$(\varepsilon, \varepsilon \vee q_k)$ – intuitionistic fuzzy bi-ideals of semigroups

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Abstract

In this paper, a new generalization of intuitionistic fuzzy bi-ideals (IFBI) of a semigroup is considered. We combine the notion of intuitionistic fuzzy point (IFP) and intuitionistic fuzzy sets (IFS) to define different types of intuitionistic fuzzy bi-ideals of a semigroups. We investigate different properties of these notions and their relationships, particularly, we define $(\varepsilon, \varepsilon \vee q_k)$ – intuitionistic fuzzy bi-ideals in semigroups.

Keywords

Semigroup, FI, FBI, IFI, IFBI, $(\varepsilon, \varepsilon \vee q_k)$ – IFBI.

AMS Subject Classification

03E72, 16Y30.

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1. Introduction

The fundamental concept of a fuzzy set, introduced by Zadeh [17], provides a natural frame-work for generalizing several basic notions of algebra. A new type of fuzzy subgroup, that is, the (α, β) -fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das [3] by using the combined notions of “belongingness” and “quasicoincidence” of a fuzzy point and a fuzzy set. In particular, the concept of an $(\varepsilon, \varepsilon \vee q)$ -fuzzy subgroup is a useful generalization of Rosenfeld’s fuzzy subgroup [13]. Davvaz and Khan [5] discussed some characterizations of regular ordered semigroups in terms of (α, β) -fuzzy generalized bi-ideals of ordered semigroups, where $\alpha, \beta \in \{\varepsilon, q, \varepsilon \vee q, \varepsilon \wedge q\}$ and $\alpha \neq \varepsilon \wedge q$. Also Khan introduced the concept of an (α, β) -fuzzy interior ideals and gave some basic properties of ordered semigroups in terms of this notion (see [11]). In [14], regular semigroups are char-

acterized by the properties of $(\varepsilon, \varepsilon \vee q)$ -fuzzy ideals. Jun [8], introduced the concept of $(\varepsilon, \varepsilon \vee q_k)$ -fuzzy subalgebras of a BCK/BCI-algebra and gave some basic properties of BCK-algebras. In [14], Shabir gave the concept of more generalized forms of (α, β) -fuzzy ideals and defined $(\varepsilon, \varepsilon \vee q_k)$ -fuzzy ideals of semigroups, by generalizing the concept of $[x; t]q_k F$ and defined $[x; t]q_k F$, as $F(x) + t + k > 1$, where $k \in [0, 1)$ (also see [9, 10]).

In 1986, Atanassov [2] premised the concept of an intuitionistic fuzzy set (IFS). An Atanassov intuitionistic fuzzy set is deliberated as a generalization of fuzzy set [17] and has been found to be useful to deal with vagueness. In the sense of Atanassov an IFS is characterized by a pair of functions valued in $[0, 1]$; the membership and the non-membership function. The evaluation degrees of membership and non-membership are independent. Thus, an Atanassov intuitionistic fuzzy set is more material and concise to describe the essence of fuzziness, and Atanassov intuitionistics fuzzy set theory may be more suitable that fuzzy set theory for dealing with imperfect knowledge in many problems. The concept has been applied to various algebraic structures. Biswas [5] introduced the notion of intuitionistic fuzzy subgroup of a group by using the notion of intuitionistic fuzzy sets. In [10], Kim and Jun defined intuitionistic fuzzy ideals of semigroups. Kim and Lee [11] studied intuitionistic fuzzy bi-ideals of semigroups. Jun [8] introduced the notion of (Φ, Ψ) -intuitionistic fuzzy subgroups, where Φ, Ψ are any two of $\{\varepsilon, q, \varepsilon \vee q, \varepsilon \wedge q\}$ with $\Phi \neq \varepsilon \wedge q$,

and related properties are investigated. Our aim in this paper is to introduce a new sort of fuzzy bi-ideals and fuzzy left (resp. right)-ideals in semigroup, called $(\varepsilon, \varepsilon \vee q_k)$ -fuzzy bi-ideals and characterize semigroups in terms of $(\varepsilon, \varepsilon \vee q_k)$ -fuzzy bi-ideals. We define the lower/upper parts of $(\varepsilon, \varepsilon \vee q_k)$ -fuzzy bi-ideals. Then some results are given in terms of $(\varepsilon, \varepsilon \vee q_k)$ -fuzzy bi-ideals in semigroups.

2. Preliminaries

Definition 2.1. A fuzzy subset f of S is called a fuzzy subsemigroup of S if for all $x, y \in S$
 $f(xy) \geq \min\{f(x), f(y)\}$.

Definition 2.2. A Fuzzy subset f of S is called a fuzzy left (resp. right) ideal of S if for all $x, y \in S$
 $f(xy) \geq f(y)$, (resp. $f(xy) \geq f(x)$).
A fuzzy subset f of S is called a fuzzy ideal of S if it is both fuzzy left and a fuzzy right ideal of S .

Definition 2.3. Let X be a non-empty fixed set. An intuitionistic fuzzy (briefly, IFS) A is object having the form
 $A = \{\langle x, \mu_A(x), \gamma_A(x) \rangle : x \in X\}$.
Where the functions $\mu_A : X \rightarrow [0, 1]$ and $\gamma_A : X \rightarrow [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\gamma_A(x)$) of each element $x \in X$ to the set A , respectively, and $\mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in S$ for the sake of simplicity, we use the symbol $A = \langle \mu_A, \lambda_A \rangle$ for the IFS $A = \{\langle x, \mu_A(x), \gamma_A(x) \rangle : x \in X\}$.

Definition 2.4. An IFS $A = \langle \mu_A, \lambda_A \rangle$ in S is called an intuitionistic fuzzy subsemigroup of S if the following conditions hold

(IF 1) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$,
(IF 2) $\gamma_A(xy) \leq \gamma_A(x) \vee \gamma_A(y)$,
for all $x, y \in S$.

Definition 2.5. An IFS $A = \langle \mu_A, \lambda_A \rangle$ in S is called an intuitionistic fuzzy right ideal of S if it satisfy $\mu_A(xy) \geq \mu_A(x)$ and $\gamma_A(xy) \leq \gamma_A(x)$ for all $x, y \in S$.

Definition 2.6. An IFS $A = \langle \mu_A, \lambda_A \rangle$ in S is called an intuitionistic fuzzy left ideal of S if it satisfy $\mu_A(xy) \geq \mu_A(y)$ and $\gamma_A(xy) \leq \gamma_A(y)$ for all $x, y \in S$.

