Periodicity of powers of intuitionistic fuzzy matrices

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Abstract

The powers of a general intuitionistic fuzzy matrix either converge or oscillate with a finite period. Many properties about a convergence of intuitionistic fuzzy matrix and convergence of powers of a transitive intuitionistic fuzzy matrix have been studied. In this paper, we consider an arbitrary $n \times n$ intuitionistic fuzzy matrix $(\langle q_{ij}, q'_{ij} \rangle)$ and define its period of oscillation d^* and index k^* . Moreover, we present some properties of d^* and k^* and show that $d^*[n]$ (where $[n]$ is the least common multiple of $1, 2, \dots, n$). Also, we develop the algorithm to find d^* and k^* .

Keywords

Intuitionistic Fuzzy Sets, Intuitionistic Fuzzy Matrices, Oscillation.

AMS Subject Classification

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1. Introduction

Fuzzy sets introduced by Zadeh [32] has shown meaningful applications in many fields of study. The idea of fuzzy set is welcome because it handles uncertainty and vagueness which classical set could not address. However, it often falls short of the expected standard when describing the neutral state. As a result, a new concept namely Intuitionistic Fuzzy Set (IFS) was introduced by Atanassov [5] and he has represented it as $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle / x \in E \}$ where E denotes a universal set, in which $\mu_A : E \rightarrow [0, 1]$ and $\nu_A : E \rightarrow [0, 1]$ denote membership and non-membership functions of A respectively and its sum is less than or equal to one. In short we write the elements of IFS as $\langle x, x' \rangle$ such that $x + x' \leq 1$. The ideas of IFS were developed later in [6–10]. Xu and Yager [31] have developed some geometric operators based on IFSs.

A Fuzzy Matrix (FM) is a matrix with elements having values in closed interval $[0, 1]$. Kim and Roush [33] have

introduced the concept of FM. FM play a vital role in various areas, in Science and Engineering and solve the problems involving various types of uncertainties [34]. Meenakshi [17] has studied minus ordering, space ordering and schur complement of FM and block FM. Buckley [39], Ran and Liu [35] and Gregory et. al, [38] after applying max-min operation on fuzzy matrix have found only two results, either the fuzzy matrix converges to idempotent matrix or oscillates to finite period. Hashimoto [37] has studied the convergence of power of a fuzzy transitive matrix. Lur et. al, [36] have studied the powers of convergence of FMs. Many authors have further, extended this study and addressed themselves to the discussion of special matrices such as symmetric fuzzy matrices, nilpotent fuzzy matrices, controllable fuzzy matrices, max-min transitive fuzzy matrices, etc., [40, 42, 44]. The periodicity of these special matrices is 1 or 2 [44]. Li [43] has studied periodicity and index of fuzzy matrices in the general case. In [41] Fan has proved that the periodicity of a fuzzy matrix is the least common multiple (l.c.m.) of periodicity of its cut matrices, and the index of a fuzzy matrix is not greater than the maximum index of its cut matrices. Liua and Ji [46] have discussed the periodicity of square fuzzy matrices based on minimal strong components. The notion Intuitionistic Fuzzy Matrix (IFM) was introduced by Atanssov [3]. The index matrix representation of the intuitionistic fuzzy graphs has been studied in [4]. Pal et. al, [24], Meenakshi and Gandhimathi [16] and Sriram and Murugadas [29, 30] have studied IFM for finding intuitionistic fuzzy linear relation equation, g-inverse

and intuitionistic fuzzy linear transformation etc.,. Shyamal and Pal [28] have studied the distance between IFM. Bhowmik and Pal [12, 13] have examined circulant IFM and generalized IFMs. Im et al., [14] have studied determinant of square IFMs. In [11] Atanassov has studied intuitionistic fuzzy index matrix and extended intuitionistic fuzzy index matrix. Several authors [1, 2, 15, 18–20, 22, 23, 25–27] have worked on IFMs and obtained various interesting results, which are very useful in handling uncertainty problems in our daily life. The purpose of this paper is to study the periodicity of powers IFM.

2. Preliminaries

Throughout the paper matrix mean IFM. Atanassov has introduced the following operations on IFS. For $\langle x, x' \rangle, \langle y, y' \rangle \in IFS$, $\langle x, x' \rangle \vee \langle y, y' \rangle = \langle \max\{x, y\}, \min\{x', y'\} \rangle$ and $\langle x, x' \rangle \wedge \langle y, y' \rangle = \langle \min\{x, y\}, \max\{x', y'\} \rangle$. $\langle x, x' \rangle \geq \langle y, y' \rangle \Rightarrow x \geq y$ and $x' \leq y'$, in this case we say $\langle x, x' \rangle$ and $\langle y, y' \rangle$ are comparable.

Definition 2.1. [23] Let $X = \{x_1, x_2, \dots, x_m\}$ be a set of alternatives and $Y = \{y_1, y_2, \dots, y_n\}$ be the attribute set of each element of X . An IFM is defined by $A = (\langle (x_i, y_j), \mu_A(x_i, y_j), \nu_A(x_i, y_j) \rangle)$ for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, where $\mu_A : X \times Y \rightarrow [0, 1]$ and $\nu_A : X \times Y \rightarrow [0, 1]$ satisfy the condition $0 \leq \mu_A(x_i, y_j) + \nu_A(x_i, y_j) \leq 1$. For simplicity we denote an IFM is a matrix of pairs $A = (\langle a_{ij}, a'_{ij} \rangle)$ of non negative real numbers satisfying $a_{ij} + a'_{ij} \leq 1$ for all i, j . We denote the set of all IFM of order $m \times n$ by $\mathcal{F}_{mn}$.

