Some fixed point theorems in generalized fuzzy metric spaces

R. Pandiselvi^{1*}, M. Jeyaraman² and S. Durga³

Abstract

In this paper, we extend and generalize the classical Banach's contraction principle on a generalized fuzzy metric spaces endowed with a binary relation. We also prove some fixed point theorem in generalized fuzzy metric spaces.

Keywords

Fixed point theorem, Generalized Fuzzy Metric Space, Contraction mappings.

AMS Subject Classification

47H10, 54H25.

¹Department of Mathematics, The Madura College, Madurai, Tamil Nadu, India.

^{2,3}PG and Research Department of Mathematics, Raja Doraisingam Government Arts College, Sivagangai, Tamil nadu, India

*Corresponding author: ¹rpselvi@gmail.com; ²jeya.math@gmail.com; ³durgasundaram96@gmail.com

Article History: Received 16 March 2019; Accepted 11 May 2019

©2019 MJM.

Contents

1	Introduction	339
2	Preliminaries	339
3	Main Results	340
	References	341

1. Introduction

The best known result from the fixed point theory is Banach's contraction Principle, which is one of the most important results of analysis and considered as the beginning and main source of metric fixed point theory. Banach contraction principle has been extended in many different directions. The concept of fuzzy set was initiated by Zadeh [12]. Thereafter, it was developed extensively by many authors which also include interesting applications of this theory in diverse areas. To use this concept, several researchers have defined fuzzy metric space in several ways. George and Veeramani [3] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [5]. Very recently Alam and Imdad [1] presented a new variant of classical Banach contraction principle in a complete metric space endowed with a binary relation which, under a universal relation reduces to Banach contraction principle. Many authors proved fixed point theorems in fuzzy metric space including the aim of this paper is to extend and generalize the, Banach contraction principle to a complete fuzzy metric space.

2. Preliminaries

In this section, we recall some definitions and basic results of generalized fuzzy metric space which will be used throughout the paper.

Definition 2.1. A 3-tuple $(X, \mathcal{M}, *)$ is called $\mathcal{M}$ -fuzzy metric space if X is an arbitrary non empty set, $*$ is a continuous t -norm and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0, \infty)$, satisfying the following conditions:

(M1) $\mathcal{M}(x, y, z, t) > 0$, for each $x, y, z, a \in X$ and $t, s > 0$.

(M2) $\mathcal{M}(x, y, z, t) = 1$ if and only if $x = y = z$,

(M3) $\mathcal{M}(x, y, z, t)$

tion function

(M4) $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$,

(M5) $\mathcal{M}(x, y, z, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous,

(M6) $\lim_{t \rightarrow \infty} \mathcal{M}(x, y, z, t) = 1$ for all $x, y, z \in X$.

Definition 2.2. Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$ -fuzzy metric space. The $\mathcal{M}$ -fuzzy metric is said to be strong (non-Archimedean) if it satisfies

$$(\mathcal{M}4') : \mathcal{M}(x, y, z, t) \geq \mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, t)$$

for each $x, y, z \in X$ and each $t > 0$. Throughout the paper $\varphi(x, y, z, t)$ is denote $\left(\frac{1}{\mathcal{M}(x, y, z, t)} - 1 \right)$.

Definition 2.3. Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$ -fuzzy metric space

1. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if

$$\lim_{n \rightarrow \infty} \mathcal{M}(x_n, x, x, t) = 1 \quad \text{for all } t > 0.$$

2. A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathcal{N}$ such that $\mathcal{M}(x_n, x_m, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$.
3. A generalized fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.
4. A generalized fuzzy metric space in which every sequence has a convergent subsequence is said to be compact.

Definition 2.4. Let $(X, \mathcal{M}, *)$ be a generalized fuzzy metric space. We will say the mapping $T : X \rightarrow X$ is fuzzy contractive if there exist $k \in (0, 1)$ such that $\varphi(Tx, Ty, Tz, t) \leq k\varphi(x, y, z, t)$ for each $x, y, z \in X$ and $t > 0$. (k is called the constant of T).

Definition 2.5. A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is an altering distance function if

- i. ψ is monotone increasing and continuous.
- ii. $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.6. A function $\phi : [0, \infty) \rightarrow [0, \infty)$ is an altering function if

- (A1). ϕ is strictly decreasing and left continuous
- (A2). $\phi(\lambda) = 0$ if and only if $\lambda = 1$.

Obviously we obtain that $\lim_{\lambda \rightarrow 1} \phi(\lambda) = \phi(1) = 0$

3. Main Results

Theorem 3.1. Let $(X, \mathcal{M}, *)$ be a complete generalized fuzzy metric space and $T : X \rightarrow X$ be a self mapping satisfying the inequality

$$\psi((Tx, T^2x, Ty, t)) \leq k\{\psi(\varphi(x, Tx, y, t)) - \phi(\varphi(x, Tx, y, t))\} \quad (3.1.1)$$

where $x, y \in X$ and for some $0 < k < 1$ and $\psi, \phi : [0, \infty) \rightarrow [0, \infty)$ are both continuous and monotone non-decreasing functions with $\psi(t) = \phi(t) = 0$, if and only if $t = 0$. Then T has a unique fixed point.

