces

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Abstract
Y. C. Kim introduced \( r \)-fuzzy semi preopen (\( r \)-fuzzy semi preclosed) sets in Šostak’s fuzzy topological space. We investigate some properties of them. Moreover, we introduce and study fuzzy semi pre-irresolute mapping and fuzzy semi pre-connectedness in Šostak’s fuzzy topological space.

Keywords
\( r \)-fuzzy semi preopen (preclosed) sets, fuzzy semi pre-irresolute (semi pre-continuous) mapping, \( r \)-fuzzy semi pre-connected.

AMS Subject Classification
54A05, 54A10, 54A40.

Throughout this paper, let \( X \) be a nonempty set, \( I = [0, 1] \) and \( I_0 = (0,1] \). For \( \alpha \in I \), \( \overline{\alpha} (x) = \alpha \) for all \( x \in X \). The family of all fuzzy sets on \( X \) denoted by \( I^X \). For \( \lambda, \mu \in I^X \), \( \lambda \) is called quasi-coincident with \( \mu \), denoted by \( \lambda q \mu \), if there exists \( x \in X \) such that \( \lambda(x) + \mu(x) > 1 \). Otherwise we denote \( \overline{\lambda q \mu} \).

Definition 1.1. [7] A function \( \tau : I^X \rightarrow I \) is called a fuzzy topology on \( X \) if it satisfies the following conditions:
\( (O1) \) \( \tau(\overline{0}) = \tau(\overline{1}) = 1 \),
\( (O2) \) \( \tau(\bigvee_{i \in I} \mu_i) \geq \bigwedge_{i \in I} \tau(\mu_i) \), for any \( \{\mu_i\}_{i \in I} \subset I^X \),
\( (O3) \) \( \tau(\mu_1 \cap \mu_2) \geq \tau(\mu_1) \land \tau(\mu_2) \), for any \( \mu_1, \mu_2 \in I^X \).

The pair \( (X, \tau) \) is called a fuzzy topological space (for short, fts).

Remark 1.2. [4] Let \( (X, \tau) \) be a fuzzy topological space. Then, for each \( r \in I \), \( \tau_r = \{\mu \in I^X : \tau(\mu) \geq r\} \) is a Chang’s fuzzy topology on \( X \).

Theorem 1.3. [2], [3] Let \( (X, \tau) \) be a fts. Then for each \( r \in I_0 \), \( \lambda \in I^X \) we define an operator \( C_\tau : I^X \times I_0 \rightarrow I^X \) as follows
\[ C_\tau(\lambda, r) = \bigwedge \{\mu \in I^X : \lambda \leq \mu, \tau(\overline{\lambda - \mu}) \geq r\} \].

For \( \lambda, \mu \in I^X \) and \( r, s \in I_0 \), the operator \( C_\tau \) satisfies the following conditions:
\( (1) \) \( C_\tau(\overline{0}, r) = \overline{0} \),
\( (2) \) \( \lambda \leq C_\tau(\lambda, r) \),
\( (3) \) \( C_\tau(\lambda, r) \lor C_\tau(\mu, r) = C_\tau(\lambda \lor \mu, r) \),
\( (4) \) \( C_\tau(\lambda, r) \leq C_\tau(\lambda, s) \) if \( r \leq s \).
(5) \( C_I(C_I(\lambda, r), r) = C_I(\lambda, r) \).

**Theorem 1.4.** [3] Let \((X, \tau)\) be a fts. Then for each \(r \in I_0, \lambda \in I^X\) we define an operator \(I_\tau : I^X \times I_0 \to I^X\) as follows:
\[
I_\tau(\lambda, r) = \{ \mu \in I^X : \lambda \geq \mu, \tau(\mu) \geq r \}.
\]
For \(\lambda, \mu \in I^X\) and \(r, s \in I_0\), the operator \(I_\tau\) satisfies the following conditions:

1. \( I_\tau(\lambda, r) = \lambda \) is \( r \)-fuzzy pre-open for short if \( \lambda \leq I_\tau(C_I(\lambda, r), r) \).
2. \( \lambda \) is called \( r \)-fuzzy pre-open for short if \( \lambda \leq I_\tau(C_I(\lambda, r), r) \).
3. \( \lambda \) is called \( r \)-fuzzy preclosed for short if \( \lambda \leq I_\tau(C_I(\lambda, r), r) \).
4. \( \lambda \) is called \( r \)-fuzzy pre-open for short if \( \lambda \leq I_\tau(C_I(\lambda, r), r) \).
5. \( \lambda \) is called \( r \)-fuzzy preclosed for short if \( \lambda \leq I_\tau(C_I(\lambda, r), r) \).