Consider the IFMs, $Q = (\langle q_{ij}, q'_{ij} \rangle)$ and $P = (\langle p_{ij}, p'_{ij} \rangle)$. We recall some basic operation.

$$QP = (\bigvee_{k=1}^n (\langle q_{ik} \wedge p_{kj} \rangle), \bigwedge_{k=1}^n (\langle q'_{ik} \vee p'_{kj} \rangle)).$$

$$Q^{K+1} = Q^K Q, \text{ for all } k = 1, 2, 3, \dots$$

$$Q^0 = I, \text{ where } I \text{ is a square unit matrix all whose diagonal entries are } (1, 0) \text{ and all other are } (0, 1).$$

$$Q^k = (\langle q_{ij}^k, q_{ij}^{k'} \rangle), \text{ for all } k = 1, 2, 3, \dots. \text{ Then,}$$

$$\langle q_{ij}, q'_{ij} \rangle = \bigvee_{j_1=1}^n \bigvee_{j_2=1}^n \dots \bigvee_{j_{k-1}=1}^n (\langle q_{ij_1}, q'_{ij_1} \rangle \wedge \langle q_{j_1 j_2}, q'_{j_1 j_2} \rangle \wedge \dots \wedge \langle q_{j_{k-1} j_k}, q'_{j_{k-1} j_k} \rangle).$$

$$Q \leq P \text{ iff } \langle q_{ij}, q'_{ij} \rangle \leq \langle p_{ij}, p'_{ij} \rangle \text{ for all } i, j \in \{1, 2, \dots, n\}.$$

$$\text{Let } I^+ = \{x | x \text{ is a positive integer}\}.$$

Remark 2.2. [30] Let Q, S, H and T be $n \times n$ intuitionistic fuzzy matrices. If $Q \leq S, T \leq H, \Rightarrow QT \leq SH$,

3. Results

Let $(\langle q_{ij}, q'_{ij} \rangle)$ be an arbitrary $n \times n$ intuitionistic fuzzy matrix. The number of different matrices is at most r^{n^2} in the following sequence,

$$(\langle q_{ij}, q'_{ij} \rangle), (\langle q_{ij}^2, q_{ij}^{2'} \rangle), (\langle q_{ij}^3, q_{ij}^{3'} \rangle), \dots, (\langle q_{ij}^m, q_{ij}^{m'} \rangle), \dots, \quad (3.1)$$

where r is the number of all the different elements that occur in intuitionistic fuzzy matrix $(\langle q_{ij}, q'_{ij} \rangle)$, which is clearly finite. Thus there exists $s, t \in I^+ (s \neq t)$ such that $(\langle q_{ij}^s, q_{ij}^{s'} \rangle) = (\langle q_{ij}^t, q_{ij}^{t'} \rangle)$. Let

$$H = \{(s, t) | (\langle q_{ij}^s, q_{ij}^{s'} \rangle) = (\langle q_{ij}^t, q_{ij}^{t'} \rangle), s \neq t, s, t \in I^+\}$$

$$D = \{d | d = |s - t|, (s, t) \in H\}.$$

Then, D has a least element d^* ($d^* \geq 1$) by using the well-ordering property of natural numbers.

Let

$$K = \{k | (\langle q_{ij}^k, q_{ij}^{k'} \rangle) = (\langle q_{ij}^{k+d^*}, q_{ij}^{k+d^*'} \rangle), k \in Z^+\}.$$

By well-ordering property, K has a least element k^* ($k^* \geq 1$).

Definition 3.1. We call d^* the period of oscillation of $(\langle q_{ij}, q'_{ij} \rangle)$, and k^* the index of $(\langle q_{ij}, q'_{ij} \rangle)$. $(\langle q_{ij}, q'_{ij} \rangle)$ is convergent if $d^* = 1$

Lemma 3.2. Let $q \in I^+$. If $q \geq k^*$, then

$$(\langle q_{ij}^q, q_{ij}^{q'} \rangle) = (\langle q_{ij}^{q+md^*}, q_{ij}^{q+md^*'} \rangle) \text{ for any } m \in I^+.$$

Proof. We will prove this lemma by induction on $m \geq 1$

For $m = 1$,

since $(\langle q_{ij}^q, q_{ij}^{q'} \rangle) = (\langle q_{ij}^{k^*+d^*}, q_{ij}^{k^*+d^*'} \rangle)$, then

$$(\langle q_{ij}^{q-k^*}, q_{ij}^{q-k^*'} \rangle) \times (\langle q_{ij}^{k^*}, q_{ij}^{k^*'} \rangle)$$

$$= (\langle q_{ij}^{q-k^*}, q_{ij}^{q-k^*'} \rangle)$$

$$\times (\langle q_{ij}^{k^*+d^*}, q_{ij}^{k^*+d^*'} \rangle),$$

and the required equality $(\langle q_{ij}^q, q_{ij}^{q'} \rangle) = (\langle q_{ij}^{k^*+d^*}, q_{ij}^{k^*+d^*'} \rangle)$, is obtained.