Proof. For any $x_0 \in X$. We construct the sequence by

$$\begin{aligned} x_{n+1} &= Tx_n, n = 1, 2, \dots \text{ Setting } x = x_{n-1}, y = x_n \\ \psi(\varphi(Tx_{n-1}, T^2x_{n-1}, Tx_n, t)) \\ &\leq k\{\psi(\varphi(x_{n-1}, Tx_{n-1}, x_n, t)) - \phi(\varphi(x_{n-1}, Tx_{n-1}, x_n, t))\} \\ \psi(\varphi(x_n, x_{n+1}, x_{n+1}, t)) \\ &\leq k\{\psi(\varphi(x_{n-1}, x_n, x_n, t)) - \phi(\varphi(x_{n-1}, x_n, x_n, t))\} \quad (3.1.2) \end{aligned}$$

Which implies, (using monotone property of ψ -function.)

$$\varphi(x_n, x_{n+1}, x_{n+1}, t) \leq \varphi(x_{n-1}, x_n, x_n, t)$$

It follows that the sequence $\{\varphi(x_n, x_{n+1}, x_{n+1}, t)\}$ is monotone decreasing and consequently there exists $r \geq 0$ such that

$$\varphi(x_n, x_{n+1}, x_{n+1}, t) \rightarrow r \text{ as } n \rightarrow \infty \quad (3.1.3)$$

Letting $n \rightarrow \infty$ in (3.1.2) we obtain

$$\psi(r) \leq k\{\psi(r) - \phi(r)\} \quad (3.1.4)$$

which is a contradiction unless $r = 0$.
Hence

$$\varphi(x_n, x_{n+1}, x_{n+1}, t) \rightarrow 0 \text{ as } n \rightarrow \infty \quad (3.1.6)$$

We next prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$,

$$\varphi(x_{m_k}, Tx_{m_k}, x_{n_k}, t) \geq \varepsilon \quad (3.1.7)$$

Further corresponding to m_k , we can choose n_k in such a way that it is the smallest Integer with $n_k > m_k$ and satisfying (3.1.7). Then

$$\varphi(x_{m_k}, Tx_{m_k}, x_{n_k-1}, t) < \varepsilon \quad (3.1.8)$$

Then we have

$$\begin{aligned} \varepsilon &\leq \varphi(x_{m_k}, Tx_{m_k}, x_{n_k}, t) \\ &= \varphi(x_{n_k}, x_{m_k}, Tx_{m_k}, t) \\ &\leq \varphi(x_{m_k}, Tx_{m_k}, x_{n_k-1}, t) + \varphi(x_{n_k-1}, x_{n_k-1}, x_{n_k}, t) \\ &< \varepsilon + \varphi(x_{n_k-1}, x_{n_k-1}, x_{n_k}, t) \quad (3.1.9) \end{aligned}$$

Letting $k \rightarrow \infty$ and using (3.1.6),

$$\lim_{k \rightarrow \infty} \varphi(x_{m_k}, Tx_{m_k}, x_{n_k}, t) = \varepsilon \quad (3.1.10)$$

Again

$$\begin{aligned} \varphi(x_{m_k}, Tx_{m_k}, x_{n_k}, t) &\leq \varphi(x_{n_k}, x_{n_k-1}, x_{n_k-1}, t) \\ &\quad + \varphi(x_{n_k-1}, x_{m_k-1}, Tx_{m_k}, t) + \varphi(x_{m_k}, x_{m_k-1}, x_{m_k-1}, t) \\ \varphi(x_{n_k}, x_{m_k-1}, Tx_{m_k}, t) &\leq \varphi(x_{n_k-1}, x_{n_k}, x_{n_k}, t) \\ &\quad + \varphi(x_{m_k-1}, x_{m_k}, x_{m_k}, t) + \varphi(x_{m_k}, Tx_{m_k}, x_{n_k}, t) \end{aligned}$$

Let $k \rightarrow \infty$ in the above two inequalities and using (3.1.6), (3.1.10), we get

$$\lim_{k \rightarrow \infty} \varphi(x_{n_k-1}, x_{m_k-1}, Tx_{m_k}, t) = \varepsilon \quad (3.1.11)$$