**Definition 1.5.** [6] Let \((X, \tau)\) be a fts. For \(\lambda \in I^X\) and \(r \in I_0\),

(1) \( \lambda \) is called \( r \)-fuzzy pre-open \((r\text{-fpo}), \) for short if \( \lambda \leq I_\tau(C_I(\lambda, r), r) \).
(2) \( \lambda \) is called \( r \)-fuzzy preclosed \((r\text{-fpc}), \) for short if \( \lambda \leq I_\tau(C_I(\lambda, r), r) \).

**Theorem 1.7.** [4] Let \((X, \tau)\) be a fts. For \(\lambda \in I^X\) and \(r, s \in I_0\),

(1) \( \lambda \) is \( r \)-fspo iff \( \lambda \leq I_\tau(C_I(\lambda, r), r) \).
(2) Any union of \( r \)-fspo sets is \( r \)-fspo.
(3) Any intersection of \( r \)-fspo sets is \( r \)-fspo.
(4) \( \lambda \) is \( r \)-fspo, then \( \lambda \) is \( r \)-fspo.

**Definition 1.8.** [3], [7] Let \((X, \tau)\) and \((Y, \eta)\) be a fts’s. Let \( f : X \to Y \) be a mapping.

1. \( f \) is called fuzzy continuous if \( \eta(\mu) \leq \tau(f^{-1}(\mu)) \) for each \( \mu \in I^Y \).
2. \( f \) is called fuzzy open if \( \tau(\lambda) \leq \eta(f(\lambda)) \) for each \( \lambda \in I^X \).
3. \( f \) is called fuzzy closed if \( \tau(f(\lambda)) \leq \eta(f^{-1}(\lambda)) \) for each \( \lambda \in I^X \).

**2. \( r \)-fuzzy semi preopen and \( r \)-fuzzy semi preclosed sets**

**Theorem 2.1.** Let \((X, \tau)\) be a fts. For each \( r \in I_0, \lambda \in I^X, \) we define an operator \( SPC_\tau : I^X \times I_0 \to I^X\) as follows:
\[
SPC_\tau(\lambda, r) = \{ \mu \in I^X : \lambda \leq \mu, \tau(\mu) \geq r \}.
\]
For \(\lambda, \mu \in I^X\) and \(r \in I_0\), \( SPC_\tau\) follows the properties.

1. \( SPC_\tau(\lambda, r) = SPC_\tau(\mu, r) \) if \( \lambda = \mu \).
2. \( \lambda \leq SPC_\tau(\lambda, r) \).
3. \( SPC_\tau(\lambda, r) = SPC_\tau(\mu, r) \)
4. \( SPC_\tau(SPC_\tau(\lambda, r), r) = SPC_\tau(\lambda, r) \).
5. \( \lambda \) is \( r \)-fspc.
6. \( SPC_\tau(SPC_\tau(\lambda, r), r) = SPC_\tau(\lambda, r) \).

**Proof.** (1), (2), (5), (7) are easily proved from the definition of \( SPC_\tau \).

(3) Since \( \lambda, \mu \leq \lambda \lor \mu \), we have
\[
SPC_\tau(\lambda, r) \lor SPC_\tau(\mu, r) \leq SPC_\tau(\lambda \lor \mu, r).
\]
(4) From (2), we only show \( SPC_\tau(\lambda, r) \geq SPC_\tau(PC_\tau(\lambda, r), r) \).