Let $(\langle q_{ij}^q, q_{ij}^{q'} \rangle) = (\langle q_{ij}^{q+(m-1)d^*}, q_{ij}^{q+(m-1)d^*'} \rangle)$, then

$$(\langle q_{ij}^q, q_{ij}^{q'} \rangle) \times (\langle q_{ij}^q, q_{ij}^{q'} \rangle) = (\langle q_{ij}^{2q}, q_{ij}^{2q'} \rangle)$$

$$\times (\langle q_{ij}^{q+(m-1)d^*}, q_{ij}^{q+(m-1)d^*'} \rangle),$$

$$\text{that is, } (\langle q_{ij}^{q+d^*}, q_{ij}^{q+d^*'} \rangle) = (\langle q_{ij}^{q+md^*}, q_{ij}^{q+md^*'} \rangle);$$

$$\text{but } (\langle q_{ij}^{q+d^*}, q_{ij}^{q+d^*'} \rangle) = (\langle q_{ij}^q, q_{ij}^{q'} \rangle), \text{ so that}$$

$$(\langle q_{ij}^q, q_{ij}^{q'} \rangle) = (\langle q_{ij}^{q+md^*}, q_{ij}^{q+md^*'} \rangle). \quad \square$$

Theorem 3.3. If there exist $t, s \in I^+$ such that $(\langle q_{ij}^t, q_{ij}^{t'} \rangle) \leq (\langle q_{ij}^s, q_{ij}^{s'} \rangle)$, then $d^* | (t - s)$, where $a | b$ means that a divides b .

Proof. (1) If $t = s$, then $t - s = 0$, and $d^* | 0$.

(2) If $t < s$, then $(\langle q_{ij}^t, q_{ij}^{t'} \rangle) \leq (\langle q_{ij}^{t+(s-t)}, q_{ij}^{t+(s-t)'} \rangle)$. Using Remark 2.2 repeatedly, we will have

$$(\langle q_{ij}^t, q_{ij}^{t'} \rangle) \leq (\langle q_{ij}^{t+(s-t)}, q_{ij}^{t+(s-t)'} \rangle) \leq (\langle q_{ij}^{t+2(s-t)}, q_{ij}^{t+2(s-t)'} \rangle) \quad (3.2)$$

$$\leq \dots \leq (\langle q_{ij}^{t+m(s-t)}, q_{ij}^{t+m(s-t)'} \rangle) \leq \dots \quad (3.3)$$

Since in the sequence

$$(\langle q_{ij}^t, q_{ij}^{t'} \rangle), (\langle q_{ij}^{t+(s-t)}, q_{ij}^{t+(s-t)'} \rangle), (\langle q_{ij}^{t+2(s-t)}, q_{ij}^{t+2(s-t)'} \rangle), \dots,$$

$$(\langle q_{ij}^{t+m(s-t)}, q_{ij}^{t+m(s-t)'} \rangle), \dots,$$

the number of different IFMs is finite, there exists $k, h \in I^+$ ($h < k$) such that

$(\langle q_{ij}^{t+h(s-t)}, q_{ij}^{t+h(s-t)} \rangle) = (\langle q_{ij}^{t+k(s-t)}, q_{ij}^{t+k(s-t)} \rangle)$. From (2), it follows that

$$(\langle q_{ij}^{t+h(s-t)}, q_{ij}^{t+h(s-t)} \rangle) = (\langle q_{ij}^{t+h(s-t)+(s-t)}, q_{ij}^{t+h(s-t)+(s-t)} \rangle). \quad (3.4)$$

Let $q \in I^+$ be such that $q \geq \max \{t + h(s-t), k^\bullet\}$.

Then, by (3), we will have

$$(\langle q_{ij}^q, q_{ij}^q \rangle) = (\langle q_{ij}^{q+(s-t)}, q_{ij}^{q+(s-t)} \rangle). \quad (3.5)$$

On the other side, Lemma 3.2 implies that

$$(\langle q_{ij}^q, q_{ij}^q \rangle) = (\langle q_{ij}^{q+d^\bullet}, q_{ij}^{q+d^\bullet} \rangle) \quad (3.6)$$

By (4), (5) and the definition of $d^\bullet$, we have $d^\bullet \leq s-t$. From the division algorithm, it follows that there are unique integers a and c such that

$$s-t = ad^\bullet + c \text{ with } 0 \leq c < d^\bullet, a > 0. \quad (3.7)$$

By Lemma 3.2, (4) and (5), we have

$$\begin{aligned} (\langle q_{ij}^{q+c}, q_{ij}^{q+c} \rangle) &= (\langle q_{ij}^{q+c+ad^\bullet}, q_{ij}^{q+c+ad^\bullet} \rangle) \\ &= (\langle q_{ij}^{q+(s-t)}, q_{ij}^{q+(s-t)} \rangle) = (\langle q_{ij}^q, q_{ij}^q \rangle). \end{aligned}$$

From the definition of $d^\bullet$ and the fact $0 \leq c < d^\bullet$, it follows that $c = 0$, and hence

$d^\bullet | (s-t)$, so $d^\bullet | (t-s)$.

(3) If $t > s$, the proof is same as of (2). $\square$

Corollary 3.4. $(\langle q_{ij}, q'_{ij} \rangle)$ is convergent iff there exist

$t_1, t_2, t_3, t_4 \in I^+ (t_1 \neq t_2)$ such that $(\langle q_{ij}^{t_1}, q_{ij}^{t_1} \rangle) \leq (\langle q_{ij}^{t_2}, q_{ij}^{t_2} \rangle)$, $(\langle q_{ij}^{t_3}, q_{ij}^{t_3} \rangle) \leq (\langle q_{ij}^{t_4}, q_{ij}^{t_4} \rangle)$ and $(t_1 - t_2, t_3 - t_4) = 1$, where (a, b) denotes the greatest common divisor of a and b .

