Fuzzy almost quasi-ideals in semigroups

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Abstract

In this paper, we define fuzzy almost quasi-ideals in semigroups and give some relationship between almost quasi-ideals and fuzzy almost quasi-ideals of semigroups.

Keywords

Quasi-ideals, almost quasi-ideals, fuzzy almost quasi-ideals, minimal almost quasi-ideals, minimal fuzzy almost quasi-ideals.

AMS Subject Classification

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1. Introduction

In 1965, the definition of fuzzy subsets was introduced by Zadeh. This theory rapidly growing and having high impact on all the areas of science. It occupied a very important place in algebraic theory too. Various kinds of fuzzy ideals in semigroups were studied and characterized by Kuroki et.al,[8]. In 1952 R.A. Good and D.R.Hunghes[3] addressed the associated group for semigroup and studied bi-ideals in semigroup. Grosek and Satko [4–6] contributed several concepts smallest A-ideal and minimal A-ideal in semigroup. In 1981, Bogdanovic[2] introduced the notion of almost bi-ideal and K.Wattanatripop et.al,[10] extended this to fuzzy concept. The notion of quasi-ideal was first introduced for semigroup by Steinfeld[9]. J.Ahsan et.al,[1] extended it to fuzzy theory. Kuroki [7] ushered semi prime concept in quasi-ideal theory.

In this paper, we first give the notion of almost quasi-ideals in semigroups and some properties are developed.

2. Preliminaries

We recall some basic definitions and results related to this study.

The notion of quasi-ideals was introduced by O.Steinfeld [9] as follows:

Definition 2.1. Let S be a semigroup and Q be a subsemigroup of S . If $QS \cap SQ \subseteq Q$, then Q is called a quasi-ideal of S .

Example 2.2. Let $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. Then $\mathbb{Z}_4$ is a semigroup under the multiplication. Let $Q = \{0, 2\}$. Then Q is a subsemigroup of $\mathbb{Z}_4$ and $Q\mathbb{Z}_4 \cap \mathbb{Z}_4 Q = Q$. Hence Q is a quasi-ideal of S .

Fuzzy sets: In 1965, Zadeh [11] introduced the concept of a fuzzy subset of a set. Let X be any set. A function f from X to the unit interval $[0,1]$ is called a fuzzy subset of X . Let f and g be any two fuzzy subset of X . The fuzzy subsets $f \cap g, f \cup g$, of X and $f \subseteq g$ are defined as follows

$$(f \cap g)(x) = f(x) \wedge g(x),$$

$$(f \cup g)(x) = f(x) \vee g(x),$$

$$f \subseteq g, \text{ if } f(x) \leq g(x)$$

for all $x \in X$. Let f be a fuzzy subset of X . The support of f is defined by $\text{supp } f := \{x \in X | f(x) \neq 0\}$.

Let A be a non-empty subset of X . The characteristic mapping of A is a fuzzy subset of X defined by

$$C_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A \end{cases}$$

Let s be any element in X . The characteristic mapping of s in a fuzzy subset of X defined by

$$C_s(x) = \begin{cases} 1 & \text{if } x \in s, \\ 0 & \text{if } x \notin s \end{cases}$$

Next, we recall the definitions and propositions of fuzzy ideals in semigroups from [11].

Definition 2.3. Let f be a fuzzy subset of a semigroup S .

1. f is called a fuzzy subsemigroup of S if and only if $f(xy) \geq f(x) \wedge f(y)$.
2. f is called a fuzzy left ideal of S if and only if $f(xy) \geq f(y)$.
3. f is called a fuzzy right ideal of S if and only if $f(xy) \geq f(x)$.
4. f is called a fuzzy two-sided ideal of S if and only if $f(xy) \geq f(x) \vee f(y)$.

Let f and g be two fuzzy subsets of a semigroup S . A product $f \circ g$ is denoted by

$$(f \circ g)(x) = \begin{cases} \bigvee_{x=yz} \{f(y) \wedge g(z)\}, & \text{if } \exists y, z \in S, \\ & \text{such that } x = yz, \\ 0, & \text{otherwise} \end{cases}$$

Proposition 2.4. Let f be a fuzzy subset of a semigroup S . Then the following properties hold:

1. f is a fuzzy subsemigroup of S if and only if $f \circ f \subseteq f$.
2. f is a fuzzy left ideal of S if and only if $S \circ f \subseteq f$.
3. f is a fuzzy right ideal of S if and only if $f \circ S \subseteq f$.
4. f is a fuzzy two-sided ideal of S if and only if $S \circ f \subseteq f$ and $f \circ S \subseteq f$.