Definition 2.7. An IFS $A = \langle \mu_A, \lambda_A \rangle$ in a semigroup S is said to be an $(\in, \in \vee q_k)$ – intuitionistic fuzzy subsemigroup of a semigroup S if the following conditions holds:
(IFSI) $(\forall x, y \in S) (\forall t_1, t_2 \in (0, 1] \text{ and } \forall s_1, s_2 \in [0, 1)) x(t_1, s_1) \in A \text{ and } y(t_2, s_2) \in A \implies (xy)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in \vee q_k A$.

3. (α, β) -IFBI

Definition 3.1. An IFS $A = \langle \mu_A, \lambda_A \rangle$ in a semigroup S is said to be an (α, β) – IFBI of a semigroup S , where α, β are any two of $\{\in, q_k, \in \vee q_k, \in \wedge q_k\}$ with $\alpha \neq \in \wedge q_k$, if for all $x, y, z \in S, t_1, t_2 \in (0, 1], s_1, s_2 \in [0, 1)$, the following conditions

are hold:

(IFB 1) $x(t_1, s_1) \alpha A$
and $y(t_2, s_2) \alpha A \implies (xy)(m\{t_1, t_2\}, M\{s_1, s_2\}) \beta A$,
(IFB 2) $x(t_1, s_1) \alpha A$
and $z(t_2, s_2) \alpha A \implies (xyz)(m\{t_1, t_2\}, M\{s_1, s_2\}) \beta A$.

Definition 3.2. An IFS $A = \langle \mu_A, \lambda_A \rangle$ in a semigroup S is said to be an (α, β) – IF (1, 2) ideal of a semigroup S if for all $x, y, z, a \in S, t_1, t_2 \in (0, 1]$ and $s_1, s_2 \in [0, 1)$, the following conditions hold.

(IFS 1) $x(t_1, s_1) \alpha A$
and $y(t_2, s_2) \alpha A \implies (xy)(m\{t_1, t_2\}, M\{s_1, s_2\}) \beta A$,
(IFS 2) $x(t_1, s_1) \alpha A$
and $z(t_2, s_2) \alpha A \implies (xa(yz))(m\{t_1, t_2\}, M\{s_1, s_2\}) \beta A$.

Theorem 3.3. Let $A = \langle \mu_A, \lambda_A \rangle$ be a non-zero (α, β) – IFSG of S . Then, the set

$A_{(0,1)} = \{x \in S : \mu_A(x) > 0 \text{ and } \lambda_A(x) < 1\}$ is a subsemigroup of S .

Proof: Let $x, y \in A_{(0,1)}$. Then, $\mu_A(x) > 0$ and $\lambda_A(x) < 1$, and $\mu_A(y) > 0$ and $\lambda_A(y) < 1$. Let us suppose that $\mu_A(xy) = 0$ and $\lambda_A(xy) = 1$. If $\alpha \in \{\in, \in \vee q_k\}$ then $x(\mu_A(x), \lambda_A(x)) \alpha A$ and $y(\mu_A(y), \lambda_A(y)) \alpha A$ but $\mu_A(xy) = 0 < m\{\mu_A(x), \mu_A(y)\}$ and $\lambda_A(xy) = 1 > M\{\lambda_A(x), \lambda_A(y)\}$ so, $(xy)(m\{\mu_A(x), \mu_A(y)\}, M\{\lambda_A(x), \lambda_A(y)\}) \bar{\beta} A$ for $\beta \in \{\in, q_k, \in \vee q_k, \in \wedge q_k\}$, which is a contradiction. Now, let $x(1, 0) q_k A$ and $y(0, 1) q_k A$ but $(xy)(1, 0) \bar{\beta} A$ for $\beta \in \{\in, q_k, \in \vee q_k, \in \wedge q_k\}$, which is contradiction.

Hence $\mu_A(xy) > 0$ and $\lambda_A(xy) < 1$, that is $xy \in A_{(0,1)}$. Thus, $A_{(0,1)}$ is a subsemigroup of S .

Theorem 3.4. Let $A = \langle \mu_A, \lambda_A \rangle$ be a non-zero (α, β) -IFBI of S . Then, the set

$A_{(0,1)} = \{x \in S : \mu_A(x) > 0 \text{ and } \lambda_A(x) < 1\}$ is a bi-ideal of S .

Proof: Let $A = \langle \mu_A, \lambda_A \rangle$ be a non-zero (α, β) -IFBI of S . Then, by Theorem 3.1, $A_{(0,1)}$ is a subsemigroup of S . Now, let $x, z \in A_{(0,1)}$ and $y \in S$. Then, $\mu_A(x) > 0$ and $\lambda_A(x) < 1$, and $\mu_A(z) > 0$ and $\lambda_A(z) < 1$.

Suppose that $\mu_A(xyz) = 0$ and $\lambda_A(xyz) = 1$. If $\alpha \in \{\in, \in \vee q_k\}$, then

$x(\mu_A(x), \lambda_A(x)) \alpha A$ and $z(\mu_A(z), \lambda_A(z)) \alpha A$ but $\mu_A(xyz) = 0 < m\{\mu_A(x), \mu_A(z)\}$ and $\lambda_A(xyz) = 1 > M\{\lambda_A(x), \lambda_A(z)\}$ which implies that $(xy)(m\{\mu_A(x), \mu_A(z)\}, M\{\lambda_A(x), \lambda_A(z)\}) \bar{\beta} A$ for $\beta \in \{\in, q_k, \in \vee q_k, \in \wedge q_k\}$,

which is a contradiction. Now, let $x(1, 0) q_k A$ and $z(1, 0) q_k A$ but $(xyz)(1, 0) \bar{\beta} A$ for $\beta \in \{\in, q_k, \in \vee q_k, \in \wedge q_k\}$, which is contradiction.

Hence $\mu_A(xyz) > 0$ and $\lambda_A(xyz) < 1$, that is $xyz \in A_{(0,1)}$. Thus, $A_{(0,1)}$ is a bi-ideal of S .

Theorem 3.5. Let $A = \langle \mu_A, \lambda_A \rangle$ be a non-zero (α, β) -IF (1, 2) ideal of S . Then, the set $A_{(0,1)} = \{x \in S : \mu_A(x) > 0 \text{ and } \lambda_A(x) < 1\}$

$\lambda_A(x) < 1\}$ is a $(1, 2)$ ideal of S .

Proof. Straightforward. $\square$

Theorem 3.6. Let L be a left (resp, right) ideal of S and let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS such that

(a) $(\forall x \in S \setminus L) (\mu_A(x) = 0 \text{ and } \lambda_A(x) = 1)$.

(b) $(\forall x \in L) (\mu_A(x) \geq \frac{1-k}{2} \text{ and } \lambda_A(x) \leq \frac{1+k}{2})$.

Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\alpha, \in \vee q_k)$ -IFLI (resp, RI) of S .