**Proof.** \( SPC_\tau(\lambda, r) \) if \( \lambda \) is \( r \)-fspc.

There exist \( x \in X \) and \( t \in (0, 1) \) such that
\[
SPC_\tau(\lambda, r)(x) < t < SPC_\tau(PC_\tau(\lambda, r), r)(x).
\]
(A)

Since \( SPC_\tau(\lambda, r)(x) < t \), by the Definition of \( SPC_\tau \), there exists \( r \)-fspc set \( \lambda_1 \) with \( \lambda \leq \lambda_1 \) such that
\[
SPC_\tau(\lambda, r)(x) \leq \lambda_1(x) < t.
\]

Since \( \lambda \leq \lambda_1 \), we have \( SPC_\tau(\lambda, r) \leq \lambda_1 \). Again, by the Definition of \( SPC_\tau \),
\[
SPC_\tau(PC_\tau(\lambda, r), r) \leq \lambda_1.
\]

Hence \( SPC_\tau(PC_\tau(\lambda, r), r)(x) \leq \lambda_1(x) < t \). It is a contradiction for (A). Thus,
\[
SPC_\tau(\lambda, r) \geq SPC_\tau(PC_\tau(\lambda, r), r).
\]

(6) From (2) and \( C_\tau(\lambda, r) \) is \( r \)-fspc we have, \( SPC_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r) \) we only show that
\[
C_\tau(SPC_\tau(\lambda, r), r) = C_\tau(\lambda, r).
\]
Since \( \lambda \leq SPC_\tau(\lambda, r) \),
\[
C_\tau(SPC_\tau(\lambda, r), r) \geq C_\tau(\lambda, r).
\]

Suppose that
\[
C_\tau(SPC_\tau(\lambda, r), r) \not\leq C_\tau(\lambda, r). 
\]

There exist \( x \in X \) and \( r \in I_0 \) such that
\[
C_\tau(SPC_\tau(\lambda, r), r)(x) > C_\tau(\lambda, r)(x). 
\]

By the definition of \( C_\tau \), there exists \( \rho \in I^X \) with \( \lambda \leq \rho \) and \( \tau(\overline{\rho} - \rho) \geq r \) such that
\[
C_\tau(SPC_\tau(\lambda, r), r)(x) > \rho(x) \geq C_\tau(\lambda, r)(x). 
\]

On the other hand, since \( \rho = C_\tau(\rho, r) \), \( \lambda \leq \rho \) implies
\[
SPC_\tau(\lambda, r) \leq SPC_\tau(\rho, r) = SPC_\tau(C_\tau(\rho, r), r) = C_\tau(\rho, r) = \rho. 
\]

Thus,
\[
C_\tau(SPC_\tau(\lambda, r), r) \leq \rho.
\]

It is a contradiction. Hence \( C_\tau(SPC_\tau(\lambda, r), r) \leq C_\tau(\lambda, r) \).

**Theorem 2.2.** Let \((X, \tau)\) be a fts. Define an operator \( SPI_\tau : I^X \times I_0 \to I^X \) as follows:
\[
SPI_\tau(\lambda, r) = \{ \mu \in I^X | \mu \leq \lambda, \mu \text{ is } r\text{-fspc} \}.
\]

Then:

1. \( SPI_\tau(\overline{\lambda}, r) = \tau - SPC_\tau(\lambda, r) \).
2. \( \lambda \leq \rho \Rightarrow SPI_\tau(\lambda, r) \subseteq SPI_\tau(\rho, r) \leq SPC_\tau(\lambda, r) \leq PC_\tau(\lambda, r) \leq C_\tau(\lambda, r) \).
3. \( \lambda \) is \( r \)-fspc iff \( SPI_\tau(\lambda, r) = \lambda \).

**Proof.** (1) For each \( \lambda \in I^X \) and \( r \in I_0 \), we have
\[
SPI_\tau(\overline{\lambda}, r) = \{ \mu \in I^X | \mu \leq \overline{\lambda}, \mu \text{ is } r\text{-fspc} \} = \tau - \{ \mu \in I^X | \mu \geq \overline{\lambda}, \mu - \overline{\lambda} \text{ is } r\text{-fspc} \} = \tau - SPC_\tau(\lambda, r).
\]

(2) and (3) are easily proved.

3. **Fuzzy semi pre-irresolute mappings**

**Definition 3.1.** Let \((X, \tau)\) and \((Y, \eta)\) be a fts's. Let \( f : X \to Y \) be a mapping.

1. \( f \) is fuzzy semi pre-irresolute (resp. fuzzy semi pre-continuous \cite{4}) iff \( f(\mu) \) is \( r\)-fspc set of \( X \) for each \( r\)-fspc set \( \mu \in I^X \) (resp. \( \eta(\mu) \geq r \)).