Setting $x = x_{m_k-1}$ and $y = u_{m_k-1}$ in (3.1.1) and using (3.1.7) we obtain

$$\begin{aligned} \psi(\varepsilon) &\leq \psi(\varphi(x_{m_k}, Tx_{m_k}, x_{n_k}, t)) \\ &\leq k\{\psi(\varphi(x_{m_k-1}, Tx_{m_k-1}, x_{n_k-1}, t)) \\ &\quad - \phi(\varphi(x_{m_k-1}, Tx_{m_k-1}, x_{n_k-1}, t))\} \quad (3.1.12) \end{aligned}$$

Letting $k \rightarrow \infty$ utilizing (3.1.10) and (3.1.11) we obtain

$$\psi(\varepsilon) \leq \{\psi(\varepsilon) - \phi(\varepsilon)\} \quad (3.1.13)$$

which is a contradiction if $\varepsilon > 0$. This shows that $\{x_n\}$ is a Cauchy sequence and hence is convergent in the complete generalized fuzzy metric space X .

$$\text{Let } x_n \rightarrow z \text{ (say) as } n \rightarrow \infty \quad (3.1.14)$$

Substitution $x = x_n$ and $y = z$ in (3.1.1) we obtain

$$\begin{aligned} \varphi(x_{n+1}, x_{n+2}, Tz, t) &= \psi(\varphi(Tx_n, T^2x_n, Tz, t)) \\ &\leq k\{\psi(\varphi(x_n, Tx_n, z, t)) - \phi(\varphi(x_n, Tx_n, z, t))\} \\ &\leq k\{\psi(\varphi(x_n, x_{n+1}, z, t)) - \phi(\varphi(x_n, x_{n+1}, z, t))\} \end{aligned} \quad (3.1.15)$$

Letting $n \rightarrow \infty$, using (3.1.14) and continuity of ψ and ϕ

$$\psi(\varphi(z, z, Tz, t)) \leq k\{\psi(0) - \phi(0)\}$$

Which implies $\psi(\varphi(z, z, Tz, t)) = 0$.

That is $\varphi(z, z, Tz, t) = 0 \Rightarrow Tz = z$.

Let us suppose that z_1 and z_2 are two fixed points of T .

Putting $x = z_1$ and $y = z_2$ in (3.1.1)

$$\begin{aligned} \psi(\varphi(Tz_1, T^2z_1, Tz_2, t)) \\ \leq k\{\psi(\varphi(z_1, Tz_1, z_2, t)) - \phi(\varphi(z_1, Tz_1, z_2, t))\} \\ \psi(\varphi(z_1, z_1, z_2, t)) \end{aligned}$$

$$\leq k\{\psi(\varphi(z_1, z_1, z_2, t)) - \phi(\varphi(z_1, z_1, z_2, t))\}$$

$$\psi(\varphi(z_1, z_1, z_2, t)) \leq 0$$

$$\varphi(z_1, z_1, z_2, t) = 0. \text{ That is } z_1 = z_2.$$

Hence T has a unique fixed point $\square$

- [12] Zadeh, L. A, Fuzzy sets, *Information and Computation*, 8(1965) 338-353.

ISSN(P):2319 – 3786

Malaya Journal of Matematik

ISSN(O):2321 – 5666

References

- [1] Alam .A, Imdad .M., Relation theoretic contraction principle, *Journal of Fixed Point Theory and Applications*, July 2015.
- [2] Ciric.L.B., A Generalization of a contraction principle, *Proc. Am. Math. Soc.*, 45 (1974), 267 - 273.
- [3] George . A.and Veeramani.P., On some result in fuzzy metric space, *Fuzzy sets system* , 64 (1994), 393 - 399.
- [4] Grabiec .M., Fixed points in fuzzy metric spaces , *fuzzy sets systems*, 27 (1988), 383-389.
- [5] Kramosil.J, Michalek.J: Fuzzy metric and statistical metric spaces. *Kibernetika*.11(1975), 336 – 344.
- [6] Mihet.D., A Generalization of a contraction principle in probabilistic Metric spaces II, *Int.J.Math Sci.*, 5(2005), 729 -736.
- [7] Mihet D., A Banch Contraction theorem in fuzzy metric spaces, *Fuzzy Sets and system*, 144(2004),431 -439.
- [8] Park J.H., Park.J.S and Kwun .Y.C., Fixed point in $\mathcal{M}$ -fuzzy metric spaces, *Optimization and Decision Making*, 7(2008),No.4,305 - 315.
- [9] Sedghi.S and Shobe.N, Fixed point theorem in $\mathcal{M}$ -fuzzy metric spaces with property (E), *Advances in Fuzzy mathematics*, 1(1),(2006),55 – 65.
- [10] Vasuki., Common fixed point for R-weakly commuting maps in fuzzy metric spaces. *Indian J. Pure Appl. Math.*, 30(4)1999, 419 – 423.
- [11] Veerapandi T., Jeyaraman M and Paul Raj Joseph J., Some fixed point and coincident point theorem in generalized M-fuzzy metric spaces *Int.Journal of Math. Analysis*, Vol.3,(2009), 627- 635.