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Monotone interval fuzzy neutrosophic soft eigenproblem

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Abstract

The Interval Fuzzy Neutrosophic Soft Eigenproblem in max-min Fuzzy Neutrosopic Soft Algebra (FNSA) is studied. A classification of Interval Fuzzy neutrosophic Soft Eigenvectors (IFNSEvs) is introduced and six types of IFNSEvs are described. Characterization of all six types is given for the case of strictly increasing FNSEvs and Hasse diagram of relations between the types are presented.

Keywords

Fuzzy Neutrosophic Soft Set (FNSS), Fuzzy Neutrosophic Soft Matrices(FNSMs), Fuzzy Neutrosophic Soft Eigenvector (FNSEvs), Interval Fuzzy Neutrosophic Soft Eigenvectors (IFNSEvs), Interval Fuzzy Neutrosophic Soft Matrices (IFNSMs) **AMS Subject Classification**

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Contents

1. Introduction

Most of our traditional tools for formal modeling, reasoning and computing are crisp, deterministic and precise in character. However, in real life, there are many complicated problems in Engineering, Economics, Environment, Social Sciences, Medical Sciences etc., that involve data which are not all always crisp, precise and deterministic in character because of various uncertainties of typical problems. Such uncertainties are being dealt with the help of the theories, like theory of Probability, theory of Fuzzy sets (Zadeh 1965) [38], theory of Intuitionistic fuzzy sets (Atanassov 1986)[3], theory of Vague set (Gau and Buehrer 1993) [5, 13], theory of Interval Mathematics (Moore, 1996) [26, 33] and theory of Rough sets (Pawlak 1982) [27]. But all these theories have their own difficulties. The reason of the difficulties is possibly, the inadequacy of the parametrization tool of the theories.

To overcome these difficulties Molodtsov [29] introduced the concept of soft set as a new mathematical tool for dealing with uncertainty which is free from the difficulties that have troubled the usual theoretical approaches. He successfully applied the soft theory into several directions, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, theory of probability, theory of measurement and so on. Maji *et al.* [22] initiated the concept of soft sets with some properties regarding fuzzy soft union, intersection, complement of fuzzy soft set and they are applied in decision making problem. Further, [24] Maji et. al, successfully extended the soft set as fuzzy soft set and intuitionistic fuzzy soft set and studied the application of these soft sets in decision-making problems. Jang et. al, studied interval valued intuitionistic fuzzy sets.

The IFS can only handle the incomplete information considering both the truth-membership (or simply membership) and falsity-membership (or non-membership) values. It does not handle the indeterminate and inconsistent information which exists in belief system. Smarandache [32] introduced the concept of neutrosophic set which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data. This theory is a powerful tool which generalize the concept of the classical set, fuzzy set, intuitionistic fuzzy set, paraconsistent sets, dialetheist sets, paradoxist sets, and tautological set and so on.

Fuzzy matrices defined first time by Thomason in 1977 [34] and discussed about the convergence of the powers of a fuzzy matrix. The theory of fuzzy matrices were developed by Kim and Roush [20] as an extension of Boolean matrices. Manoj Bora *et al.* [28] have applied intuitionistic fuzzy soft matrices in the medical diagnosis problem. Arockiarani and Sumathi [1, 2, 36] introduced Fuzzy Neutrosophic Soft Matrix (FNSM) and used them in decision making problems also in which they defined new type of operations. Broumi *et al.* [4] proposed the concept of generalized interval neutrosophic soft set and studied their operations. Also, they presented an application of it in decision making problem. Kavitha *et al.* [16–18] introduced the concept of unique solvability of max-min operation through FNSM equation $Ax = b$ and explained strong regularity of FNSMs over fuzzy neutrosophic soft algebra and computing the greatest X-eigenvector of fuzzy neutrosophic soft matrix. They also addressed on the power of fuzzy neutrosophic soft matrix. In [37], Uma et. al, introduced the concept of fuzzy neutrosophic soft matrices of Type-1 and Type-2.

In practice, the values of vector or matrix inputs are not exact numbers and often they are rather contained in some intervals. Considering matrices and vectors with interval coefficients is therefore of great practical importance, see [7, 8, 10, 14, 30]. This paper investigates monotone IFNSEvs of IFNSMs in max-min FNSA.

By max-min FNSA we understand a triple $(\mathscr{N}, \oplus, \otimes)$, where $\mathscr{N}$ is a linearly ordered set and $\oplus = \max$, $\otimes = \min$ are binary operations on $\mathscr{N}$. The notation $\mathscr{N}_{(m,n)}, \mathscr{N}_{(n)}$ denotes the set of all FNSMs, FNSVs of given dimension over $\mathscr{N}$. Operations $\otimes, \otimes$ are extended to FNSMs and FNSVs in a formal way. The linear ordering on $\mathscr{N}$ induces partial ordering on $\mathscr{N}_{(m,n)}$ and $\mathscr{N}_{(n)}$, the notations $\wedge(\vee)$ and $\bigwedge(\bigvee)$ are used for the operation of meet (join) in these sets.

The FNSE problem for a given FNSM $A \in \mathscr{N}_{(n,n)}$ in max-min FNSA consists of finding a value $\langle \lambda^T, \lambda^I, \lambda^F \rangle \in \mathscr{N}$ (FNSE value) and a FNSV $\langle x^T, x^I, x^F \rangle \in \mathscr{N}_{(n)}$ (FNSEv) such that the equation $A \otimes \langle x^T, x^I, x^F \rangle = \langle \lambda^T, \lambda^I, \lambda^F \rangle \otimes \langle x^T, x^I, x^F \rangle$ holds true. It is well-known that the above problem in max-min FNSA can be reduced to solving the equation. $A \otimes \langle x^T, x^I, x^F \rangle = \langle x^T, x^I, x^F \rangle$. Th FNSE problem in max-min FNSA has been studied by many authors. Interesting results were found in describing the structure of the FNSE space (the set of all FNSEvs), and algorithms for computing the largest FNSEv of a given FNSM were suggested in [18].