Definition 2.5. A fuzzy subset f of a semigroup S is called a fuzzy bi-ideal of S if

$$f(xyz) \geq f(x) \wedge f(z)$$

for all $x, y, z \in S$.

Proposition 2.6. Let f be any fuzzy subsemigroup of a semigroup S . Then f is a fuzzy quasi-ideal of S if and only if $f \circ S \cap S \circ f \subseteq f$

3. Almost quasi-ideals in semigroups

Definition 3.1. A non-empty subset Q of a semigroup S is called an almost quasi-ideal of S if $(QS \cap SQ) \cap Q \neq \emptyset$ for all $s \in S$.

Example 3.2. Consider the semigroup $\mathbb{Z}_4$ under the usual addition. Let $Q = \{0, 2, 3\}$.

Consider,

$$\begin{aligned} (Q + 0 \cap 0 + Q) \cap Q &= \{0, 2, 3\} \cap \{0, 2, 3\} = \{0, 2, 3\} \neq \emptyset \\ (Q + 1 \cap 1 + Q) \cap Q &= \{1, 3, 0\} \cap \{0, 2, 3\} = \{0, 3\} \neq \emptyset \\ (Q + 2 \cap 2 + Q) \cap Q &= \{2, 0, 1\} \cap \{0, 2, 3\} = \{0, 2\} \neq \emptyset \\ (Q + 3 \cap 3 + Q) \cap Q &= \{3, 1, 2\} \cap \{0, 2, 3\} = \{2, 3\} \neq \emptyset \end{aligned}$$

Then Q is an almost quasi-ideal of $\mathbb{Z}_4$.

Example 3.3. Consider the semigroup $S = \{a, b, c, d\}$ with the multiplication table.

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let $Q = \{a, b\}$. Then

$$\begin{aligned} (Qa \cap aQ) \cap Q &= [\{a, b\} \cap \{a, a\}] \cap \{a, b\} = \{a\} \neq \emptyset. \\ (Qb \cap bQ) \cap Q &= [\{a, a\} \cap \{b, a\}] \cap \{a, b\} = \{a\} \neq \emptyset. \\ (Qc \cap cQ) \cap Q &= [\{b, b\} \cap \{a, b\}] \cap \{a, b\} = \{b\} \neq \emptyset. \\ (Qd \cap dQ) \cap Q &= [\{b, b\} \cap \{a, a\}] \cap \{a, b\} = \{b\} \neq \emptyset. \end{aligned}$$

Q is an almost quasi-ideal of S .

Theorem 3.4. Every quasi-ideal of a semigroup S is an almost quasi-ideal of S .

Proof. Assume Q is a quasi-ideal of a semigroup S . Then $Qs \cap sQ \neq \emptyset$ and $Qs \cap sQ \subseteq Qs \cap sQ \subseteq Q$ for all $s \in S$. Hence $(Qs \cap sQ) \cap Q = Qs \cap sQ \neq \emptyset$. Therefore Q is an almost quasi-ideal of S . $\square$

Theorem 3.5. Let Q be an almost quasi-ideal of a semigroup S . Let Q' be a non-empty subset of S such that $Q \subset Q' \subseteq S$. Then Q' is an almost quasi-ideal of S .

Proof. Let Q be an almost quasi-ideal of a semigroup S . Let $Q' \subseteq S$ such that $Q \subseteq Q' \subseteq S$. Then $(Qs \cap sQ) \cap Q \subseteq (Q' \cap sQ') \cap Q' \subseteq S$ for all $s \in S$. Therefore $(Q' \cap sQ') \cap Q' \neq \emptyset$. Then Q' is an almost quasi-ideal of S . $\square$

Theorem 3.6. The union of two almost quasi-ideals of a semigroup S is an almost quasi-ideal of S .