Proof. The part of the necessity is easy to verify. We prove the sufficiency. If $(\langle q_{ij}^{t_1}, q_{ij}^{t_1} \rangle) \leq (\langle q_{ij}^{t_2}, q_{ij}^{t_2} \rangle)$, $(\langle q_{ij}^{t_3}, q_{ij}^{t_3} \rangle) \leq (\langle q_{ij}^{t_4}, q_{ij}^{t_4} \rangle)$ and $(t_1 - t_2, t_3 - t_4) = 1$, then by Theorem 3.3, we have

$$d^\bullet | (t_1 - t_2), d^\bullet | (t_3 - t_4),$$

which means that $d^\bullet | 1$, so, $d^\bullet = 1$. $\Rightarrow (\langle q_{ij}, q'_{ij} \rangle)$ is convergent. $\square$

Theorem 3.5. Let $t, h \in I^+$ and $t \neq h$. Then,

$$(\langle q_{ij}^h, q_{ij}^h \rangle) = (\langle q_{ij}^t, q_{ij}^t \rangle) \text{ iff } d^\bullet | (t-h), \text{ and } k^\bullet \leq \min \{h, t\}$$

Proof. We may assume without loss of generality that $h < t$.

We first prove the necessity.

$$\text{Let } (\langle q_{ij}^h, q_{ij}^h \rangle) = (\langle q_{ij}^t, q_{ij}^t \rangle).$$

Then $(\langle q_{ij}^h, q_{ij}^h \rangle) = (\langle q_{ij}^{h+(t-h)}, q_{ij}^{h+(t-h)} \rangle)$, and by Theorem 3.3, we have $d^\bullet | (t-h)$. Hence there exists a positive integer q such that $t-h = qd^\bullet$, then

$$(\langle q_{ij}^h, q_{ij}^h \rangle) = (\langle q_{ij}^{h+qd^\bullet}, q_{ij}^{h+qd^\bullet} \rangle). \quad (3.8)$$

If

$$(\langle q_{ij}^h, q_{ij}^h \rangle) \neq (\langle q_{ij}^{h+d^\bullet}, q_{ij}^{h+d^\bullet} \rangle), \quad (3.9)$$

then, using (7) repeatedly, we get

$$\begin{aligned} (\langle q_{ij}^h, q_{ij}^h \rangle) &= (\langle q_{ij}^{h+qd^\bullet}, q_{ij}^{h+qd^\bullet} \rangle) \\ &= (\langle q_{ij}^h, q_{ij}^h \rangle) \times (\langle q_{ij}^{qd^\bullet}, q_{ij}^{qd^\bullet} \rangle) \\ &= (\langle q_{ij}^{h+qd^\bullet}, q_{ij}^{h+qd^\bullet} \rangle) \times (\langle q_{ij}^{qd^\bullet}, q_{ij}^{qd^\bullet} \rangle) \\ &= (\langle q_{ij}^h, q_{ij}^h \rangle) \times (\langle q_{ij}^{2qd^\bullet}, q_{ij}^{2qd^\bullet} \rangle) \\ &= \dots = (\langle q_{ij}^h, q_{ij}^h \rangle) \times (\langle q_{ij}^{mqd^\bullet}, q_{ij}^{mqd^\bullet} \rangle) \\ &= (\langle q_{ij}^{h+mqd^\bullet}, q_{ij}^{h+mqd^\bullet} \rangle), \end{aligned}$$

for any $m \in I^+$. And

$$\begin{aligned} (\langle q_{ij}^{h+d^\bullet}, q_{ij}^{h+d^\bullet} \rangle) &= (\langle q_{ij}^h, q_{ij}^h \rangle) \times (\langle q_{ij}^{d^\bullet}, q_{ij}^{d^\bullet} \rangle) \\ &= (\langle q_{ij}^{h+qd^\bullet}, q_{ij}^{h+qd^\bullet} \rangle) \times (\langle q_{ij}^{qd^\bullet}, q_{ij}^{qd^\bullet} \rangle) \\ &= (\langle q_{ij}^h, q_{ij}^h \rangle) \times (\langle q_{ij}^{(q+1)d^\bullet}, q_{ij}^{(q+1)d^\bullet} \rangle) \\ &= \dots = (\langle q_{ij}^h, q_{ij}^h \rangle) \times (\langle q_{ij}^{(mq+1)d^\bullet}, q_{ij}^{(mq+1)d^\bullet} \rangle) \\ &= (\langle q_{ij}^{h+mqd^\bullet+d^\bullet}, q_{ij}^{h+mqd^\bullet+d^\bullet} \rangle), \end{aligned}$$

for any $m \in I^+$.

Then (8) implies that

$$(\langle q_{ij}^{h+mqd^\bullet}, q_{ij}^{h+mqd^\bullet} \rangle) \neq (\langle q_{ij}^{h+mqd^\bullet+d^\bullet}, q_{ij}^{h+mqd^\bullet+d^\bullet} \rangle) \quad (3.10)$$

for any $m \in I^+$

Otherwise, take $m \in I^+$ such that $h+mqd^\bullet \geq k^\bullet$.