2. \( f \) is fuzzy semi pre-irresolute open (resp. fuzzy semi pre-open \cite{4}) iff \( f(\mu) \) is \( r\)-fspc set of \( Y \) for each \( r\)-fspc set \( \mu \in I^X \) (resp. \( \tau(\mu) \geq r \)).
(4) \(\Rightarrow\) (5) It is easily proved from Theorem 3(1)

(5) \(\Rightarrow\) (1) Let \(\mu\) be \(r\)-fspo set of \(Y\). From Theorem 3(3), 

\[ \mu = SPI_\tau I_\tau (\mu, r) \]  

By (5),

\[ f^{-1}(\mu) \leq SPI_\tau (f^{-1}(\mu), r) \]

On the other hand, by Theorem 3(2),

\[ f^{-1}(\mu) \geq SPI_\tau (f^{-1}(\mu), r) \]

Thus, \(f^{-1}(\mu) = SPI_\tau (f^{-1}(\mu), r)\), that is, \(f^{-1}(\mu)\) is \(r\)-fspo.

The following theorem is similarly proved as Theorem 3.2

**Theorem 3.3.** Let \((X, \tau_1)\) and \((Y, \tau_2)\) be a fts’s. Let \(f : X \rightarrow Y\) be a mapping. The following statements are equivalent:

1. A map \(f\) is fuzzy semi pre-continuous.
2. \(f(SPC_\tau (\lambda, r)) \leq C_\tau (f(\lambda), r)\), for each \(\lambda \in \lambda^X\) and \(r \in I_0\).
3. \(SPC_\tau (f^{-1}(\mu), r) \leq f^{-1}(C_\tau (\mu, r))\), for each \(\mu \in \mu^Y\) and \(r \in I_0\).
4. \(f^{-1}(I_\tau (\mu, r)) \leq SPI_\tau (f^{-1}(\mu), r)\), for each \(\mu \in \mu^Y\) and \(r \in I_0\).

**Theorem 3.4.** Let \((X, \tau)\) and \((Y, \eta)\) be a fts’s. Let \(f : X \rightarrow Y\) be a bijective mapping. The following statements are equivalent:

1. A map \(f\) is fuzzy semi pre-irresolute.
2. \(SPI_\eta (f(\lambda), r) \leq f(SPI_\tau (\lambda, r))\), for each \(\lambda \in \lambda^X\) and \(r \in I_0\).

**Proof.** (1) \(\Rightarrow\) (2): Let \(f\) be fuzzy semi pre-irresolute mapping and \(\lambda \in \lambda^X\), \(r \in I_0\). Then \(f^{-1}(SPI_\eta (f(\lambda), r))\) is \(r\)-fspo set in \(X\). By Theorem 3.2 and the fact that \(f\) is one-to-one we have

\[ f^{-1}(SPI_\eta (f(\lambda), r)) \leq SPI_\tau (f^{-1}(f(\lambda), r)) = SPI_\tau (\lambda, r) \]

Again since \(f\) is onto we have

\[ SPI_\eta (f(\lambda), r) = f SPI_\tau (f(\lambda), r) \leq f(SPI_\tau (\lambda, r)) \]

(2) \(\Rightarrow\) (1): Let \(\mu\) be \(r\)-fspo set of \(Y\). Then by Theorem 3(3), 

\[ \mu = SPI_\eta (\mu, r) \]

By (2)

\[ f(SPI_\tau (f^{-1}(\mu), r)) \geq SPI_\eta (f f^{-1}(\mu), r) = SPI_\eta (\mu, r) \]

It implies

\[ f^{-1}(\mu) = SPI_\eta (f^{-1}(\mu), r) \geq f^{-1}(\mu) \]

Hence, \(f^{-1}(\mu) = SPI_\tau (f^{-1}(\mu), r)\), that is, \(f^{-1}(\mu)\) is \(r\)-fspo set of \(X\).

**Theorem 3.5.** Let \((X, \tau)\) and \((Y, \eta)\) be fts’s. Let \(f : X \rightarrow Y\) be a mapping. The following statements are equivalent:

1. \(f\) is called fuzzy semi-pre-irresolute open.
2. \(f(SPI_\tau (\lambda, r)) \leq SPI_\eta (f(\lambda), r)\), for each \(\lambda \in \lambda^X\) and \(r \in I_0\).
3. \(SPI_\tau (f^{-1}(\mu), r) \leq f^{-1}(SPI_\eta (\mu, r))\), for each \(\mu \in \mu^Y\) and \(r \in I_0\).
4. For any \(\mu \in \mu^Y\) and any \(r\)-fspc \(\lambda \in \lambda^X\) with \(f^{-1}(\mu) \leq \lambda\), there exists a \(r\)-fspc \(\rho \in \rho^X\) with \(\mu \leq \rho\) such that \(f^{-1}(\rho) \leq \lambda\).