A classification consisting of six different types of IFNSEvs in presented in this paper and detailed characterization of all described types is given for strictly increasing IFNSEvs using the methods provided in [11].

2. Preliminaries

This section basically describes Neutrosophic Set (NS), Fuzzy Neutrosophic Soft Set (FNSS), Fuzzy Neutrosophic Soft Matrix (FNSM) and Fuzzy Neutrosophic Soft Matrices of type-I.

Definition 2.1. [35] A neutrosophic set A on the universe of discourse X is defined as $A = \{\langle x,\ T_A(x),\ I_A(x),\ F_A(x)\rangle, x \in X\}$, where $T, I, F : X \to]^{-}0, 1^{+}[$ and

$$^{-}0 \le T_A(x) + I_A(x) + F_A(x) \le 3^{+}. \tag{2.1}$$

From philosophical point of view the neutrosophic set takes the value from real standard or non-standard subsets of $]^{-}0, 1^{+}[$, But in real life application especially in Scientific and Engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^{-}0, 1^{+}[$. Hence we consider the neutrosophic set which takes the value from the subset of $[0, 1]$. Therefore we can rewrite equation (2.1) as $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$. In short an element $\widetilde{a}$ in the neutrosophic set A, can be written as $\widetilde{a} = \langle a^T, a^I, a^F \rangle$, where a^T denotes degree of truth, a^I denotes degree of indeterminacy, a^F denotes degree of falsity such that $0 \le a^T + a^I + a^F \le 3$.

Example 2.2. Assume that the universe of discourse $X = \{x_1,\ x_2,\ x_3\}$ where $x_1,\ x_2$ and x_3 characterize the quality, reliability, and the price of the objects. It may be further assumed that the values of $\{x_1,\ x_2,\ x_3\}$ are in [0,1] and they are obtained from some investigations of some experts. The experts may impose their opinion in three components viz; the degree of goodness, the degree of indeterminacy and the degree of poorness to explain the characteristics of the objects. Suppose A is a Neutrosophic Set (NS) of X, such that $A = \{\langle x_1,\ 0.4,\ 0.5,\ 0.3\rangle, \langle x_2,\ 0.7,\ 0.2,\ 0.4\rangle, \langle x_3, 0.8,\ 0.3,\ 0.4\rangle\}$ where for x_1 the degree of goodness of quality is 0.4, degree of indeterminacy of quality is 0.5 and degree of falsity of quality is 0.3 etc,.

Definition 2.3. [29] Let U be the initial universe set and E be a set of parameter. Consider a non-empty set $A, A \subset E$. Let $P(U)$ denotes the set of all fuzzy neutrosophic sets of U. The collection (F, A) is termed to the fuzzy neutrosophic soft set over U, where F is a mapping given by $F : A \to\ P(U)$. Here after we simply consider A as FNSS over U instead of (F, A).

Definition 2.4. [2] Let $U = \{c_1, c_2, ..., c_m\}$ be the universal set and E be the set of parameters given by $E = \{e_1, e_2, ..., e_m\}$. Let $A \subset E$. A pair (F, A) be a FNSS over U. Then the subset of $U \times E$ is defined by $R_A = \{(u, e);\ e \in A,\ u \in F_A(e)\}$ which is called a relation form of $(F_A,\ E)$. The membership function, indeterminacy membership function and non membership function are written by $T_{R_A} : U \times E \to [0, 1]$, $I_{R_A} : U \times E \to [0, 1]$ and $F_{R_A} : U \times E \to [0, 1]$ where $T_{R_A}(u, e) \in [0, 1]$, $I_{R_A}(u, e) \in [0, 1]$ and $F_{R_A}(u, e) \in$

$[0,1]$ are the membership value, indeterminacy value and non membership value respectively of $u \in U$ for each $e \in E$.
If $[(T_{ij},\ I_{ij},\ F_{ij})] = [T_{ij}(u_i,e_j),\ I_{ij}(u_i,e_j)\ ,F_{ij}(u_i,e_j)]$ we define a matrix

$$[\langle T_{ij}\ ,I_{ij},\ F_{ij}\rangle]_{m\times n} = \begin{bmatrix} \langle T_{11},\ I_{11},\ F_{11}\rangle & \cdots & \langle T_{1n},\ I_{1n},\ F_{1n}\rangle \\ \langle T_{21},\ I_{21},\ F_{21}\rangle & \cdots & \langle T_{2n},\ I_{2n},\ F_{2n}\rangle \\ \vdots & \vdots & \vdots \\ \langle T_{m1},\ I_{m1},\ F_{m1}\rangle & \cdots & \langle T_{mn},\ I_{mn},\ F_{mn}\rangle \end{bmatrix}$$

Which is called an $m \times n$ $FNSM$ of the FNSS (F_A, E) over U

Definition 2.5. [37] Let $A = (\langle a^T_{ij},\ a^I_{ij},\ a^F_{ij}\rangle)$, $B = \langle (b^T_{ij},\ b^I_{ij},\ b^F_{ij})\rangle \in \mathscr{N}_{m\times n}$. The component wise addition and component wise multiplication is defined as
$A \oplus B = (sup\{a^T_{ij},\ b^T_{ij}\},\quad sup\{a^I_{ij},\ b^I_{ij}\},\quad inf\{a^F_{ij},\ b^F_{ij}\})$
$A \otimes B = (inf\{a^T_{ij},\ b^T_{ij}\},\quad inf\{a^I_{ij},\ b^I_{ij}\},\quad sup\{a^F_{ij},\ b^F_{ij}\})$

Definition 2.6. Let $A \in \mathscr{N}_{m\times n},\ B \in \mathscr{N}_{n\times p}$, the composition of A and B is defined as.

$$A \circ B = \left(\sum_{k=1}^{n}(a^T_{ik} \wedge b^T_{kj}),\quad \sum_{k=1}^{n}(a^I_{ik} \wedge b^I_{kj}),\quad \prod_{k=1}^{n}(a^F_{ik} \vee b^F_{kj})\right)$$

equivalently we can write the same as

$$= \left(\bigvee_{k=1}^{n}(a^T_{ik} \wedge b^T_{kj}),\quad \bigvee_{k=1}^{n}(a^I_{ik} \wedge b^I_{kj}),\quad \bigwedge_{k=1}^{n}(a^F_{ik} \vee b^F_{kj})\right).$$

The product $A \circ B$ is defined if and only if the number of columns of A is same as the number of rows of B. Then A and B are said to be conformable for multiplication. We shall use AB instead of $A \circ B$. Where $\sum(a^T_{ik} \wedge\ b^T_{kj})$ means max-min operation and $\prod_{k=1}^{n}(a^F_{ik} \vee\ b^F_{kj})$ means min-max operation.