Proof. i) (For $\alpha = q$), let $x, y \in S$ and $t \in (0, 1]$ and $s \in [0, 1]$ such that $y(t, s) q_k A$. Then, $\mu_A(y) + t > 1 - k$ and $\lambda_A(y) + s < 1 + k$. So, $y \in L$. Therefore, $xy \in L$. Thus, if $t \leq \frac{1-k}{2}$ and $s \geq \frac{1+k}{2}$, Then $\mu_A(xy) \geq \frac{1-k}{2} \geq t$ and $\lambda_A(xy) \leq \frac{1+k}{2} \leq s$ and so $(xy)(t, s) \in A$. If $t > \frac{1-k}{2}$ and $s < \frac{1+k}{2}$, then $\mu_A(xy) + t > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$ and $\lambda_A(xy) + s < \frac{1+k}{2} + \frac{1+k}{2} = 1 + k$ and so $(x, y)(t, s) \in q_k A$. Therefore, $(x, y)(t, s) \in \vee q_k A$. Hence, $A = \langle \mu_A, \lambda_A \rangle$ is an $(q, \in \vee q_k)$ - IFLI of S .

(ii) (For $\alpha = \in$), let $x, y \in S$ and $t \in (0, 1]$ and $s \in [0, 1]$ such that $y(t, s) \in A$. Then, $\mu_A(y) \geq t$ and $\lambda_A(y) \leq s$. So, $y \in L$. Therefore, $xy \in L$. Thus, if $t \leq \frac{1-k}{2}$ and $s \geq \frac{1+k}{2}$, then $\mu_A(xy) \geq \frac{1-k}{2} \geq t$ and $\lambda_A(y) \leq \frac{1+k}{2} \leq s$ and so $(xy)(t, s) \in A$. If $t > \frac{1-k}{2}$ and $s < \frac{1+k}{2}$, then $\mu_A(xy) + t > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$ and $\lambda_A(y) + s < \frac{1+k}{2} + \frac{1+k}{2} = 1 + k$ and so $(xy)(t, s) q_k A$. Therefore, $(xy)(t, s) \in \vee q_k A$. Hence, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q_k)$ - IFLI of S .

iii) (For $\alpha = \in \vee q_k$), follows from (i) and (ii). This completes the proof. $\square$

Theorem 3.7. Let B be a subsemigroup of S and let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS such that

a) $(\forall x \in s \setminus B) (\mu_A(x) = 0 \text{ and } \lambda_A(x) = 1)$,

b) $(\forall x \in B) (\mu_A(x) \geq \frac{1-k}{2} \text{ and } \lambda_A(x) \leq \frac{1+k}{2})$.

Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\alpha, \in \vee q_k)$ - IFSG of S .

Proof. Straightforward. $\square$

Theorem 3.8. Let B be a bi-ideal of S and let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS of S such that

a) $(\forall x \in s \setminus B) (\mu_A(x) = 0 \text{ and } \lambda_A(x) = 1)$,

b) $(\forall x \in B) (\mu_A(x) \geq \frac{1-k}{2} \text{ and } \lambda_A(x) \leq \frac{1+k}{2})$.

Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\alpha, \in \vee q_k)$ - IFBI of S .

Proof. i) (For $\alpha = q_k$), let $x, y \in S$ and $t_1, t_2 \in (0, 1]$ and $s_1, s_2 \in [0, 1]$ be such that $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A$. Then, $x, y \in B$. Since B is subsemigroup. So, $xy \in B$. If $m(t_1, t_2) > \frac{1-k}{2}$ and $M(s_1, s_2) < \frac{1+k}{2}$, then $\mu_A(xy) + m(t_1, t_2) > 1 - k$ and $\lambda_A(xy) + M(s_1, s_2) < 1 + k$,
So, $(xy)(m(t_1, t_2), M(s_1, s_2)) q_k A$.

If $m(t_1, t_2) \leq \frac{1-k}{2}$ and $M(s_1, s_2) \geq \frac{1+k}{2}$,

then $(xy)(m(t_1, t_2), M(s_1, s_2)) \in A$. Hence, $A = \langle \mu_A, \lambda_A \rangle$ is a $(q_k, \in \vee q_k)$ - IFSG of S . Let $x, y, z \in S$ and $t_1, t_2 \in (0, 1]$ and $s_1, s_2 \in [0, 1]$ be such that $x(t_1, s_1) \in A$ and $z(t_2, s_2) \in A$. Then $x, z \in B$. Since B is a bi-ideal. So, $xyz \in B$. If $m(t_1, t_2) > \frac{1-k}{2}$ and $M(s_1, s_2) < \frac{1+k}{2}$, then $\mu_A(xyz) + m(t_1, t_2) > 1 - k$ and $\lambda_A(xyz) + M(s_1, s_2) < 1 + k$.

So, $(xyz)(m(t_1, t_2), M(s_1, s_2)) q_k A$.

If $m(t_1, t_2) \leq \frac{1-k}{2}$ and $M(s_1, s_2) \geq \frac{1+k}{2}$,

then $(xyz)(m(t_1, t_2), M(s_1, s_2)) \in A$.

Therefore, $(xyz)(m(t_1, t_2), M(s_1, s_2)) \vee \in q_k A$. Hence, $A = \langle \mu_A, \lambda_A \rangle$ is a $(q_k, \in \vee q_k)$ - IFBI of S .

iii) (For $\alpha = \in$ and $\in \vee q_k$), straightforward. $\square$

Theorem 3.9. Let B be a $(1, 2)$ ideal of a semigroup S and let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS of S such that

a) $(\forall x \in s \setminus B) (\mu_A(x) = 0 \text{ and } \lambda_A(x) = 1)$,

b) $(\forall x \in B) (\mu_A(x) \geq \frac{1-k}{2} \text{ and } \lambda_A(x) \leq \frac{1+k}{2})$.

Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\alpha, \in \vee q_k)$ - IF $(1, 2)$ ideal of S .

Proof. Proof follow from theorem 3.6 $\square$

4. IFBI of type $(\in, \in \vee q_k)$

Definition 4.1. An IFS $A = \langle \mu_A, \lambda_A \rangle$ in semigroup S is said to be an $(\in, \in \vee q_k)$ - IFBI of semigroup S if $\forall x, y, a \in S, t_1, t_2 \in (0, 1]$ and $s_1, s_2 \in [0, 1]$, the following conditions hold.

(IFB3) $x(t_1, s_1) \in A$

and $y(t_2, s_2) \in A \implies (xy)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in \vee q_k A$.

(IFB4) $x(t_1, s_1) \in A$

and $y(t_2, s_2) \in A \implies (xay)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in \vee q_k A$.