Then by Lemma 3.2, we have

$$(\langle q_{ij}^{h+mqd^\bullet}, q_{ij}^{h+mqd^\bullet} \rangle) = (\langle q_{ij}^{h+mqd^\bullet+d^\bullet}, q_{ij}^{h+mqd^\bullet+d^\bullet} \rangle), \text{ a contradiction to (9).}$$

Hence, $(\langle q_{ij}^h, q_{ij}^h \rangle) = (\langle q_{ij}^{h+d^\bullet}, q_{ij}^{h+d^\bullet} \rangle)$, and by the definition of $k^\bullet$, it follows that $k^\bullet \leq h = \min \{h, t\}$.

Now, we prove the sufficiency.

If $d^\bullet | (t-h)$, $k^\bullet \leq \min \{h, t\}$, then there exists a positive integer q such that $t-h = qd^\bullet$. Since $h \geq k^\bullet$, by Lemma 3.2, we have

$$\begin{aligned} (\langle q_{ij}^h, q_{ij}^h \rangle) &= (\langle q_{ij}^{h+qd^\bullet}, q_{ij}^{h+qd^\bullet} \rangle) \\ &= (\langle q_{ij}^{h+(t-h)}, q_{ij}^{h+(t-h)} \rangle) \\ &= (\langle q_{ij}^t, q_{ij}^t \rangle). \end{aligned}$$

$\square$

Theorem 3.6. In the sequence

$(\langle q_{ij}, q'_{ij} \rangle), (\langle q_{ij}^2, q_{ij}^2 \rangle), (\langle q_{ij}^3, q_{ij}^3 \rangle), \dots$, the number of different IFMs is $k^\bullet + d^\bullet - 1$. Also, the set of the $k^\bullet + d^\bullet - 1$ different IFMs is

$$\{(\langle q_{ij}, q'_{ij} \rangle), (\langle q_{ij}^2, q_{ij}^2 \rangle), \dots, (\langle q_{ij}^{k^\bullet}, q_{ij}^{k^\bullet} \rangle), (\langle q_{ij}^{k^\bullet+1}, q_{ij}^{k^\bullet+1} \rangle), \dots, (\langle q_{ij}^{k^\bullet+d^\bullet-1}, q_{ij}^{k^\bullet+d^\bullet-1} \rangle)\}.$$

Proof. $T = \{(\langle q_{ij}, q'_{ij} \rangle), (\langle q_{ij}^2, q'_{ij}{}^2 \rangle), \dots, (\langle q_{ij}^{k^*}, q'_{ij}{}^{k^*} \rangle),$
 $(\langle q_{ij}^{k^*+1}, q'_{ij}{}^{k^*+1} \rangle), \dots, (\langle q_{ij}^{k^*+d^*-1}, q'_{ij}{}^{k^*+d^*-1} \rangle)\}$,

$$W = \{(\langle q_{ij}^m, q'_{ij}{}^m \rangle) | m \in I^+\}.$$

We first show $W = T$.

Let $(\langle q_{ij}^m, q'_{ij}{}^m \rangle) \in W$.

If $m - k^* < d^*$, then $m < k^* + d^*$,

and it follows that $(\langle q_{ij}^m, q'_{ij}{}^m \rangle) \in T$.

If $m - k^* \geq d^*$ then $m \geq k^* + d^*$.

Let $t = m - (k^* + d^*)$, clearly $t \geq 0$, $m = t + k^* + d^* > k^*$. We have two cases.

case (1) $t < d^*$. Then by Lemma 3.2 we have

$$(\langle q_{ij}^m, q'_{ij}{}^m \rangle) = (\langle q_{ij}^{k^*+t+d^*}, q'_{ij}{}^{k^*+t+d^*} \rangle) = (\langle q_{ij}^{k^*+t}, q'_{ij}{}^{k^*+t} \rangle) \in T.$$

case (2) $t \geq d^*$. Then there exists two positive integers a, c such that $t = ad^* + c$, $0 \leq c < d^*$, and hence

$$\begin{aligned} (\langle q_{ij}^m, q'_{ij}{}^m \rangle) &= (\langle q_{ij}^{k^*+t+d^*}, q'_{ij}{}^{k^*+t+d^*} \rangle) \\ &= (\langle q_{ij}^{k^*+ad^*+c+d^*}, q'_{ij}{}^{k^*+ad^*+c+d^*} \rangle) \\ &= (\langle q_{ij}^{(k^*+c)+(a+1)d^*}, q'_{ij}{}^{(k^*+c)+(a+1)d^*} \rangle) \\ &= (\langle q_{ij}^{k^*+c}, q'_{ij}{}^{k^*+c} \rangle) \end{aligned}$$

(by Lemma 3.2).

So $(\langle q_{ij}^m, q'_{ij}{}^m \rangle) \in T$.

Thus $W \subseteq T$, and $W = T$.

Now, we show that the number of different elements of T is $k^* + d^* - 1$.

Suppose that $t, m \in \{1, 2, \dots, k^* + d^* - 1\}$, $t \neq m$

and $(\langle q_{ij}^t, q'_{ij}{}^t \rangle) = (\langle q_{ij}^m, q'_{ij}{}^m \rangle)$.