3. Interval Fuzzy Neutrosophic Soft Eigenvectors Classification

In this section we define six types of IFNSEvs of IFNSMs and describe the necessary and sufficient conditions for these types of monotone IFNSEvs.
Let n be a given natural number. We shall use the notation $N = \{1,2,...,n\}$. Similarly to [8, 10, 14], we define IFNSM with bounds $\underline{A},\overline{A} \in \mathscr{N}_{(n,n)}$ and IFNSV with bounds $\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle, \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle \in \mathscr{N}_{(n)}$, as follows
$[\underline{A},\overline{A}] = \{A \in \mathscr{N}_{(n,n)};\ \underline{A} \le A \le \overline{A}\}$,
$[\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle, \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle] = \{\langle x^T,x^I,x^F\rangle \in \mathscr{N}_{(n)};\ \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle \le \langle x^T,x^I,x^F\rangle \le \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle\}$.
We assume in this section that an IFNSM $\mathbf{A} = [\underline{A},\overline{A}]$ and an IFNSEv
$\mathbf{X} = [\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle, \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle]$ are fixed. The IFNSE problem for $\mathbf{A}$ and $\mathbf{X}$ consists in recognizing whether $A \otimes \langle x^T,x^I,x^F\rangle = \langle x^T,x^I,x^F\rangle$
holds true for $A \in \mathbf{A}, \langle x^T,x^I,x^F\rangle \in \mathbf{X}$. In dependence on the applied quantifiers, we get six types of IFNSEvs.

Definition 3.1. If IFNSM $\mathbf{A}$ is given, then IFNSV $\mathbf{X}$ is called
1. Strong FNSEv of $\mathbf{A}$ if
$(\forall A \in \mathbf{A})\ (\forall \langle x^T,x^I,x^F\rangle \in \mathbf{X})$
$[A \otimes \langle x^T,x^I,x^F\rangle = \langle x^T,x^I,x^F\rangle]$,
2. Strong universal FNSEv of $\mathbf{A}$ if
$(\exists \langle x^T,x^I,x^F\rangle \in \mathbf{X})(\forall A \in \mathbf{A})[A \otimes \langle x^T,x^I,x^F\rangle = \langle x^T,x^I,x^F\rangle]$,
3. Universal FNSEv of $\mathbf{A}$ if
$(\forall A \in \mathbf{A})(\exists \langle x^T,x^I,x^F\rangle \in \mathbf{X})[A \otimes \langle x^T,x^I,x^F\rangle = \langle x^T,x^I,x^F\rangle]$,
4. Strong tolerance FNSEv of $\mathbf{A}$ if
$(\exists A \in \mathbf{A})(\forall \langle x^T,x^I,x^F\rangle \in \mathbf{X})[A \otimes \langle x^T,x^I,x^F\rangle = \langle x^T,x^I,x^F\rangle]$,
5. Tolerance FNSEv of $\mathbf{A}$ if
$(\forall \langle x^T,x^I,x^F\rangle \in \mathbf{X})(\exists A \in \mathbf{A})[A \otimes \langle x^T,x^I,x^F\rangle = \langle x^T,x^I,x^F\rangle]$,
6. Weak FNSEv of $\mathbf{A}$ if
$(\exists A \in \mathbf{A})(\exists \langle x^T,x^I,x^F\rangle \in \mathbf{X})[A \otimes \langle x^T,x^I,x^F\rangle = \langle x^T,x^I,x^F\rangle]$.
Analogously as in [11], we denote the set of all strictly increasing FNSVs of dimension n as

$$\mathscr{N}^{<}_{(n)} = \{\langle x^T,x^I,x^F\rangle \in \mathscr{N}_{(n)}; (\forall i,j,\in N)[i \le j \Rightarrow \langle x^T_i,x^I_i,x^F_i\rangle < \langle x^T_j,x^I_j,x^F_j\rangle]\},$$

and the set of all increasing FNSV as

$$\mathscr{N}^{\le}_{(n)} = \{\langle x^T,x^I,x^F\rangle \in \mathscr{N}_{(n)}; (\forall i,j,\in N)[i < j \Rightarrow \langle x^T_i,x^I_i,x^F_i\rangle \le \langle x^T_j,x^I_j,x^F_j\rangle]\}.$$

Further we denote the FNSE space of a FNSM $A \in \mathscr{N}_{(n,n)}$ as

$$\mathscr{F}(A) = \{\langle x^T,x^I,x^F\rangle \in \mathscr{N}_{(n)}; A \otimes \langle x^T,x^I,x^F\rangle = \langle x^T,x^I,x^F\rangle\},$$

and the FNSE space of all strictly increasing FNSEvs (increasing FNSEvs) as

$$\mathscr{F}^{<}(A) = \mathscr{F}(A) \cap \mathscr{N}^{<}_{(n)},\ \mathscr{F}^{\le}(A) = \mathscr{F}(A) \cap \mathscr{N}^{\le}_{(n)}.$$

it is clear that any FNSV $\langle x^T,x^I,x^F\rangle \in \mathscr{N}_{(n)}$ can be permuted to an increasing FNSV. Therefore, in view of the next theorem, the structure of the FNSE space $\mathscr{F}(A)$ of a given $n \times n$ max-min FNSM A can be described by investigating the structure of monotone FNSE spaces $\mathscr{F}^{<}(A_{\varphi\varphi})$ and $\mathscr{F}^{\le}(A_{\varphi\varphi})$, for all permutations φ on N.