Definition 4.2. An IFS $A = \langle \mu_A, \lambda_A \rangle$ in a semigroup S is said to be an $(\in, \in \vee q_k)$ - IF $(1, 2)$ ideal of a semigroup S if for all $x, y, z, a \in S, t_1, t_2 \in (0, 1]$ and $s_1, s_2 \in [0, 1]$, the following conditions hold:

(IFB3) $x(t_1, s_1) \in A$

and $y(t_2, s_2) \in A \implies (xy)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in \vee q_k A$.

(IFB4) $x(t_1, s_1) \in A$

and $y(t_2, s_2) \in A \implies (xa(yz))(m\{t_1, t_2\}, M\{s_1, s_2\}) \in \vee q_k A$.

Proposition 4.3. An IFS $A = \langle \mu_A, \lambda_A \rangle$ of a semigroup S is an IF subsemigroup if and only if it satisfy for all $x, y \in S$ and $t_1, t_2 \in (0, 1], x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A \implies (xy)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in A$.

Proof. Let us suppose that $A = \langle \mu_A, \lambda_A \rangle$ is an IFSG of S . Let $x, y \in S, t_1, t_2 \in (0, 1]$ and $s_1, s_2 \in [0, 1]$ let $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A$. Then, $\mu_A(x) \geq t_1$ and $\lambda_A(x) \leq s_1$ and $\mu_A(y) \geq t_2$ and $\lambda_A(y) \leq s_2$. Since by given condition
 $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$ and
 $\lambda_A(xy) \leq \max\{\lambda_A(x), \lambda_A(y)\}$.
 $\mu_A(xy) \geq m\{t_1, t_2\}$ and $\lambda_A(xy) \leq \{s_1, s_2\}$.

So, $(xy)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in A$. Thus, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in)$ - IFSG of S .

Conversely, suppose that $A = \langle \mu_A, \lambda_A \rangle$ is satisfy the given condition. We show that $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\lambda_A(xy) \leq \max\{\lambda_A(x), \lambda_A(y)\}$. On contrary assume that there exist $x, y \in S$ such that $\mu_A(xy) < \min\{\mu_A(x), \mu_A(y)\}$ and $\lambda_A(xy) > \max\{\lambda_A(x), \lambda_A(y)\}$. Let $t \in (0, 1]$ and $s \in [0, 1)$ be such that $\mu_A(xy) < t < \min\{\mu_A(x), \mu_A(y)\}$ and $\lambda_A(xy) > s > \max\{\lambda_A(x), \lambda_A(y)\}$. Then, $x(t, s) \in A$ and $y(t, s) \in A$ but $(xy)(t, s) \notin A$, which contradicts our hypothesis. Hence, $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\lambda_A(xy) \leq \max\{\lambda_A(x), \lambda_A(y)\}$. $\square$

Proposition 4.4. An IFS $A = \langle \mu_A, \lambda_A \rangle$ of a semigroup S is an IFBI of S if and only if it satisfy for all $x, y, z \in S$ and $t_1, t_2 \in (0, 1]$ and $s_1, s_2 \in [0, 1)$,

- (a) $x(t_1, s_1) \in A$
and $y(t_2, s_2) \in A \implies (xy)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in A$,
(b) $x(t_1, s_1) \in A$
and $z(t_2, s_2) \in A \implies (xyz)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in A$.

Proof. Proof follows from proposition 4.1 $\square$

Theorem 4.5. Let $A = \langle \mu_A, \lambda_A \rangle$ be IFS in semigroup S . Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q_k)$ IFBI of semigroup S if and only if the following condition are hold;

- (a) $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\}$
and $\lambda_A(xy) \leq \max\{\lambda_A(x), \lambda_A(y), \frac{1+k}{2}\}$.
(b) $\mu_A(xay) \geq \min\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\}$
and $\lambda_A(xay) \leq \max\{\lambda_A(x), \lambda_A(y), \frac{1+k}{2}\}$.

Proof. Suppose that $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q_k)$ - IFBI of semigroup S .

(a) Let $x, y \in S$. We consider the following cases:

1. $\min\{\mu_A(x), \mu_A(y)\} < \frac{1-k}{2}$
and $\max\{\lambda_A(x), \lambda_A(y)\} > \frac{1+k}{2}$
2. $\min\{\mu_A(x), \mu_A(y)\} \geq \frac{1-k}{2}$
and $\max\{\lambda_A(x), \lambda_A(y)\} \leq \frac{1+k}{2}$.

case(1)

Assume that $\mu_A(xy) < \min\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\}$
and $\lambda_A(xy) > \max\{\lambda_A(x), \lambda_A(y), \frac{1+k}{2}, \frac{1+k}{2}\}$.

Then, $\mu_A(xy) < \min\{\mu_A(x), \mu_A(y)\}$
and $\lambda_A(xy) > \max\{\lambda_A(x), \lambda_A(y)\}$.

choose $t \in (0, 1]$ and $s \in [0, 1)$

such that $\mu_A(xy) < t < \min\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\}$
and $\lambda_A(xy) > s > \max\{\lambda_A(x), \lambda_A(y)\}$.

Then $x(t, s) \in A$ and $y(t, s) \in A$, but $(xy)(t, s) \notin \overline{\vee q_k A}$, a contadiction,

case(2)

Assume that $\mu_A(xy) < \frac{1-k}{2}$ and $\lambda_A(xy) > \frac{1+k}{2}$.

Then, $x(\frac{1-k}{2}, \frac{1+k}{2}) \in A$ and $y(\frac{1-k}{2}, \frac{1+k}{2}) \in A$,
but $(xy)(\frac{1-k}{2}, \frac{1+k}{2}) \notin \overline{\vee q_k A}$, a contradiction.

Therefore, $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\}$
and $\lambda_A(xy) \leq \max\{\lambda_A(x), \lambda_A(y), \frac{1+k}{2}\}$.

(b) Now, let $x, y, a \in S$. We consider the following cases

1. $\min\{\mu_A(x), \mu_A(y)\} < \frac{1-k}{2}$
and $\max\{\lambda_A(x), \lambda_A(y)\} > \frac{1+k}{2}$.
2. $\min\{\mu_A(x), \mu_A(y)\} \geq \frac{1-k}{2}$
and $\max\{\lambda_A(x), \lambda_A(y)\} \leq \frac{1+k}{2}$.

Case(1) Assume that

$\mu_A(xay) < \min\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\}$
and $\lambda_A(xay) > \max\{\lambda_A(x), \lambda_A(y), \frac{1+k}{2}\}$

$\mu_A(xay) < \min\{\mu_A(x), \mu_A(y)\}$
and $\lambda_A(xay) > \max\{\lambda_A(x), \lambda_A(y)\}$,
choose $t \in (0, 1]$ and $s \in [0, 1)$ such that

$\mu_A(xay) < t < \min\{\mu_A(x), \mu_A(y)\}$
and $\lambda_A(xay) > s > \max\{\lambda_A(x), \lambda_A(y)\}$,

Then $x(t, s) \in A$ and $y(t, s) \in A$,

but $(xay)(t, s) \notin \overline{\vee q_k A}$, which is a contradiction.