If $t > m$, then by Theorem 3.5 we get $t > m \geq k^*$ and $d^* | (t - m)$. But $t > m \geq k^*$, so we get

$$t - m \leq (k^* + d^* - 1) - k^* = d^* - 1,$$

and $d^* > t - m$. But, from the fact that $d^* | (t - m)$, we get $d^* \leq t - m$, a contradiction. $\square$

Theorem 3.7. (1) $d^* | [n]$

$$(2) (\langle q_{ij}^n, q'_{ij}{}^n \rangle) \leq (\langle q_{ij}^{n+[n]}, q'_{ij}{}^{n+[n]} \rangle).$$

Proof. We first prove (2). For $(\langle q_{ij}^k, q'_{ij}{}^k \rangle)$, $k = 1, 2, \dots$, let the (i, j) entry of $(\langle q_{ij}, q'_{ij} \rangle)$

$$\langle q_{ij}^n, q'_{ij}{}^n \rangle = \langle q_{h_0 h_1}, q'_{h_0 h_1} \rangle \wedge \langle q_{h_1 h_2}, q'_{h_1 h_2} \rangle \wedge \dots \wedge \langle q_{h_{n-1} h_n}, q'_{h_{n-1} h_n} \rangle,$$

where $h_0 = i$ and $h_n = j$. We have $h_a = h_b$ for some $a, b \in \{0, 1, 2, \dots, n\}$ ($a < b$) since $h_i \in \{1, 2, \dots, n\}$ for every $i = 0, 1, 2, \dots, n$. Then

$$\begin{aligned} \langle q_{ij}^n, q'_{ij}{}^n \rangle &= \langle q_{h_0 h_1}, q'_{h_0 h_1} \rangle \\ &\wedge \dots \wedge \langle q_{h_{a-1} h_a}, q'_{h_{a-1} h_a} \rangle \wedge \\ &\langle q_{h_a h_{a+1}}, q'_{h_a h_{a+1}} \rangle \wedge \dots \wedge \langle q_{h_{b-1} h_b}, q'_{h_{b-1} h_b} \rangle \wedge \langle q_{h_b h_{b+1}}, q'_{h_b h_{b+1}} \rangle \wedge \\ &\langle q_{h_{b+1} h_{b+2}}, q'_{h_{b+1} h_{b+2}} \rangle \wedge \dots \wedge \langle q_{h_{n-1} h_n}, q'_{h_{n-1} h_n} \rangle \\ &= (\langle q_{h_0 h_1}, q'_{h_0 h_1} \rangle \wedge \dots \wedge \langle q_{h_{a-1} h_a}, q'_{h_{a-1} h_a} \rangle) \wedge \\ &(\langle q_{h_a h_{a+1}}, q'_{h_a h_{a+1}} \rangle \wedge \dots \wedge \langle q_{h_{b-1} h_b}, q'_{h_{b-1} h_b} \rangle) \wedge (\langle q_{h_b h_{b+1}}, q'_{h_b h_{b+1}} \rangle \\ &\wedge \langle q_{h_{b+1} h_{b+2}}, q'_{h_{b+1} h_{b+2}} \rangle \wedge \dots \wedge \langle q_{h_{n-1} h_n}, q'_{h_{n-1} h_n} \rangle) \end{aligned}$$

$$\langle q_{ij}^n, q'_{ij}{}^n \rangle \leq \langle q_{h_0 h_a}, q'_{h_0 h_a} \rangle \wedge \langle q_{h_a h_b}, q'_{h_a h_b} \rangle \wedge \langle q_{h_b h_n}, q'_{h_b h_n} \rangle \quad (3.11)$$

where $1 \leq b - a \leq n$.

There exists $t \in I^+$ such that $[n] = t(b - a)$,

Hence

$$\langle q_{h_a h_a}^{[n]}, q'_{h_a h_a}{}^{[n]} \rangle \geq \langle q_{h_a h_a}^{b-a}, q'_{h_a h_a}{}^{b-a} \rangle \wedge \langle q_{h_a h_a}^{b-a}, q'_{h_a h_a}{}^{b-a} \rangle \wedge \dots \wedge \langle q_{h_a h_a}^{b-a}, q'_{h_a h_a}{}^{b-a} \rangle$$

and so

$$\langle q_{ij}^n, q'_{ij}{}^n \rangle \leq \langle q_{h_0 h_a}^a, q'_{h_0 h_a}{}^a \rangle \wedge \langle q_{h_a h_a}^{b-a}, q'_{h_a h_a}{}^{b-a} \rangle \wedge \langle q_{h_a h_a}^{[n]}, q'_{h_a h_a}{}^{[n]} \rangle \wedge \langle q_{h_a h_a}^{n-b}, q'_{h_a h_a}{}^{n-b} \rangle \quad (3.12)$$

$$\leq \langle q_{h_0 h_n}^{n+[n]}, q'_{h_0 h_n}{}^{n+[n]} \rangle = \langle q_{ij}^{n+[n]}, q'_{ij}{}^{n+[n]} \rangle. \quad (3.13)$$

By Theorem 3.3 and (2) we have $d^* | [n]$, that is (1) holds. $\square$

Corollary 3.8. $1 \leq d^* \leq [n]$

Theorem 3.9. If there exists a positive integer h such that $\langle q_{ii}^k, q'_{ii}{}^k \rangle \leq \langle q_{ii}^h, q'_{ii}{}^h \rangle$ for all $i, j \in \{1, 2, \dots, n\}$, then we have

(1) $(\langle q_{ij}^n, q'_{ij}{}^n \rangle) \leq (\langle q_{ij}^{n+h}, q'_{ij}{}^{n+h} \rangle)$;

(2) $d^* | h$;

(3) if $(h, [n]) = 1$, then $(\langle q_{ij}, q'_{ij} \rangle)$ is convergent.