Theorem 3.2. Let $A \in \mathscr{N}_{(n,n)},\ \langle x^T,x^I,x^F\rangle \in \mathscr{N}_{(n)}$ and let φ be a permutation on N. Then $\langle x^T,x^I,x^F\rangle \in \mathscr{F}(A)$ if and only if $\langle x^T,x^I,x^F\rangle_\varphi \in \mathscr{F}(A_{\varphi\varphi})$.
Proof. Let ε be the identical permutation on N. It is easy to see that the following formulas are equivalent:
$A \otimes \langle x^T,x^I,x^F\rangle = \langle x^T,x^I,x^F\rangle, A_{\varphi\varepsilon} \otimes \langle x^T,x^I,x^F\rangle = \langle x^T,x^I,x^F\rangle_\varphi$, $A_{\varphi\varphi} \otimes \langle x^T,x^I,x^F\rangle_\varphi = \langle x^T,x^I,x^F\rangle_\varphi$. By this, the proof is complete.
For $A \in \mathscr{N}_{(n,n)}$ the structure of $\mathscr{F}^{<}(A)$ has been described in [11] as an interval of strictly increasing FNSVs. FNSVs $m^*(A),\ M^*(A) \in \mathscr{N}_{(n)}$ are defined as follows.
For any $i \in N$, we put

$$m^*_i(A) := \max_{j \le i}\max_{k > j}\langle a^T_{jk}, a^I_{jk}, a^F_{jk}\rangle,\quad M^*_i(A) := \min_{j \ge i}\max_{k \ge j}\langle a^T_{jk}, a^I_{jk}, a^F_{jk}\rangle.$$

Remark 3.3. If a maximum of an empty set should be computed in the above definition of $m^*(A)$, then we use the fact that, by usual definition, max ϕ is the least element in $\mathscr{N}$.

Theorem 3.4. [11] Let $A \in \mathscr{N}_{(n,n)}$ and $\langle x^T, x^I, x^F \rangle \in \mathscr{N}_{(n)}$ be a strictly increasing FNSV. Then $\langle x^T, x^I, x^F \rangle \in \mathscr{F}(A)$ if and only if $m^*(A) \leq \langle x^T, x^I, x^F \rangle \leq M^*(A)$. In formal notation,

$$\mathscr{F}^{<}(A) = \langle m^*(A),\ M^*(A) \rangle \cap \mathscr{N}^{<}_{(n)}.$$

In this paper our considerations will be restricted to strictly increasing FNSEvs in $\mathbf{X}$. The restricted IFNSEvs will be called monotone IFNSEvs and similarly, the name of the restricted types will be extended by 'monotone', i.e. monotone strong FNSEv,...,monotone weak FNSEv.

Formally, we denote by $\mathbf{X}^{<} = [\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle] \cap \mathscr{N}^{<}_{(n)}$ the set of all strictly increasing FNSVs in $[\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle]$. Then all the above general definitions concerning IFNSEvs and their types are modified by substituting $\mathbf{X}$ by $\mathbf{X}^{<}$.

Using Theorem 3.4, we describe necessary and sufficient conditions for six types of monotone IFNSEvs defined in Definition 3.1, which will be referred to as $T1,\ T2,\ T3,\ T4,\ T5,\ T6$.

Theorem 3.5. $(T1)$ Let IFNSM $\mathbf{A} = [\underline{A}, \overline{A}]$ and monotone IFNSV $\mathbf{X}^{<}$ with strictly increasing bounds $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle$, $\langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle$ be given. Then $\mathbf{X}^{<}$ is a monotone strong FNSEv of $\mathbf{A}$ if and only if

$$m^*(\overline{A}) \leq \langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle,\ \ \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle \leq M^*(\underline{A}). \tag{3.1}$$

Proof. Let $\mathbf{X}^{<}$ is a monotone strong FNSEv of $\mathbf{A}$. Then $A \otimes \langle x^T, x^I, x^F \rangle = \langle x^T, x^I, x^F \rangle$ holds for every $A \in \mathbf{A}$ and every strictly increasing FNSV $\langle x^T, x^I, x^F \rangle \in \mathbf{X}^{<}$. In particular,

$$\overline{A} \otimes \langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle = \langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle,\ \underline{A} \otimes \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle = \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle.$$

In view of Theorem 3.4 we immediately get the inequalities in (3.1).

To prove the converse implication, let us assume that the inequalities in (3.1) hold true. Then for every $A \in \mathbf{A}$ and every strictly increasing FNSV $\langle x^T, x^I, x^F \rangle \in \mathbf{X}^{<}$ we get by the monotonicity results

$$m^*(A) \leq m^*(\overline{A}) \leq \langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle \leq \langle x^T, x^I, x^F \rangle \leq \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle \leq M^*(\underline{A}) \leq M^*(A).$$

Hence, $m^*(A) \leq \langle x^T, x^I, x^F \rangle \leq M^*(A)$ and $A \otimes \langle x^T, x^I, x^F \rangle = \langle x^T, x^I, x^F \rangle$, in view of Theorem 3.4. In other words $\mathbf{X}^{<}$ is a monotone strong FNSEv of $\mathbf{A}$.

Theorem 3.6. $(T2)$ Let IFNSM $\mathbf{A} = [\underline{A}, \overline{A}]$ and monotone IFNSV $\mathbf{X}^{<} = [\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle]$ with strictly increasing bounds $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle$ be given. Then $\mathbf{X}^{<}$ is a monotone strong universal FNSEv of $\mathbf{A}$ if and only if

$$m^*(\overline{A}) \leq \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle,\ \langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle \leq M^*(\underline{A}),\ \langle m^*(\overline{A}), M^*(\underline{A}) \rangle \cap \mathscr{N}^{<}_{(n)} \neq \phi. \tag{3.2}$$

Proof. Let us assume that $\mathbf{X}^{<}$ is a monotone strong universal FNSEv of $\mathbf{A}$, i.e. there exists strictly increasing FNSV $\langle x^T, x^I, x^F \rangle \in \mathbf{X}^{<}$ such that $A \otimes \langle x^T, x^I, x^F \rangle = \langle x^T, x^I, x^F \rangle$ holds for every $A \in \mathbf{A}$, in particular,
$\underline{A} \otimes \langle x^T, x^I, x^F \rangle = \langle x^T, x^I, x^F \rangle$ and $\overline{A} \otimes \langle x^T, x^I, x^F \rangle = \langle x^T, x^I, x^F \rangle$. In view of Theorem 3.4, we get the inequalities $m^*(\overline{A}) \leq \langle x^T, x^I, x^F \rangle \leq M^*(\underline{A})$, which directly imply all three conditions in (3.2).