Case(2) Assume that $\mu_A(xay) < \frac{1-k}{2}$

and

$\lambda_A(xay) > \frac{1+k}{2}$. Then, $x(\frac{1-k}{2}, \frac{1+k}{2}) \in A$ and $y(\frac{1-k}{2}, \frac{1+k}{2}) \in A$,
but $(xay)(\frac{1-k}{2}, \frac{1+k}{2}) \notin \overline{\vee q_k A}$, which is a contradiction.

Therefore, $\mu_A(xay) \geq \min\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\}$

and $\lambda_A(xay) \leq \max\{\lambda_A(x), \lambda_A(y), \frac{1+k}{2}\}$

Conversely, assume that $A = \langle \mu_A, \lambda_A \rangle$ satisfy (a) and (b). Let for any $x, y \in S$ and $t_1, t_2 \in (0, 1]$ and $s_1, s_2 \in [0, 1)$, such that $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A$. Then, $\mu_A(x) \geq t_1$ and $\lambda_A(x) \leq s_1$, and $\mu_A(y) \geq t_2$ and $\lambda_A(y) \leq s_2$. Now we have

$\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\}$

and $\lambda_A(xy) \leq \max\{\lambda_A(x), \lambda_A(y), \frac{1-k}{2}\}$

$\mu_A(xy) \geq \min\{t_1, t_2, \frac{1-k}{2}\}$ and $\lambda_A(xy) \leq \max\{s_1, s_2, \frac{1+k}{2}\}$

Then, we have the following case's

1. $\min\{t_1, t_2\} \leq \frac{1-k}{2}$ and $\max\{s_1, s_2\} \geq \frac{1+k}{2}$

2. $\min\{t_1, t_2\} > \frac{1-k}{2}$ and $\max\{s_1, s_2\} < \frac{1+k}{2}$

Case (1) If $\min\{t_1, t_2\} \leq \frac{1-k}{2}$ and $\max\{s_1, s_2\} \geq \frac{1+k}{2}$. Then,
 $\mu_A(xy) \geq \min\{t_1, t_2\}$ and $\lambda_A(xy) \leq \max\{s_1, s_2\}$,
which implies that $(xy)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in A$.

Case (2) If $\min\{t_1, t_2\} > \frac{1-k}{2}$ and $\max\{s_1, s_2\} < \frac{1+k}{2}$.

Then, $\mu_A(xy) \geq \frac{1-k}{2}$ and

$\lambda_A(xy) \leq \frac{1+k}{2}$, which implies that $\mu_A(xy) + \min\{t_1, t_2\} > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$ and $\lambda_A(xy) + \max\{s_1, s_2\} < \frac{1+k}{2} + \frac{1+k}{2} = 1 + k$. Therefore, $(xy)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in q_k A$.

Hence, $(xy)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in \overline{\vee q_k A}$.

Let $x, y, a \in S$ and $t_1, t_2 \in (0, 1]$ and $s_1, s_2 \in [0, 1)$ such that $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A$. Then, $\mu_A(x) \geq t_1$ and $\lambda_A(x) \leq s_1$, and $\mu_A(y) \geq t_2$ and $\lambda_A(y) \leq s_2$. Now we have

$\mu_A(xay) \geq \min\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\}$

and $\lambda_A(xay) \leq \max\{\lambda_A(x), \lambda_A(y), \frac{1+k}{2}\}$

$\mu_A(xay) \geq \min\{t_1, t_2, \frac{1-k}{2}\}$

and $\lambda_A(xay) \leq \max\{s_1, s_2, \frac{1+k}{2}\}$

Then, we have the following case's

3. $\min\{t_1, t_2\} \leq \frac{1-k}{2}$ and $\max\{s_1, s_2\} \geq \frac{1+k}{2}$

4. $\min\{t_1, t_2\} > \frac{1-k}{2}$ and $\max\{s_1, s_2\} < \frac{1+k}{2}$.

Case (3) If $\min\{t_1, t_2\} \leq \frac{1-k}{2}$ and $\max\{s_1, s_2\} \geq \frac{1+k}{2}$. Then,
 $\mu_A(xay) \geq \min\{t_1, t_2\}$ and $\lambda_A(xay) \leq \max\{s_1, s_2\}$,

which implies that $(xay)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in A$.

Case (4) If $\min\{t_1, t_2\} > \frac{1-k}{2}$ and $\max\{s_1, s_2\} < \frac{1+k}{2}$. Then, $\mu_A(xay) \geq \frac{1-k}{2}$ and $\lambda_A(xay) \leq \frac{1+k}{2}$,

which implies that $\mu_A(xay) + \min\{t_1, t_2\} > \frac{1-k}{2} + \frac{1-k}{2} = 1-k$ and $\lambda_A(xay) + \max\{s_1, s_2\} < \frac{1+k}{2} + \frac{1+k}{2} = 1+k$. Therefore, $(xay)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in q_k A$.

Hence, $(xay)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in \vee q_k A$.

This completes the Proof. $\square$

Remark 4.6. Every IFBI is an $(\in, \in \vee q_k)$ – IFBI of semi-group S . But the converse need not be true, see example.

Example 4.7. Let $S = \{1, 2, 3, 4\}$ be a semigroup defined by the following Cayley table.

.	1	2	3	4
1	1	1	1	1
2	1	1	4	1
3	1	1	1	1
4	1	1	1	1

Let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS in a semigroup S defined by

$\mu_A(1) = 0.3, \mu_A(2) = 0.8, \mu_A(3) = \mu_A(4) = 0.6, k = 0.6$ and

$\lambda_A(1) = 0.7, \lambda_A(2) = 0.2,$

$\lambda_A(3) = \lambda_A(4) = 0.5$. Then,

$A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q_k)$ – IFBI of S but not $(\in, \in \vee q)$ – IFBI.

Remark 4.8. From above Remark 4.1 and Example 4.1, we can say that $(\in, \in \vee q_k)$ – IFBI of S is a generalization of an IFBI of S .

Theorem 4.9. Let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS in a semigroup S . Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q_k)$ – IF $(1, 2)$ ideal of a semigroup S if and only if the following conditions hold;

1) $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\}$

and $\lambda_A(xy) \leq \max\{\lambda_A(x), \lambda_A(y), \frac{1+k}{2}\}$

2) $\mu_A(xa(yz)) \geq \min\{\mu_A(x), \mu_A(y), \mu_A(z), \frac{1-k}{2}\}$ and

$\lambda_A(xa(yz)) \leq \max\{\lambda_A(x), \lambda_A(y), \lambda_A(z), \frac{1+k}{2}\}$.