**Proof.** (1) \(\Rightarrow\) (2): For each \(\lambda \in \lambda^X\), since \(SPI_\tau (\lambda, r) \leq \lambda\) from Theorem (2), we have \(f(SPI_\tau (\lambda, r)) \leq \lambda\). From (1), \(f(SPI_\tau (\lambda, r))\) is \(r\)-fspo. Hence \(f(SPI_\tau (\lambda, r)) \leq SPI_\eta (f(\lambda), r)\).

(2) \(\Rightarrow\) (3): For all \(\mu \in \mu^Y\), \(r \in I_0\), put \(\lambda = f^{-1}(\mu)\) from (2). Then

\[ f(SPI_\tau (f^{-1}(\mu), r)) = SPI_\eta (f^{-1}(\mu), r) \leq SPI_\eta (\mu, r) \]

It implies \(SPI_\tau (f^{-1}(\mu), r) = f^{-1}(SPI_\eta (\mu, r))\).

(3) \(\Rightarrow\) (4): Let \(\lambda\) be \(r\)-fspc set of \(X\) such that \(f^{-1}(\mu) \leq \lambda\). Since \(\lambda - \mu \leq f^{-1}(\lambda - \mu)\) and

\[ SPI_\tau (\lambda - \mu, r) = SPI_\tau (\lambda, r) - SPI_\tau (\mu, r) \]

From(3),

\[ \lambda - \lambda \leq SPI_\tau (f^{-1}(\lambda - \mu), r) \leq f^{-1}(SPI_\eta (\lambda - \mu, r)) \]

It implies

\[ f^{-1}(\lambda) \geq f^{-1}(SPI_\eta (\lambda - \mu, r)) = f^{-1}(f SPI_\tau (\lambda - \mu, r)) \geq f^{-1}(SPI_\tau (\lambda, r)) \]

Hence there exists a \(r\)-fspc \(SPC_\eta (\lambda, r) \in I_0^Y\) with \(\mu \leq SPI_\tau (\lambda, r)\) such that \(f^{-1}(\mu) \leq \lambda\).

(4) \(\Rightarrow\) (1): Let \(\omega\) be \(r\)-fspo set \(X\). Put \(\mu = T - f(\omega)\) and \(\lambda = T - \omega\) such that \(\lambda\) is \(r\)-fspc. We obtain

\[ f^{-1}(\mu) = f^{-1}(T - f(\omega)) \leq T - f^{-1}(f(\omega)) \leq T - \omega = \lambda. \]

From(4), there exists \(r\)-fspc set \(\rho\) with \(\mu \leq \rho\) such that \(f^{-1}(\rho) \leq \lambda = T - \omega\). It implies \(\omega \leq T - f^{-1}(\rho) = f^{-1}(\lambda - \rho)\). Thus, \(f(\omega) \leq f(\lambda - \rho) \leq \mu - \rho\). On the other hand, since \(\mu \leq \rho\).
Theorem 3.7. Let \( \lambda \) be a mapping. The following statements are equivalent:

1. \( \lambda \) is a fuzzy semi pre-irresolute closed map.
2. \( \lambda \) is a fuzzy continuous and fuzzy semi pre-continuous.
3. \( \lambda \) is fuzzy semi pre-irresolute and fuzzy semi pre-irresolute open.