To prove the converse implication, let us assume that the conditions in (3.2) hold true, i.e. $m^*(\overline{A}) \leq \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle$, $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle \leq M^*(\underline{A})$ and there is a strictly increasing FNSV $\langle x^T, x^I, x^F \rangle'$ with $m^*(\overline{A}) \leq \langle x^T, x^I, x^F \rangle' \leq M^*(\underline{A})$. Let us denote
$\langle x^T, x^I, x^F \rangle = (\langle x^T, x^I, x^F \rangle' \vee \langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle) \wedge \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle$. Using distributivity of operations $\wedge, \vee$, it is easy to show that
$\langle x^T, x^I, x^F \rangle = (\langle x^T, x^I, x^F \rangle' \wedge \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle) \vee \langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle$ and it is also clear that $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle \leq \langle x^T, x^I, x^F \rangle \leq \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle$. By assumption, FNSVs
$\langle x^T, x^I, x^F \rangle'$, $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle$, $\langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle$ are strictly increasing. As the operations $\wedge, \vee$ preserve strict monotonicity, FNSV $\langle x^T, x^I, x^F \rangle$ is strictly increasing, too. Further, we have

$$m^*(\overline{A}) \leq \langle x^T, x^I, x^F \rangle' \wedge \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle \leq \langle x^T, x^I, x^F \rangle \leq \langle x^T, x^I, x^F \rangle' \vee \langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle \leq M^*(\underline{A}).$$

As a consequence, for any FNSM $A \in \mathbf{A}$ we get

$$m^*(A) \leq m^*(\overline{A}) \leq \langle x^T, x^I, x^F \rangle \leq M^*(\underline{A}) \leq M^*(A),$$

i.e. $A \otimes \langle x^T, x^I, x^F \rangle = \langle x^T, x^I, x^F \rangle$. Thus, $\mathbf{X}^{<}$ is a monotone strong universal FNSEv of $\mathbf{A}$.

Theorem 3.7. $(T3)$ Let IFNSM $\mathbf{A} = [\underline{A}, \overline{A}]$ and monotone IFNSV
$\mathbf{X}^{<} = [\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle]$ with strictly increasing bounds $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle$, $\langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle$ be given. Then $\mathbf{X}^{<}$ is a monotone universal
FNSEv of $\mathbf{A}$ if and only if

$$m^*(\overline{A}) \leq \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle, \langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle \leq M^*(\underline{A}), (\forall A \in \mathbf{A})[\langle m^*(\overline{A}), M^*(\underline{A}) \rangle \cap \mathscr{N}^{<}_{(n)} \neq \phi]. \tag{3.3}$$

Proof. Let us assume that $\mathbf{X}^{<}$ is a monotone universal FNSEv of $\mathbf{A}$, i.e. for every $A \in \mathbf{A}$ there exists strictly increasing FNSV $\langle x^T, x^I, x^F \rangle \in \mathbf{X}^{<}$ such that
$A \otimes \langle x^T, x^I, x^F \rangle = \langle x^T, x^I, x^F \rangle$ holds. In particular, there exists $\langle x^T, x^I, x^F \rangle', \langle x^T, x^I, x^F \rangle'' \in \mathbf{X}^{<}$ such that
$\underline{A} \otimes \langle x^T, x^I, x^F \rangle' = \langle x^T, x^I, x^F \rangle'$ and
$\overline{A} \otimes \langle x^T, x^I, x^F \rangle'' = \langle x^T, x^I, x^F \rangle''$. In view of Theorem 3.4, we get the inequalities $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle \leq \langle x^T, x^I, x^F \rangle' \leq M^*(\underline{A})$,
$m^*(\overline{A}) \leq \langle x^T, x^I, x^F \rangle'' \leq \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle$. The third condition in (3.3) follows directly from the assumption and from Theorem 3.4.

To prove the converse implication, let us assume that the conditions in (3.3) hold true, i.e.
$m^*(\overline{A}) \leq \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle, \langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle \leq M^*(\underline{A})$ and for every $A \in \mathbf{A}$, there is a strictly increasing FNSV $\langle x^T, x^I, x^F \rangle'$ with $m^*(A) \leq \langle x^T, x^I, x^F \rangle' \leq M^*(A)$. Let FNSM A and FNSV $\langle x^T, x^I, x^F \rangle'$

be fixed and let us denote
$\langle x^T,x^I,x^F\rangle = (\langle x^T,x^I,x^F\rangle' \vee \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle) \wedge \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle$. Similarly as in the above proof, we show easily that $\langle x^T,x^I,x^F\rangle = (\langle x^T,x^I,x^F\rangle' \wedge \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle) \vee \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle$ and $\langle x^T,x^I,x^F\rangle \in \mathbf{X}^<$. By the assumption we have $m^*(A) \leq m^*(\overline{A}) \leq \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle$ and $\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle \leq M^*(\underline{A}) \leq M^*(A)$, which implies

$$m^*(A) \leq \langle x^T,x^I,x^F\rangle' \wedge \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle \leq \langle x^T,x^I,x^F\rangle \leq \langle x^T,x^I,x^F\rangle' \vee \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle \leq M^*(\underline{A}).$$

As a consequence, we get $m^*(A) \leq \langle x^T,x^I,x^F\rangle \leq M^*(A)$, i.e. $A \otimes \langle x^T,x^I,x^F\rangle = \langle x^T,x^I,x^F\rangle$. As the fixed FNSM $A \in \mathbf{A}$ is arbitrary, we have proved that $\mathbf{X}^<$ is a monotone universal FNSEv of $\mathbf{A}$.