Proof. (1) By (10), we have

$$\langle q_{ij}^n, q'_{ij}{}^n \rangle \leq \langle q_{ih_a}^a, q'_{ih_a}{}^a \rangle \wedge \langle q_{h_a h_a}^{b-a}, q'_{h_a h_a}{}^{b-a} \rangle \wedge \langle q_{h_a j}^{n-b}, q'_{h_a j}{}^{n-b} \rangle$$

$$= \langle q_{ih_a}^a, q'_{ih_a}{}^a \rangle \wedge \langle q_{h_a h_a}^{b-a}, q'_{h_a h_a}{}^{b-a} \rangle \wedge \langle q_{h_a h_a}^h, q'_{h_a h_a}{}^h \rangle \wedge \langle q_{h_a j}^{n-b}, q'_{h_a j}{}^{n-b} \rangle$$

$$\leq \langle q_{ij}^{n+h}, q'_{ij}{}^{n+h} \rangle$$

$$\text{and so } (\langle q_{ij}^n, q'_{ij}{}^n \rangle) \leq (\langle q_{ij}^{n+h}, q'_{ij}{}^{n+h} \rangle)$$

(2) By (1) and Theorem 3.3, we have $d^* | h$.

(3) By Theorem 3.7 and (2), we get $d^* | [n]$;

since $d^* | h$ and thus $d^* | 1$ we find $d^* = 1$.

$\Rightarrow (\langle q_{ij}, q'_{ij} \rangle)$ is convergent. $\square$

Corollary 3.10. If $\langle q_{ii}^k, q'_{ii}{}^k \rangle \leq \langle q_{ii}^n, q'_{ii}{}^n \rangle$ for all $i, k \in \{1, 2, \dots, n\}$, then (1) $d^* | n$ and (2) $(\langle q_{ij}^n, q'_{ij}{}^n \rangle) \leq (\langle q_{ij}^{2n}, q'_{ij}{}^{2n} \rangle)$.

Corollary 3.11. If there exists a positive integer h such that $\langle q_{ii}^k, q'_{ii}{}^k \rangle \leq \langle q_{ii}^n, q'_{ii}{}^n \rangle \leq \langle q_{ii}^h, q'_{ii}{}^h \rangle$ for all $i, k \in \{1, 2, \dots, n\}$, and $(n, h) = 1$, then $(\langle q_{ij}, q'_{ij} \rangle)$ is convergent.

Proof. By Corollary 3.10 and Theorem 3.9, we have $d^* | n$, and $d^* | h$. Thus $d^* | 1$, and so $d^* = 1$. $\square$

4. An algorithm to find the period of oscillation

Let $(\langle q_{ij}, q'_{ij} \rangle)$ be an $n \times n$ IFM, and d^*, k^* be the period of oscillation and the index of $(\langle q_{ij}, q'_{ij} \rangle)$, respectively.

We develop the following algorithm to find the d^* and k^* .
 Step 1. Compute successively

$$((\langle q_{ij}, q'_{ij} \rangle), (\langle q_{ij}^{1+[n]}, q'_{ij}{}^{1+[n]} \rangle), (\langle q_{ij}^2, q'_{ij}{}^2 \rangle), \quad (4.1)$$

$$((\langle q_{ij}^{2+[n]}, q'_{ij}{}^{2+[n]} \rangle), \dots, ((\langle q_{ij}^m, q'_{ij}{}^m \rangle), (\langle q_{ij}^{m+[n]}, q'_{ij}{}^{m+[n]} \rangle), \dots,$$

In (12) find k such that

$$(\langle q_{ij}^{k-1}, q_{ij}^{k-1} \rangle) \neq (\langle q_{ij}^{(k-1)+[n]}, q_{ij}^{(k-1)+[n]} \rangle) \quad (4.3)$$

but

$$(\langle q_{ij}^k, q_{ij}^k \rangle) = (\langle q_{ij}^{k+[n]}, q_{ij}^{k+[n]} \rangle). \quad (4.4)$$

Then $k^\bullet = k$. Go to Step 2.

Step 2. Compute successively

$$(\langle q_{ij}^{k^\bullet}, q_{ij}^{k^\bullet} \rangle), (\langle q_{ij}^{k^\bullet+1}, q_{ij}^{k^\bullet+1} \rangle), (\langle q_{ij}^{k^\bullet+2}, q_{ij}^{k^\bullet+2} \rangle), \dots, \quad (4.5)$$

$$(\langle q_{ij}^{k^\bullet+[n]}, q_{ij}^{k^\bullet+[n]} \rangle), \quad (4.6)$$

In (13) find d such that

$$(\langle q_{ij}^{k^\bullet}, q_{ij}^{k^\bullet} \rangle) \neq (\langle q_{ij}^{(k^\bullet+j)}, q_{ij}^{(k^\bullet+j)} \rangle) \text{ for } j = 1, 2, \dots, d-1 \quad (4.7)$$

$$\text{but } (\langle q_{ij}^{k^\bullet}, q_{ij}^{k^\bullet} \rangle) = (\langle q_{ij}^{k^\bullet+d}, q_{ij}^{k^\bullet+d} \rangle). \quad (4.8)$$

Then $d^\bullet = d$. Stop

Proof. Step 1. By Theorem 3.7 and Lemma 3.2, we have $(\langle q_{ij}^{k^\bullet}, q_{ij}^{k^\bullet} \rangle) = (\langle q_{ij}^{k^\bullet+[n]}, q_{ij}^{k^\bullet+[n]} \rangle)$.