Proof. (1) \( \Rightarrow \) Let \( \lambda \) be fuzzy semi pre-irresolute closed. From Theorem 3.6(2), for each \( \lambda \in l^X \) and \( r \in I_0 \),

\[
f(\text{SPC}_r(\lambda, r)) \geq \text{SPC}_r(f(\lambda), r).
\]

For all \( \mu \in I^Y \) and \( r \in I_0 \). Put \( \lambda = f^{-1}(\mu) \) from (2) since \( f \) is onto, \( f(f^{-1}(\mu)) = \mu \). Thus,

\[
f(\text{SPC}_r(f^{-1}(\mu), r)) \geq f^{-1}(\text{SPC}_r(\mu, r)).
\]

It implies

\[
\text{SPC}_r(f^{-1}(\mu, r)) = f^{-1}(f(\text{SPC}_r(f^{-1}(\mu), r)))
\]

(\( \Leftarrow \)) Put \( \mu = f(\lambda) \). Since \( f \) is injective,

\[
f^{-1}(\text{SPC}_r(f(\lambda), r)) \leq f^{-1}(\text{SPC}_r(f(\lambda), r)) = \text{SPC}_r(\lambda, r).
\]

Since \( f \) is onto,

\[
\text{SPC}_r(f(\lambda), r) \leq f(\text{SPC}_r(\lambda, r)).
\]

(2) It is easily proved from:

\[
f^{-1}(\text{SPC}_r(\mu, r)) \leq f^{-1}(\text{SPC}_r(f^{-1}(\mu), r))
\]

\[
f^{-1}(\mu, r) \leq f^{-1}(\text{SPC}_r(f^{-1}(\mu), r))
\]

\[
f^{-1}(\text{SPC}_r(f^{-1}(\mu, r))) \geq f^{-1}(\text{SPC}_r(f^{-1}(\mu, r)))
\]

From the above Theorems we obtain the following theorem.

Theorem 3.8. Let \( \tau : (X, \tau) \rightarrow (Y, \eta) \) be a bijective mapping from a fts \( (X, \tau) \) into a fts \( (Y, \eta) \). For each \( \lambda \in I^X \), \( \mu \in I^Y \) and \( r \in I_0 \). The following statements are equivalent:

1. \( \lambda \) is a fuzzy semi pre-irresolute homeomorphism.
2. \( \lambda \) is a fuzzy semi pre-irresolute and fuzzy semi pre-irresolute open.
3. \( \lambda \) is fuzzy semi pre-irresolute and fuzzy semi pre-irresolute closed.
4. \( f(\text{SPI}_r(\lambda, r)) = \text{SPI}_r(f(\lambda), r)).
5. \( f(\text{SPC}_r(\lambda, r)) = \text{SPC}_r(f(\lambda), r).
6. \( \text{SPI}_r(f^{-1}(\mu, r)) = f^{-1}(\text{SPI}_r(\mu, r)).
7. \( \text{SPC}_r(f^{-1}(\mu, r)) = f^{-1}(\text{SPC}_r(\mu, r)).

Remark 3.9. For the mapping \( f : X \rightarrow Y \), the following statements are valid:

1. \( f \) is fuzzy semi pre-irresolute \( \Rightarrow \) \( f \) is fuzzy semi pre-continuous.
2. \( f \) is fuzzy continuous \( \Rightarrow \) \( f \) is fuzzy semi pre-continuous.

The converses of Remark 3.9(1,2) need not be true from the following example.

Example 3.10. Let \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) be fuzzy subsets of \( X = \{a, b, c\} \) defined as follows:

\[
\begin{align*}
\lambda_1(a) & = 0.2, \ 
\lambda_1(b) & = 0.4, \ 
\lambda_1(c) & = 0.5, \\
\lambda_2(a) & = 0.2, \ 
\lambda_2(b) & = 0.2, \ 
\lambda_2(c) & = 0.2, \\
\lambda_3(a) & = 0.5, \ 
\lambda_3(b) & = 0.5, \ 
\lambda_3(c) & = 0.5, \\
\lambda_4(a) & = 0.6, \ 
\lambda_4(b) & = 0.6, \ 
\lambda_4(c) & = 0.6.
\end{align*}
\]

Then \( \tau, \eta : I^X \rightarrow I \) defined as

\[
\begin{align*}
\tau(\lambda) & = \begin{cases} 
1, & \text{if } \lambda = \overline{0, \overline{T}} \\
0, & \text{otherwise}
\end{cases} \\
\eta(\lambda) & = \begin{cases} 
1, & \text{if } \lambda = \overline{0, \overline{T}} \\
0, & \text{otherwise}
\end{cases}
\end{align*}
\]

are fuzzy topologies on \( X \). Then, the identity mapping \( id_X : (X, \tau) \rightarrow (X, \eta) \) is fuzzy continuous and fuzzy semi pre-continuous but not fuzzy semi pre-irresolute, because \( \lambda_4 \) is fuzzy semi pre-irresolute if not \( \frac{1}{2} \)-fspo but \( f^{-1}(\lambda_4) \) is not \( \frac{1}{2} \)-fspo from \( \lambda_4 \leq C_T(\overline{\tau_1(\lambda_4)^{1/2}, 1/2)} = \overline{\tau_1(\lambda_2^{1/2})^{1/2}} = \overline{\tau_1(\lambda_3^{1/2})^{1/2}} \leq \lambda_3 \).