Theorem 3.8. $(T4)$ Let IFNSM $\mathbf{A} = [\underline{A},\overline{A}]$ and monotone IFNSV
$\mathbf{X}^< = [\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle, \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle]$ with strictly increasing bounds $\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle$, $\langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle$ be given. Further let us denote

$$\tilde{\mathbf{A}} = \{A \in \mathbf{A};\ m^*(A) \leq \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle\},\ \ \tilde{A} = \bigvee \tilde{\mathbf{A}}.$$

Then $\mathbf{X}^<$ be monotone strong tolerance FNSEv of $\mathbf{A}$ if and only if

$$m^*(\underline{A}) \leq \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle,\ \ \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle \leq M^*(\tilde{A}). \qquad (3.4)$$

Proof. Let $\mathbf{X}^<$ be monotone strong tolerance FNSEv of $\mathbf{A}$. Then there exists FNSM $A \in \mathbf{A}$ such that $A \otimes \langle x^T,x^I,x^F\rangle = \langle x^T,x^I,x^F\rangle$ holds for every FNSV $\langle x^T,x^I,x^F\rangle \in \mathbf{X}^<$. In particular we have $A \otimes \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle = \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle$, $A \otimes \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle = \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle$ and $m^*(A) \leq \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle \leq \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle \leq M^*(A)$. Hence $A \in \tilde{\mathbf{A}}$, which implies $\underline{A} \leq A \leq \tilde{A}$, $m^*(\underline{A}) \leq m^*(A) \leq \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle$ and $\langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle \leq M^*(A) \leq M^*(\tilde{A})$.
Conversely, let $m^*(\underline{A}) \leq \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle$ $(i.e. \underline{A} \in \tilde{\mathbf{A}})$, and $\langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle \leq M^*(\tilde{A})$. It is easy to verify that for any two FNSMs $A', A'' \in \tilde{\mathbf{A}}$ also their join $A' \vee A''$ belongs to $\mathbf{A}$. In other words, $\tilde{\mathbf{A}}$ is closed under the operation $\vee$. As a consequence, $\tilde{A} \in \tilde{\mathbf{A}}$ and

$$m^*(\tilde{A}) \leq \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle \leq \langle x^T,x^I,x^F\rangle \leq \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle \leq M^*(\tilde{A}),\quad \tilde{A} \otimes \langle x^T,x^I,x^F\rangle = \langle x^T,x^I,x^F\rangle$$

holds for every $\langle x^T,x^I,x^F\rangle \in \mathbf{X}^<$.
Hence, $\mathbf{X}^<$ is monotone strong tolerance FNSEv of $\mathbf{A}$.

Theorem 3.9. $(T5)$ Let IFNSM $\mathbf{A} = [\underline{A},\overline{A}]$ and monotone IFNSV
$\mathbf{X}^< = [\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle, \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle]$ with strictly increasing bounds $\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle$, $\langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle$ be given. Then $\mathbf{X}^<$ is a monotone tolerance FNSEv of $\mathbf{A}$ if and only if

$$m^*(\underline{A}) \leq \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle,\ \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle \leq M^*(\overline{A}). \qquad (3.5)$$

Proof. Let $\mathbf{X}^<$ be monotone tolerance FNSEv of $\mathbf{A}$. Then for every FNSV $\langle x^T,x^I,x^F\rangle \in \mathbf{X}^<$ there exists FNSM $A \in \mathbf{A}$ such that $A \otimes \langle x^T,x^I,x^F\rangle = \langle x^T,x^I,x^F\rangle$. In particular, there are FNSMs $A', A'' \in \mathbf{A}$ with $A' \otimes \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle = \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle$,
$A'' \otimes \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle = \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle$, which gives $m^*(\underline{A}) \leq m^*(A') \leq \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle$ and $\langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle \leq M^*(A'') \leq M^*(\overline{A})$.

Conversely, let conditions (3.5) be fulfilled and let $\langle x^T,x^I,x^F\rangle \in \mathbf{X}^<$ be arbitrary, but fixed.
Define FNSM $A \in \mathcal{N}_{(n,n)}$ by putting, for every $j,k \in N$,

$$\langle a^T_{jk},a^I_{jk},a^F_{jk}\rangle = \begin{cases} \langle \overline{a}^T_{jk},\overline{a}^I_{jk},\overline{a}^F_{jk}\rangle \wedge \langle x^T_j,x^I_j,x^F_j\rangle & j<k, \\ \langle \overline{a}^T_{jk},\overline{a}^I_{jk},\overline{a}^F_{jk}\rangle & j \leq k \end{cases}$$

For $i \in N$ we have, in view of the strict monotonicity of $\langle x^T,x^I,x^F\rangle$,

$$\begin{aligned} m^*_i(A) &= \max_{j\leq i}\max_{k>j}(\langle \overline{a}^T_{jk},\overline{a}^I_{jk},\overline{a}^F_{jk}\rangle \wedge \langle x^T_j,x^I_j,x^F_j\rangle) \\ &\leq \max_{j\leq i}\max_{k>j}(\langle \overline{a}^T_{jk},\overline{a}^I_{jk},\overline{a}^F_{jk}\rangle \wedge \langle x^T_i,x^I_i,x^F_i\rangle) \\ &= (\max_{j\leq i}\max_{k>j}\langle \overline{a}^T_{jk},\overline{a}^I_{jk},\overline{a}^F_{jk}\rangle) \wedge \langle x^T_i,x^I_i,x^F_i\rangle \\ &= m^*_i(\overline{A}) \wedge \langle x^T_i,x^I_i,x^F_i\rangle. \end{aligned}$$