Proof: Straightforward.

Proposition 4.10. Every $(\in, \in \vee q_k)$ – IF left(right) ideal of S is an $(\in, \in \vee q_k)$ – IFBI of S .

Proof: Let $A = \langle \mu_A, \lambda_A \rangle$ be an $(\in, \in \vee q_k)$ – IFBI of a semi-group S . Then for any $x, y \in S$.

$$\begin{aligned} \mu_A(xy) &\geq \min\left\{\mu_A(y), \frac{1-k}{2}\right\} \\ &\geq \min\left\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\right\}, \end{aligned}$$

and

$$\begin{aligned} \lambda_A(xy) &\leq \max\left\{\lambda_A(y), \frac{1+k}{2}\right\} \\ &\leq \max\left\{\lambda_A(x), \lambda_A(y), \frac{1+k}{2}\right\} \end{aligned}$$

Now let $x, y, z \in S$

$$\begin{aligned} \mu_A(xyz) &= \mu_A((xy)z) \\ &\geq \min\left\{\mu_A(z), \frac{1-k}{2}\right\} \\ &\geq \min\left\{\mu_A(x), \mu_A(z), \frac{1-k}{2}\right\}, \end{aligned}$$

and

$$\begin{aligned} \lambda_A(xyz) &= \lambda_A((xy)z) \\ &\leq \max\left\{\lambda_A(z), \frac{1+k}{2}\right\} \\ &\leq \max\left\{\lambda_A(x), \lambda_A(z), \frac{1+k}{2}\right\} \end{aligned}$$

Therefore $A = \langle \mu_A, \lambda_A \rangle$ be an $(\in, \in \vee q_k)$ – IFBI of S .

Remark 4.11. If $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q_k)$ – IFBI of S , then $A = \langle \mu_A, \lambda_A \rangle$ need not to be an $(\in, \in \vee q_k)$ – IF- left (right) ideal of S . It is illustrated in the following example.

Example 4.12. Let $S = \{1, 2, 3, 4, 5\}$ be a semigroup with the following Cayley table.

.	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	3	3	5
4	1	1	3	4	5
5	1	1	3	3	5

Let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS defined as $\mu_A(1) = 0.3,$

$\mu_A(2) = 0.2, \mu_A(3) = 0.6, \mu_A(4) = 0.6, \mu_A(5) = 0.2, k = 0.2$

and $\lambda_A(1) = 0.7, \lambda_A(3) = 0.7, \lambda_A(2) = 0.8, \lambda_A(4) = 0.7,$

$\lambda_A(5) = 0.7$.

Then, clearly $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q_k)$ – IFBI of S but not $(\in, \in \vee q_k)$ – IFBI of S because

$\mu_A(23) \not\geq \min\{\mu_A(3), 0.4\}$.

Proposition 4.13. 1. Every $(\in, \vee q_k, \in \vee q_k)$ – IFBI of S is $(\in, \in \vee q_k)$ – IFBI.

2. Every $(\in, \in)$ – IFBI of S

$(\in, \in \vee q_k)$ – IFBI.

proof: Straightforward.

Example 4.1 shows that the converse of proposition 4.4, is not true in general.

Theorem 4.14. If $\{A_i\}_{i \in \Lambda}$ is a family of

$(\in, \in \vee q_k)$ – IFBI of S , then $\bigcap_{i \in \Lambda} A_i$ is an

$(\in, \in \vee q_k)$ – IFBI of S . Where $\bigcap_{i \in \Lambda} A_i = \langle \bigwedge_{i \in \Lambda} \mu_{A_i}, \bigvee_{i \in \Lambda} \lambda_{A_i} \rangle$.

Proof: Let $x, y \in S$. Then, we have

$$\begin{aligned}
 (\wedge_{i \in \Lambda} \mu_{A_i})(xy) &= \wedge_{i \in \Lambda} (\mu_{A_i}(xy)) \\
 &\geq \wedge_{i \in \Lambda} \left(\min \left\{ \mu_{A_i}(x), \mu_{A_i}(y), \frac{1-k}{2} \right\} \right) \\
 &= \min \left\{ \wedge_{i \in \Lambda} \mu_{A_i}(x), \wedge_{i \in \Lambda} \mu_{A_i}(y), \frac{1-k}{2} \right\} \\
 (\wedge_{i \in \Lambda} \mu_{A_i})(xy) &\geq \min \left\{ (\wedge_{i \in \Lambda} \mu_{A_i})(x), (\wedge_{i \in \Lambda} \mu_{A_i})(y), \frac{1-k}{2} \right\} \\
 (\vee_{i \in \Lambda} \lambda_{A_i})(xy) &= \vee_{i \in \Lambda} (\lambda_{A_i}(xy)) \\
 &\leq \vee_{i \in \Lambda} \left(\max \left\{ \lambda_{A_i}(x), \lambda_{A_i}(y), \frac{1+k}{2} \right\} \right) \\
 &= \max \left\{ \vee_{i \in \Lambda} \lambda_{A_i}(x), \vee_{i \in \Lambda} \lambda_{A_i}(y), \frac{1+k}{2} \right\} \\
 (\vee_{i \in \Lambda} \lambda_{A_i})(xy) &\leq \max \left\{ (\vee_{i \in \Lambda} \lambda_{A_i})(x), (\vee_{i \in \Lambda} \lambda_{A_i})(y), \frac{1+k}{2} \right\}
 \end{aligned}$$

Now let $x, y, a \in S$. Then, we have

$$\begin{aligned}
 (\wedge_{i \in \Lambda} \mu_{A_i})(xay) &= \wedge_{i \in \Lambda} (\mu_{A_i}(xay)) \\
 &\geq \wedge_{i \in \Lambda} \left(\min \left\{ \mu_{A_i}(x), \mu_{A_i}(y), \frac{1-k}{2} \right\} \right) \\
 &= \min \left\{ \wedge_{i \in \Lambda} \mu_{A_i}(x), \wedge_{i \in \Lambda} \mu_{A_i}(y), \frac{1-k}{2} \right\} \\
 (\wedge_{i \in \Lambda} \mu_{A_i})(xay) &\geq \min \left\{ (\wedge_{i \in \Lambda} \mu_{A_i})(x), (\wedge_{i \in \Lambda} \mu_{A_i})(y), \frac{1-k}{2} \right\} \\
 (\vee_{i \in \Lambda} \lambda_{A_i})(xay) &= \vee_{i \in \Lambda} (\lambda_{A_i}(xay)) \\
 &\leq \vee_{i \in \Lambda} \left(\max \left\{ \lambda_{A_i}(x), \lambda_{A_i}(y), \frac{1+k}{2} \right\} \right) \\
 &= \max \left\{ \vee_{i \in \Lambda} \lambda_{A_i}(x), \vee_{i \in \Lambda} \lambda_{A_i}(y), \frac{1+k}{2} \right\} \\
 (\vee_{i \in \Lambda} \lambda_{A_i})(xay) &\leq \max \left\{ (\vee_{i \in \Lambda} \lambda_{A_i})(x), (\vee_{i \in \Lambda} \lambda_{A_i})(y), \frac{1+k}{2} \right\}
 \end{aligned}$$

Hence $\cap_{i \in \Lambda} A_i = \langle \wedge_{i \in \Lambda} \mu_{A_i}, \vee_{i \in \Lambda} \lambda_{A_i} \rangle$ is an $(\in, \in \vee q_k)$ – IFBI of S .