4. Fuzzy semi pre-connectedness

Definition 4.1. Let \( X \) be a fts and \( \lambda, \mu \in I^X \), \( r \in I_0 \). The two fuzzy sets \( \lambda \) and \( \mu \) are said to be \( r \)-fuzzy semi pre-separated iff \( \lambda \overline{\tau_1(\lambda, r)} \) and \( \mu \overline{\tau_1(\lambda, r)} \).
Definition 4.2. A fuzzy set which cannot be expressed as the union of two $r$-fuzzy semi pre-separated sets is said to be $r$-fuzzy semi pre-connected set.

Theorem 4.3. Let $(X, r)$ be a fts and $\lambda, \mu \in I^X$, $r \in I_0$.

(i) If $\lambda$, $\mu$ are $r$-fuzzy semi-pre-separated and $\nu$, $\eta$ are non-null fuzzy sets such that $\nu \leq \lambda$, $\eta \leq \mu$, then $\nu$, $\eta$ are also $r$-fuzzy semi pre-separated.

(ii) If $\lambda \mu$ and either both are $r$-fspo or both $r$-fspc, then $\lambda$ and $\mu$ are $r$-fuzzy semi pre separated.

(iii) If $\lambda$, $\mu$ are either both $r$-fspo or both $r$-fspc, then $\lambda \wedge (\overline{\lambda} - \mu)$ and $\mu \wedge (\overline{\lambda} - \mu)$ are $r$-fuzzy semi pre-separated.

Proof. (1) and (2) are obvious.

(3) Let $\lambda$ and $\mu$ be both $r$-spso. Since $\lambda \wedge (\overline{\lambda} - \mu) \leq \overline{\lambda} - \mu$, $SPC_r(\lambda \wedge (\overline{\lambda} - \mu), r) \leq \overline{\lambda} - \mu$ and hence $SPC_r(\lambda \wedge (\overline{\lambda} - \mu), r) \leq \overline{\lambda} - \mu$ and hence $SPC_r(\lambda \wedge (\overline{\lambda} - \mu)) \leq \overline{\lambda} - \mu$.

Then
\[
SPC_r(\lambda \wedge (\overline{\lambda} - \mu), r) \leq \overline{\lambda} - \mu.
\]

Again, since $\mu \wedge (\overline{\lambda} - \mu) \leq \overline{\lambda} - \mu$, $SPC_r(\mu \wedge (\overline{\lambda} - \mu), r) \leq \overline{\lambda} - \mu$ and hence $SPC_r(\mu \wedge (\overline{\lambda} - \mu), r) \leq \overline{\lambda} - \mu$.

Thus $\lambda \wedge (\overline{\lambda} - \mu)$ and $\mu \wedge (\overline{\lambda} - \mu)$ are $r$-fuzzy semi-pre-separated.

Similarly we can prove when $\lambda$ and $\mu$ are $r$-fspc.

Theorem 4.4. Let $(X, r)$ be a fts and $r \in I_0$. The two non-null fuzzy sets $\lambda$ and $\mu$ are $r$-fuzzy semi pre-separated iff there exists two $r$-fspo sets $\nu$, $\omega$ such that $\lambda \leq \nu$, $\mu \leq \omega$, $\lambda \eta \omega$ and $\mu \eta \nu$.

Proof. For two $r$-fuzzy semi pre-separated sets $\lambda$ and $\mu$, $\mu \leq \overline{\lambda} - SPC_r(\lambda, r) = \omega$ (say) and $\lambda \leq \overline{\lambda} - SPC_r(\mu, r) = \nu$ (say), where $\omega$ and $\nu$ are clearly $r$-fspo, then $\omega \nu SPC_r(\lambda, r)$ and $\nu \omega SPC_r(\mu, r)$.

Thus, $\lambda \eta \omega$ and $\mu \eta \nu$.