Proof. Let Q_1 and Q_2 are almost quasi-ideals of a semigroup S . Then $(Q_1 \cap sQ_1) \cap Q_1 \neq \emptyset$ and $(Q_2 \cap sQ_2) \cap Q_2 \neq \emptyset$. As $\emptyset \neq (Q_1 \cap sQ_1) \cap Q_1 \subseteq ((Q_1 \cup Q_2) \cap s(Q_1 \cup Q_2)) \cap (Q_1 \cup Q_2)$. Then $((Q_1 \cup Q_2) \cap s(Q_1 \cup Q_2)) \cap (Q_1 \cup Q_2) \neq \emptyset$. Hence $Q_1 \cup Q_2$ is an almost quasi-ideal of S . $\square$

Example 3.7. Consider a semigroup $(\mathbb{Z}_5, +)$. We have $Q_1 = \{1, 3, 4\}$ and $Q_2 = \{0, 1, 4\}$ are almost quasi-ideals but $Q = Q_1 \cap Q_2 = \{1, 4\}$ is not an almost quasi-ideal of $\mathbb{Z}_5$. Because $(Q + 1 \cap 1 + Q) \cap Q = \{2, 0\}$. So, in general the intersection of two almost quasi-ideals need not be an almost quasi-ideal.

4. Fuzzy almost quasi-ideals

Definition 4.1. Let f be a fuzzy subset of a semigroup S such that $f \neq 0$. f is called a fuzzy almost quasi-ideal of S if for all $s \in S$, $(f \circ C_s \cap C_s \circ f) \cap f \neq 0$.

Theorem 4.2. Let f be a fuzzy almost quasi-ideal of a semigroup S and g be a fuzzy subset of S such that $f \subseteq g$, then g is a fuzzy almost quasi-ideal of S .

Proof. Assume f is a fuzzy almost quasi-ideal of a semigroup S and g is a fuzzy subset of S such that $f \subseteq g$. Then for all $s \in S$, $(f \circ C_s \cap C_s \circ f) \cap f \subseteq (g \circ C_s \cap C_s \circ g) \cap g$. Since $(f \circ C_s \cap C_s \circ f) \cap f \neq 0$, $(g \circ C_s \cap C_s \circ g) \cap g \neq 0$ for all $s \in S$. Therefore g is a fuzzy almost quasi-ideal of S . $\square$

Theorem 4.3. Let f and g be a fuzzy almost quasi-ideal of a semigroup S . Then $f \cup g$ is a fuzzy almost quasi-ideal of S .

Proof. Let f and g be a fuzzy almost quasi-ideal of S . $(f \circ C_s \cap C_s \circ f) \cap f \neq 0$ and $(g \circ C_s \cap C_s \circ g) \cap g \neq 0$. Since $f \subseteq f \cup g$, $(f \circ C_s \cap C_s \circ f) \subseteq ((f \cup g) \circ C_s \cap C_s \circ (f \cup g))$ and $(f \circ C_s \cap C_s \circ f) \cap f \neq 0$. This implies $((f \cup g) \circ C_s \cap C_s \circ (f \cup g)) \cap (f \cup g) \neq 0$ for all $s \in S$. Therefore $f \cup g$ is a fuzzy almost quasi-ideal of S . $\square$

Example 4.4. Consider the Semigroup $\mathbb{Z}_5$ under the usual addition. Let $f : \mathbb{Z}_5 \rightarrow [0, 1]$ be defined by $f(0) = 0, f(1) = 0.5, f(2) = 0, f(3) = 0.1, f(4) = 0.5$ and $g : \mathbb{Z}_5 \rightarrow [0, 1]$ be defined by $g(0) = 0.2, g(1) = 0.1, g(2) = 0, g(3) = 0, g(4) = 0.2$. We have f and g are fuzzy almost quasi-ideal of $\mathbb{Z}_5$. But $(f \cap g)$ is not a fuzzy almost quasi-ideal of $\mathbb{Z}_5$.

Theorem 4.5. Let Q be a non-empty subset of a semigroup S . Then Q is an almost quasi-ideal of S if and only if C_Q is a fuzzy almost quasi-ideal of S .

Proof. Assume Q is an almost quasi-ideal of semigroup S . Then $Qs \cap sQ \cap Q \neq \emptyset$ for all $s \in S$. Thus $\exists x \in Qs \cap sQ$ and $x \in Q$. So, $(C_Q \circ C_s \cap C_s \circ C_Q)(x) = 1$ and $C_Q(x) = 1$. Hence $(C_Q \circ C_s \cap C_s \circ C_Q) \cap C_Q \neq 0$ for all $s \in S$. Therefore C_Q is a fuzzy almost quasi-ideal of S .