Remark 4.15. If $\{A_i\}_{i \in \Lambda}$ is a family of $(\in, \in \vee q_k)$ – IFBI of S , then $\cup_{i \in \Lambda} A_i$ is not an $(\in, \in \vee q_k)$ – IFBI of S .
Where $\cup_{i \in \Lambda} A_i = \langle \vee_{i \in \Lambda} \mu_{A_i}, \wedge_{i \in \Lambda} \lambda_{A_i} \rangle$. It is illustrated by an example.

Example 4.16. Let $S = \{1, 2, 3, 4\}$ be a semigroup with the following Cayley table.

.	1	2	3	4
1	1	1	1	1
2	1	1	4	1
3	1	1	1	1
4	1	1	1	1

Let $A = \langle \mu_A, \lambda_A \rangle$ and $B = \langle \mu_B, \lambda_B \rangle$ be IFS's of semigroup S defined by $\mu_A(1) = 0.4, \mu_A(2) = 0.4, \mu_A(3) = 0, \mu_A(4) = 0$ and $\mu_B(1) = 0.4, \mu_B(2) = 0, \mu_B(3) = 0.4, \mu_B(4) = 0$ and $k = 0.2$ and $\lambda_A(1) = 0.5, \lambda_A(2) = 0.5, \lambda_A(3) = 0.8, \lambda_A(4) = 0.8$ and $\lambda_B(1) = 0.5, \lambda_B(2) = 0.7, \lambda_B(3) = 0.5, \lambda_B(4) = 0.7$.

Then both $A = \langle \mu_A, \lambda_A \rangle$ and $B = \langle \mu_B, \lambda_B \rangle$ are $(\in, \in \vee q_k)$ – IFBI of S . But $A \cup B$ is not an $(\in, \in \vee q_k)$ – IFBI of S .

(i.e) $(\mu_A \vee \mu_B)(23) = 0$

and $\min \{(\mu_A \vee \mu_B)(2), (\mu_A \vee \mu_B)(3), 0.4\} = 0.4$

Hence $(\mu_A \vee \mu_B)(23) \not\geq \min \{(\mu_A \vee \mu_B)(2), (\mu_A \vee \mu_B)(3), 0.4\}$.

Theorem 4.17. If $\{A_i\}_{i \in \Lambda}$ is a family of

$(\in, \in \vee q_k)$ – IFBI of S such that

$A_i \subseteq A_j$ or

$A_j \subseteq A_i$ for all $i, j \in I$, then $\cup_{i \in \Lambda} A_i = \langle \vee_{i \in \Lambda} \mu_{A_i}, \wedge_{i \in \Lambda} \lambda_{A_i} \rangle$ is an $(\in, \in \vee q_k)$ – IFBI of S .

Proof. : For all $x, y \in S$, we have

$$\begin{aligned}
 \left(\bigvee_{i \in \Lambda} \mu_{A_i} \right)(xy) &= \bigvee_{i \in \Lambda} (\mu_{A_i}(xy)) \\
 &\geq \bigvee_{i \in \Lambda} \left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge \frac{1-k}{2} \right] \\
 &= \left[\bigvee_{i \in \Lambda} \mu_{A_i}(x) \wedge \bigvee_{i \in \Lambda} \mu_{A_i}(y) \wedge \frac{1-k}{2} \right] \\
 &= \left[\left(\bigvee_{i \in \Lambda} \mu_{A_i} \right)(x) \wedge \left(\bigvee_{i \in \Lambda} \mu_{A_i} \right)(y) \wedge \frac{1-k}{2} \right] \\
 \left(\bigvee_{i \in \Lambda} \mu_{A_i} \right)(xy) &\geq \left[\left(\bigvee_{i \in \Lambda} \mu_{A_i} \right)(x) \wedge \left(\bigvee_{i \in \Lambda} \mu_{A_i} \right)(y) \wedge \frac{1-k}{2} \right]
 \end{aligned}$$

It is clear that

$$\begin{aligned}
 &\bigvee_{i \in \Lambda} \left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge \frac{1-k}{2} \right] \\
 &\leq \left[\left(\bigvee_{i \in \Lambda} \mu_{A_i} \right)(x) \wedge \left(\bigvee_{i \in \Lambda} \mu_{A_i} \right)(y) \wedge \frac{1-k}{2} \right]
 \end{aligned}$$

Assume that

$$\begin{aligned}
 &\bigvee_{i \in \Lambda} \left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge \frac{1-k}{2} \right] \\
 &\neq \left[\left(\bigvee_{i \in \Lambda} \mu_{A_i} \right)(x) \wedge \left(\bigvee_{i \in \Lambda} \mu_{A_i} \right)(y) \wedge \frac{1-k}{2} \right]
 \end{aligned}$$

Then, there exist t such that

$$\begin{aligned}
 &\bigvee_{i \in \Lambda} \left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge \frac{1-k}{2} \right] \\
 &< t < \left[\left(\bigvee_{i \in \Lambda} \mu_{A_i} \right)(x) \wedge \left(\bigvee_{i \in \Lambda} \mu_{A_i} \right)(y) \wedge \frac{1-k}{2} \right]
 \end{aligned}$$

Since $\mu_{A_i} \subseteq \mu_{A_j}$ or $\mu_{A_j} \subseteq \mu_{A_i}$ for all $i, j \in I$, so there exist $s \in I$ such that $t < \mu_{A_s}(x) \wedge \mu_{A_s}(y) \wedge \frac{1-k}{2}$. On the other hand $\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge \frac{1-k}{2} < t$ for all $i \in I$, a contradiction.