(Conversely) Let $\nu$ and $\omega$ be $r$-fspo sets such that $\lambda \leq \nu$, $\mu \leq \omega$, $\lambda \eta \omega$ and $\mu \eta \nu$. Then $\lambda \leq \overline{\lambda} - \omega$, $\mu \leq \overline{\lambda} - \nu$. Hence $SPC_r(\lambda, r) \leq \overline{\lambda} - \omega$, $SPC_r(\mu, r) \leq \overline{\lambda} - \nu$, which in turn imply that $SPC_r(\lambda, r) \eta \omega$ and $SPC_r(\mu, r) \eta \nu$. Thus $\lambda$ and $\mu$ are $r$- fuzzy semi pre-separated.

Theorem 4.5. Let $(X, r)$ be a fts, $r \in I_0$ and $\lambda$ be a non-null $r$-fuzzy semi pre-connected set. If $\lambda \leq \mu \leq SPC_r(\lambda, r)$ then $\mu$ is also $r$-fuzzy semi pre-connected.

Proof. Suppose that $\mu$ is not $r$-fuzzy semi pre-connected. Then there exists $r$-fuzzy semi pre-separated sets $\omega_1$ and $\omega_2$ in $X$ such that $\mu = \omega_1 \lor \omega_2$. Let $\nu = \lambda \land \omega_1$ and $\omega = \lambda \land \omega_2$. Then $\lambda = \nu \lor \omega$. Since $\nu \leq \omega_1$ and $\omega \leq \omega_2$, by Theorem 4.3(1), $\nu$ and $\omega$ are $r$-fuzzy semi pre-separated, contradicting the $r$-fuzzy semi pre-connectedness of $\lambda$. Thus $\mu$ is $r$-fuzzy semi pre-connected.

Theorem 4.6. Let $f : X \rightarrow Y$ be a fuzzy semi pre-irresolute mapping, $\lambda \in I^X$ and $r \in I_0$. If $\lambda$ is $r$-fuzzy semi pre-connected set in $X$, then so is $f(\lambda)$ in $Y$.

Proof. Suppose that $f(\lambda)$ is not $r$-fuzzy semi pre-connected in $Y$. Then there exists $r$-fuzzy semi pre-separated sets $\mu$ and $\nu$ in $Y$ such that $f(\lambda) = \mu \lor \nu$. Since $\mu$ and $\nu$ are $r$-fuzzy semi pre-separated, by Theorem 4.4 there exists two $r$-fspo sets $\omega_1$ and $\omega_2$ such that $\mu \leq \omega_1$, $\nu \leq \omega_2$, $\mu \eta \omega_2$ and $\nu \eta \omega_1$. Now, $f$ being semi pre-irresolute, $f^{-1}(\omega_1)$ and $f^{-1}(\omega_2)$ are $r$-fspo sets in $X$ and
\[
\mu \leq f^{-1}(\lambda) = f^{-1}(\mu \lor \nu) = f^{-1}(\mu) \lor f^{-1}(\nu).
\]

For $\mu \eta \omega_2$ and $\nu \eta \omega_1$, we have $\mu \leq \overline{\lambda} - \omega_2$ and $\nu \leq \overline{\lambda} - \omega_1$ i.e., $f^{-1}(\mu) \leq \overline{\lambda} - f(\omega_2)$ and $f^{-1}(\nu) \leq \overline{\lambda} - f(\omega_1)$. Hence $f^{-1}(\mu) \eta f^{-1}(\omega_2)$ and $f^{-1}(\nu) \eta f^{-1}(\omega_1)$. By Theorem 4.4, $f^{-1}(\mu)$ and $f^{-1}(\nu)$ are $r$-fuzzy semi pre-separated in $X$. Since $\lambda = (\lambda \land f^{-1}(\mu)) \lor (\lambda \land f^{-1}(\nu))$ and $\lambda \land f^{-1}(\mu)$ and $\lambda \land f^{-1}(\nu)$ are $r$-fuzzy semi pre-separated in $X$ from Theorem 4.1(1), $\lambda$ is not $r$-fuzzy semi pre-connected set. It is a contradiction.

Corollary 4.7. Let $f : X \rightarrow Y$ be a fuzzy semi pre-irresolute mapping and $r \in I_0$. If $X$ is $r$-fuzzy semi pre-connected, then so is $f(X)$.

References