Conversely, assume that C_Q is a fuzzy almost quasi-ideal of S . Let $s \in S$, then $(C_Q \circ C_s \cap C_s \circ C_Q) \cap C_Q \neq 0$. Then $\exists x \in S$, such that $[(C_Q \circ C_s \cap C_s \circ C_Q) \cap C_Q](x) \neq 0$. Hence $x \in (Qs \cap sQ) \cap Q$. So $(Qs \cap sQ) \cap Q \neq \emptyset$ for all $s \in S$. Consequently, Q is an almost quasi-ideal of S . $\square$

Theorem 4.6. Let f be a fuzzy subset of a semigroup S . Then f is a fuzzy almost quasi-ideal of S if and only if $\text{supp } f$ is an almost quasi-ideal of S .

Proof. Assume that f is a fuzzy almost quasi-ideal of a semigroup S . Let $s \in S$, then $(f \circ C_s \cap C_s \circ f) \cap f \neq 0$. $\exists x \in S$ such that $[(f \circ C_s \cap C_s \circ f) \cap f](x) \neq 0$. $\exists y_1, y_2 \in S$ such that $x = y_1 y_2, f(x) \neq 0$, and $f(y_1) \neq 0$ and $f(y_2) \neq 0$. (i.e.,) $x, y_1, y_2 \in \text{supp } f$. Thus $(C_{\text{supp } f} \circ C_s \cap C_s \circ C_{\text{supp } f})(x) \neq 0$ and $C_{\text{supp } f}(x) \neq 0$.

Therefore $(C_{\text{supp } f} \circ C_s \cap C_s \circ C_{\text{supp } f}) \cap C_{\text{supp } f} \neq 0$. Hence $C_{\text{supp } f}$ is a fuzzy almost quasi-ideal of S . By Theorem 4.5, $\text{supp } f$ is an almost quasi-ideal of S .

Conversely, assume that $\text{supp } f$ is an almost quasi-ideal of S . By Theorem 4.5, $C_{\text{supp } f}$ is a fuzzy almost quasi-ideal of S . Then $(C_{\text{supp } f} \circ C_s \cap C_s \circ C_{\text{supp } f}) \cap C_{\text{supp } f} \neq 0$ for all $s \in S$. Then there exists $x \in S$ such that $[(C_{\text{supp } f} \circ C_s \cap C_s \circ C_{\text{supp } f}) \cap C_{\text{supp } f}](x) \neq 0$. Hence $(C_{\text{supp } f} \circ C_s \cap C_s \circ C_{\text{supp } f})(x) \neq 0$ and $C_{\text{supp } f}(x) \neq 0$. Then there exist $y_1, y_2 \in S$ such that $x = y_1 y_2, f(x) \neq 0, f(y_1) \neq 0$ and $f(y_2) \neq 0$. This means $(f \circ C_s \cap C_s \circ f) \cap f \neq 0$. Therefore f is a fuzzy almost quasi-ideal of S . $\square$

5. Minimal fuzzy almost quasi-ideal

Definition 5.1. A fuzzy almost quasi-ideal f is called minimal if for each fuzzy almost quasi-ideal g of S such that $g \subseteq f$, we have $\text{supp } g = \text{supp } f$.

Theorem 5.2. Let Q be a non-empty subset of a semigroup S . Then Q is minimal almost quasi-ideal of S if and only if C_Q is a minimal fuzzy almost quasi-ideal of S .

Proof. Assume Q be a minimal almost quasi-ideal of S . C_Q is fuzzy almost quasi-ideal of S . Let g be a fuzzy almost quasi-ideal of S such that $g \subseteq C_Q$. Then $\text{supp } g \subseteq \text{supp } C_Q = Q$. Since $g \subseteq C_{\text{supp } g}$. We have $(g \circ C_Q \cap C_Q \circ g) \cap g \subseteq (C_{\text{supp } g} \circ C_Q \cap C_Q \circ C_{\text{supp } g}) \cap C_{\text{supp } g}$. Thus $C_{\text{supp } g}$ is a fuzzy almost quasi-ideal of S . By theorem 4.5, $\text{supp } g$ is an almost quasi-ideal of S . Since Q is minimal, $\text{supp } g = Q = \text{supp } C_Q$. Therefore C_Q is minimal.

Conversely, assume that C_Q is minimal fuzzy almost quasi-ideal of S . Let Q' be almost quasi-ideal of S such that $Q' \subseteq Q$. Then $C_{Q'}$ is a fuzzy almost quasi-ideal of S such that $C_{Q'} \subseteq C_Q$. Hence $Q' = \text{supp } C_{Q'} = \text{supp } C_Q = Q$. Therefore Q is minimal. $\square$

6. Conclusion

In this paper we have presented the notion of fuzzy almost quasi-ideals and minimal almost quasi-ideals in semigroups and derived some properties of these ideals.

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