Hence, the integer k does exist.

Since $(\langle q_{ij}^{k-1}, q_{ij}^{k-1} \rangle) \neq (\langle q_{ij}^{(k-1)+[n]}, q_{ij}^{(k-1)+[n]} \rangle)$, then by Theorem 3.5, and Theorem 3.7, we get $k-1 < k^\bullet$, and $k \leq k^\bullet$. Since $(\langle q_{ij}^k, q_{ij}^k \rangle) \neq (\langle q_{ij}^{(k)+[n]}, q_{ij}^{(k)+[n]} \rangle)$, using Theorem 3.5 again, we have $k \geq k^\bullet$, so $k = k^\bullet$.

Step 2. From the fact that $(\langle q_{ij}^{k^\bullet}, q_{ij}^{k^\bullet} \rangle) = (\langle q_{ij}^{k^\bullet+d}, q_{ij}^{k^\bullet+d} \rangle)$, we know the integer d does not exist. Since $(\langle q_{ij}^{k^\bullet}, q_{ij}^{k^\bullet} \rangle) \neq (\langle q_{ij}^{k^\bullet+j}, q_{ij}^{k^\bullet+j} \rangle)$ for $j = 1, 2, \dots, d-1$, then by the definition of $d^\bullet$, we have $d^\bullet > d-1$.

Therefore, $d^\bullet \geq d$. But $(\langle q_{ij}^{k^\bullet}, q_{ij}^{k^\bullet} \rangle) = (\langle q_{ij}^{k^\bullet+d}, q_{ij}^{k^\bullet+d} \rangle)$ and by Theorem 3.5, we have $d^\bullet \leq d$. So, $d^\bullet = d$ $\square$

Example 4.1. Here, $[n] = [3] = 6$.

$$\begin{aligned} Q &= \begin{bmatrix} \langle 0.2, 0.7 \rangle & \langle 0.3, 0.6 \rangle & \langle 0.6, 0.3 \rangle \\ \langle 0.8, 0.1 \rangle & \langle 0.0, 1.0 \rangle & \langle 0.5, 0.4 \rangle \\ \langle 0.4, 0.5 \rangle & \langle 0.7, 0.2 \rangle & \langle 0.1, 0.8 \rangle \end{bmatrix} \\ Q^2 &= \begin{bmatrix} \langle 0.4, 0.5 \rangle & \langle 0.6, 0.3 \rangle & \langle 0.3, 0.6 \rangle \\ \langle 0.4, 0.5 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle \\ \langle 0.7, 0.2 \rangle & \langle 0.3, 0.6 \rangle & \langle 0.5, 0.4 \rangle \end{bmatrix} \\ &\neq \begin{bmatrix} \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle \end{bmatrix} = Q^{2+[3]} = Q^6 \end{aligned}$$

$$\begin{aligned} (4.2) \quad Q^3 &= \begin{bmatrix} \langle 0.6, 0.3 \rangle & \langle 0.3, 0.6 \rangle & \langle 0.5, 0.4 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle \\ \langle 0.4, 0.5 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle \end{bmatrix} \\ &\neq \begin{bmatrix} \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle \end{bmatrix} = Q^{3+[3]} = Q^9 \\ Q^4 &= \begin{bmatrix} \langle 0.4, 0.5 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle \\ \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle \end{bmatrix} \\ &\neq \begin{bmatrix} \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle \\ \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle \end{bmatrix} = Q^{3+[3]} = Q^{10} \\ Q^5 &= \begin{bmatrix} \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle \\ \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle \end{bmatrix} \\ &= \begin{bmatrix} \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle \\ \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle \end{bmatrix} = Q^{3+[3]} = Q^{11}. \end{aligned}$$

So, $k^\bullet = 5$. Further,

$$\begin{aligned} Q^5 &= \begin{bmatrix} \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle \\ \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle \end{bmatrix} \\ &\neq \begin{bmatrix} \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle \end{bmatrix} = Q^{5+1} = Q^6. \\ Q^5 &= \begin{bmatrix} \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle \\ \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle \end{bmatrix} \\ &\neq \begin{bmatrix} \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle \\ \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle \end{bmatrix} = Q^{5+2} = Q^7, \\ &\text{but,} \\ Q^5 &= \begin{bmatrix} \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle \\ \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle \end{bmatrix} \\ &= \begin{bmatrix} \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle \\ \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.6, 0.3 \rangle \\ \langle 0.6, 0.3 \rangle & \langle 0.5, 0.4 \rangle & \langle 0.5, 0.4 \rangle \end{bmatrix} = Q^{5+3} = Q^8. \\ \text{So, } d^\bullet &= 3 \end{aligned}$$

5. Conclusion

In this paper, we considered an arbitrary square IFM and defined its period of oscillation $d^\bullet$ and index $k^\bullet$. Moreover, we have developed some properties of $d^\bullet$ and $k^\bullet$ and proved that $d^\bullet | [n]$ (where $[n]$ is the least common multiple of $1, 2, \dots, n$). Also, we have given the algorithm to find $d^\bullet$ and $k^\bullet$.

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