Hence,

$$\begin{aligned}
 &\bigvee_{i \in \Lambda} \left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge \frac{1-k}{2} \right] \\
 &= \left[\left(\bigvee_{i \in \Lambda} \mu_{A_i} \right)(x) \wedge \left(\bigvee_{i \in \Lambda} \mu_{A_i} \right)(y) \wedge \frac{1-k}{2} \right]
 \end{aligned}$$

and

$$\begin{aligned} \left(\bigwedge_{i \in \Lambda} \lambda_{A_i} \right) (xy) &= \bigwedge_{i \in \Lambda} (\lambda_{A_i}(xy)) \\ &\geq \bigwedge_{i \in \Lambda} \left[\lambda_{A_i}(x) \vee \lambda_{A_i}(y) \vee \frac{1+k}{2} \right] \\ &= \left[\bigwedge_{i \in \Lambda} \lambda_{A_i}(x) \vee \bigwedge_{i \in \Lambda} \lambda_{A_i}(y) \vee \frac{1+k}{2} \right] \\ &= \left[\left(\bigwedge_{i \in \Lambda} \lambda_{A_i} \right) (x) \vee \left(\bigwedge_{i \in \Lambda} \lambda_{A_i} \right) (y) \vee \frac{1+k}{2} \right] \\ \left(\bigwedge_{i \in \Lambda} \lambda_{A_i} \right) (xy) &\leq \left[\left(\bigwedge_{i \in \Lambda} \lambda_{A_i} \right) (x) \vee \left(\bigwedge_{i \in \Lambda} \lambda_{A_i} \right) (y) \vee \frac{1+k}{2} \right] \end{aligned}$$

It is clear that

$$\begin{aligned} &\bigwedge_{i \in \Lambda} \left[\lambda_{A_i}(x) \vee \lambda_{A_i}(y) \vee \frac{1+k}{2} \right] \\ &\geq \left[\left(\bigwedge_{i \in \Lambda} \lambda_{A_i} \right) (x) \vee \left(\bigwedge_{i \in \Lambda} \lambda_{A_i} \right) (y) \vee \frac{1+k}{2} \right] \end{aligned}$$

Assume that

$$\begin{aligned} &\bigwedge_{i \in \Lambda} \left[\lambda_{A_i}(x) \vee \lambda_{A_i}(y) \vee \frac{1+k}{2} \right] \\ &\neq \left[\left(\bigwedge_{i \in \Lambda} \lambda_{A_i} \right) (x) \vee \left(\bigwedge_{i \in \Lambda} \lambda_{A_i} \right) (y) \vee \frac{1+k}{2} \right] \end{aligned}$$

Then, there exist t such that

$$\begin{aligned} &\bigwedge_{i \in \Lambda} \left[\lambda_{A_i}(x) \vee \lambda_{A_i}(y) \vee \frac{1+k}{2} \right] \\ &> t > \left[\left(\bigwedge_{i \in \Lambda} \lambda_{A_i} \right) (x) \vee \left(\bigwedge_{i \in \Lambda} \lambda_{A_i} \right) (y) \vee \frac{1+k}{2} \right] \end{aligned}$$

Since $\lambda_{A_i} \subseteq \lambda_{A_j}$ or $\lambda_{A_j} \subseteq \lambda_{A_i}$ for all $i, j \in I$, so there exist $s \in I$ such that $t > \lambda_{A_s}(x) \vee \lambda_{A_s}(y) \vee \frac{1+k}{2}$. On the other hand $\lambda_{A_i}(x) \vee \lambda_{A_i}(y) \vee \frac{1+k}{2} < t$ for all $i \in I$, a contradiction.

Hence,

$$\begin{aligned} &\bigwedge_{i \in \Lambda} \left[\lambda_{A_i}(x) \vee \lambda_{A_i}(y) \vee \frac{1+k}{2} \right] \\ &= \left[\left(\bigwedge_{i \in \Lambda} \lambda_{A_i} \right) (x) \vee \left(\bigwedge_{i \in \Lambda} \lambda_{A_i} \right) (y) \vee \frac{1+k}{2} \right]. \end{aligned}$$

Let $x, a, y \in S$, we obtain

$$\begin{aligned} \left(\bigvee_{i \in \Lambda} \mu_{A_i} \right) (xay) &= \bigvee_{i \in \Lambda} (\mu_{A_i}(xay)) \\ &\geq \bigvee_{i \in \Lambda} \left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge \frac{1-k}{2} \right] \\ &= \left[\bigvee_{i \in \Lambda} \mu_{A_i}(x) \wedge \bigvee_{i \in \Lambda} \mu_{A_i}(y) \wedge \frac{1-k}{2} \right] \\ &= \left[\left(\bigvee_{i \in \Lambda} \mu_{A_i} \right) (x) \wedge \left(\bigvee_{i \in \Lambda} \mu_{A_i} \right) (y) \wedge \frac{1-k}{2} \right] \\ \left(\bigvee_{i \in \Lambda} \mu_{A_i} \right) (xay) &\geq \left[\left(\bigvee_{i \in \Lambda} \mu_{A_i} \right) (x) \wedge \left(\bigvee_{i \in \Lambda} \mu_{A_i} \right) (y) \wedge \frac{1-k}{2} \right] \end{aligned}$$

and

$$\begin{aligned} \left(\bigwedge_{i \in \Lambda} \lambda_{A_i} \right) (xay) &= \bigwedge_{i \in \Lambda} (\lambda_{A_i}(xay)) \\ &\geq \bigwedge_{i \in \Lambda} \left[\lambda_{A_i}(x) \vee \lambda_{A_i}(y) \vee \frac{1+k}{2} \right] \\ &= \left[\bigwedge_{i \in \Lambda} \lambda_{A_i}(x) \vee \bigwedge_{i \in \Lambda} \lambda_{A_i}(y) \vee \frac{1+k}{2} \right] \\ &= \left[\left(\bigwedge_{i \in \Lambda} \lambda_{A_i} \right) (x) \vee \left(\bigwedge_{i \in \Lambda} \lambda_{A_i} \right) (y) \vee \frac{1+k}{2} \right] \\ \left(\bigwedge_{i \in \Lambda} \lambda_{A_i} \right) (xay) &\leq \left[\left(\bigwedge_{i \in \Lambda} \lambda_{A_i} \right) (x) \vee \left(\bigwedge_{i \in \Lambda} \lambda_{A_i} \right) (y) \vee \frac{1+k}{2} \right] \end{aligned}$$

Hence $\bigcup_{i \in \Lambda} A_i = \langle \bigvee_{i \in \Lambda} \mu_{A_i}, \bigwedge_{i \in \Lambda} \lambda_{A_i} \rangle$, is an $(\in, \in \vee q_k)$ – IFBI of S . □

5. Conclusion

In this paper we have presented the notion of $(\varepsilon, \varepsilon \vee q_k)$ – IFBI in semigroups and derived some properties of these ideals.

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