Hence $m^*(A) \leq m^*(\overline{A}) \wedge \langle x^T,x^I,x^F\rangle \leq \langle x^T,x^I,x^F\rangle$. Further we have

$$\begin{aligned} M^*_i(A) &= \min_{j\geq i}\max_{k\geq j}\langle a^T_{jk},a^I_{jk},a^F_{jk}\rangle \\ &= \min_{j\geq i}(\langle \overline{a}^T_{kk},\overline{a}^I_{kk},\overline{a}^F_{kk}\rangle \vee \max_{k>j}(\langle \overline{a}^T_{jk},\overline{a}^I_{jk},\overline{a}^F_{jk}\rangle \wedge \langle x^T_j,x^I_j,x^F_j\rangle)) \\ &\geq \min_{j\geq i}(\langle \overline{a}^T_{kk},\overline{a}^I_{kk},\overline{a}^F_{kk}\rangle \vee \max_{k>j}(\langle \overline{a}^T_{jk},\overline{a}^I_{jk},\overline{a}^F_{jk}\rangle \wedge \langle x^T_i,x^I_i,x^F_i\rangle)) \\ &\geq \min_{j\geq i}\max_{k\geq j}(\langle \overline{a}^T_{jk},\overline{a}^I_{jk},\overline{a}^F_{jk}\rangle \wedge \langle x^T_i,x^I_i,x^F_i\rangle) \\ &= (\min_{j\geq i}\max_{k\geq j}\langle \overline{a}^T_{jk},\overline{a}^I_{jk},\overline{a}^F_{jk}\rangle) \wedge \langle x^T_i,x^I_i,x^F_i\rangle \\ &= M^*(\overline{A}) \wedge \langle x^T_i,x^I_i,x^F_i\rangle. \end{aligned}$$

Hence $M^*(A) \geq M^*(\overline{A}) \wedge \langle x^T,x^I,x^F\rangle \geq \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle \wedge \langle x^T,x^I,x^F\rangle = \langle x^T,x^I,x^F\rangle$ and the inequalities $m^*(A) \leq \langle x^T,x^I,x^F\rangle \leq M^*(A)$ imply $A \otimes \langle x^T,x^I,x^F\rangle = \langle x^T,x^I,x^F\rangle$, in view of Theorem 3.4.
We have shown that $\mathbf{X}^<$ is a monotone tolerance FNSEv of $\mathbf{A}$.

Theorem 3.10. $(T6)$ Let IFNSM $\mathbf{A} = [\underline{A},\overline{A}]$ and monotone IFNSV
$\mathbf{X}^< = [\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle, \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle]$ with strictly increasing bounds $\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle$, $\langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle$ be given. Then $\mathbf{X}^<$ is a monotone weak FNSEv of $\mathbf{A}$ if and only if

$$\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle \leq M^*(\overline{A}),\ m^*(\underline{A}) \leq \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle,\ \ \langle m^*(\underline{A}),M^*(\overline{A})\rangle \cap \mathcal{N}^<_{(n)} \neq \phi. \qquad (3.6)$$

Proof. Let us assume that $\mathbf{X}^<$ is a monotone weak FNSEv of $\mathbf{A}$, i.e. there exists a FNSM $A \in \mathbf{A}$ and a strictly increasing FNSV $\langle x^T,x^I,x^F\rangle \in \mathbf{X}^<$ such that $A \otimes \langle x^T,x^I,x^F\rangle = \langle x^T,x^I,x^F\rangle$ holds. In view of Theorem 3.4, we have $m^*(\underline{A}) \leq m^*(A) \leq \langle x^T,x^I,x^F\rangle \leq M^*(A) \leq M^*(\overline{A})$,
hence $\langle m^*(\underline{A}),M^*(\overline{A})\rangle \cap \mathcal{N}^<_{(n)} \neq \phi$. Moreover, the above inequalities imply

$$\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle \leq \langle x^T,x^I,x^F\rangle \leq M^*(A) \leq M^*(\overline{A}),$$
$$m^*(\underline{A}) \leq m^*(A) \leq \langle x^T,x^I,x^F\rangle \leq \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle.$$

To prove the converse implication, let us assume that the conditions in (3.6) hold true, i.e. $\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle \leq M^*(\overline{A}), m^*(\underline{A}) \leq \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle$ and there is a strictly increasing FNSV $\langle x^T,x^I,x^F\rangle'$ with $m^*(\underline{A}) \leq \langle x^T,x^I,x^F\rangle' \leq M^*(\overline{A})$. We denote $\langle x^T,x^I,x^F\rangle = (\langle x^T,x^I,x^F\rangle' \vee \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle) \wedge \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle$. Similarly as in the proof of Theorem 3.6, it can be easily shown that $\langle x^T,x^I,x^F\rangle = (\langle x^T,x^I,x^F\rangle' \wedge \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle) \vee \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle$ and $\langle x^T,x^I,x^F\rangle \in \mathbf{X}^<$. Moreover, the inequalities $m^*(\underline{A}) \leq \langle x^T,x^I,x^F\rangle' \leq M^*(\overline{A})$ together with assumption $\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle \leq M^*(\overline{A})$,
$m^*(\underline{A}) \leq \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle$ imply

$$m^*(\underline{A}) \leq (\langle x^T,x^I,x^F\rangle' \wedge \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle) \leq (\langle x^T,x^I,x^F\rangle' \wedge \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle) \vee \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle = \langle x^T,x^I,x^F\rangle = ((\langle x^T,x^I,x^F\rangle' \vee \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle) \wedge \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle \leq \langle x^T,x^I,x^F\rangle' \vee \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle \leq M^*(\overline{A}).$$

Hence, $m^*(\underline{A}) \leq \langle x^T,x^I,x^F\rangle \leq M^*(\overline{A})$. Denoting

$$\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle' = \bigwedge\{\langle y^T,y^I,y^F\rangle \in \mathbf{X}^<; m^*(\underline{A}) \leq \langle y^T,y^I,y^F\rangle\}\langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle' = \bigvee\{\langle \underline{y}^T,y^I,y^F\rangle \in \mathbf{X}^<; \langle y^T,y^I,y^F\rangle \leq M^*(\overline{A})\},$$

we get $\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle' \leq \langle x^T,x^I,x^F\rangle \leq \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle'$.
Clearly, the inequalities $m^*(\underline{A}) \leq \langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle'$ and $\langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle' \leq M^*(\overline{A})$ holds true, which implies that the IFNSV $\mathbf{X}' = [\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle', \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle']$ is monotone tolerance FNSEv of $\mathbf{A}$, in view of Theorem 3.9. As we have shown above,
$\langle x^T,x^I,x^F\rangle$ belongs to $\mathbf{X}'$, hence there exists $A \in \mathbf{A}$ with
$m^*(A) \leq \langle x^T,x^I,x^F\rangle \leq M^*(A)$,
i.e. $A \otimes \langle x^T,x^I,x^F\rangle = \langle x^T,x^I,x^F\rangle$.

4. Relations between various types of monotone FNSEvs

Theorem 4.1. Let IFNSM $\mathbf{A} = [\underline{A},\overline{A}]$ and monotone IFNSV $\mathbf{X}^< = [\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle, \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle]$ with strictly increasing bounds $\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle$, $\langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle$ be given. Then the following implications hold true.
$(T1) \Rightarrow (T2)$ If $\mathbf{X}^<$ is a monotone strong FNSEv of $\mathbf{A}$, then $\mathbf{X}^<$ is a monotone strong universal FNSEv of $\mathbf{A}$,
$(T1) \Rightarrow (T4)$ If $\mathbf{X}^<$ is a monotone strong FNSEv of $\mathbf{A}$, then $\mathbf{X}^<$ is a monotone strong tolerance FNSEv of $\mathbf{A}$,
$(T3) \Rightarrow (T6)$ If $\mathbf{X}^<$ is a monotone universal FNSEv of $\mathbf{A}$, then $\mathbf{X}^<$ is a monotone weak FNSEv of $\mathbf{A}$,
$(T5) \Rightarrow (T6)$ if $\mathbf{X}^<$ is a monotone tolerance FNSEv of $\mathbf{A}$, then $\mathbf{X}^<$ is a monotone weak universal FNSEv of $\mathbf{A}$,
Proof. The implications follow directly from Definition 3.1.

Remark 4.2. It is easy to shown that converse implications to those in Theorem 4.1 do not hold true.

Theorem 4.3. $(T2 \Rightarrow T3)$ Let IFNSM $A = [\underline{A},\overline{A}]$ and monotone IFNSV $\mathbf{X}^< = [\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle, \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle]$ with strictly increasing bounds $\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle, \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle$ be given. If $\mathbf{X}^<$ is a monotone strong universal FNSEv of $\mathbf{A}$, then $\mathbf{X}^<$ is a monotone universal FNSEv of $\mathbf{A}$.
Proof. The implication follows directly from Definition 3.1.

The converse implication is not true as the next example shows.

Example 4.4. $(T3 \nRightarrow T2)$ Let $\overline{A}$, $\underline{A}$ and $\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle, \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle$ have the form

$$\underline{A} = \begin{bmatrix} \langle 0.2,0.1,0.8\rangle & \langle 0.2,0.1,0.8\rangle \\ \langle 0.3,0.2,0.7\rangle & \langle 0.4,0.3,0.6\rangle, \end{bmatrix},$$
$$\overline{A} = \begin{bmatrix} \langle 0.2,0.1,0.8\rangle & \langle 0.3,0.2,0.7\rangle \\ \langle 0.3,0.2,0.7\rangle & \langle 0.5,0.4,0.6\rangle \end{bmatrix}$$
$$\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle = \begin{bmatrix} \langle 0.2,0.1,0.8\rangle \\ \langle 0.3,0.2,0.7\rangle \end{bmatrix},$$
$$\langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle = \begin{bmatrix} \langle 0.3,0.2,0.7\rangle \\ \langle 0.4,0.3,0.6\rangle \end{bmatrix}.$$

Then for every FNSM $A \in \mathbf{A}$ of the form

$$A = \begin{bmatrix} \langle 0.2,0.1,0.8\rangle & \langle k^T,k^I,k^F\rangle \\ \langle 0.3,0.2,0.7\rangle & \langle s^T,s^I,s^F\rangle \end{bmatrix},$$

$$\langle 0.2,0.1,0.8\rangle \leq \langle k^T,k^I,k^F\rangle \leq \langle 0.3,0.2,0.7\rangle,$$
$$\langle 0.4,0.3,0.6\rangle \leq \langle s^T,s^I,s^F\rangle \leq \langle 0.5,0.4,0.6\rangle,$$

strictly increasing FNSV $\begin{bmatrix} \langle k^T,k^I,k^F\rangle \\ \langle 0.4,0.3,0.6\rangle \end{bmatrix}$. belongs to

$$\left\langle \begin{bmatrix} \langle k^T,k^I,k^F\rangle \\ \langle k^T,k^I,k^F\rangle \end{bmatrix}, \begin{bmatrix} \langle k^T,k^I,k^F\rangle \\ \langle s^T,s^I,s^F\rangle \end{bmatrix} \right\rangle = \langle m^*(A), M^*(A)\rangle,$$

that means $\mathbf{X}^<$ is a monotone universal FNSEv of $\mathbf{A}$ but $\mathbf{X}^<$ is not a monotone strong universal FNSEv of $\mathbf{A}$ because

$$\langle m^*(\overline{A}), M^*(\underline{A})\rangle = \left\langle \begin{bmatrix} \langle 0.3,0.2,0.7\rangle \\ \langle 0.3,0.2,0.7\rangle \end{bmatrix}, \begin{bmatrix} \langle 0.2,0.1,0.8\rangle \\ \langle 0.4,0.3,0.6\rangle \end{bmatrix} \right\rangle \cap \mathscr{N}^<_{(n)} = \phi.$$

Remark 4.5. Monotone interval FNSV $\mathbf{X}^<$ in the previous example fails to be a monotone strong universal FNSEv of $\mathbf{A}$ because the third condition in (3.2) is not fulfilled. We may note that in the example the first two conditions in (3.2) are satisfied, hence these two conditions together are not sufficient for the monotone strong universality of $\mathbf{X}^<$.

Theorem 4.6. $(T4 \Rightarrow T5)$ Let interval FNSM $A = [\underline{A},\overline{A}]$ and monotone IFNSV $\mathbf{X}^< = [\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle, \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle]$ with strictly increasing bounds $\langle \underline{x}^T,\underline{x}^I,\underline{x}^F\rangle, \langle \overline{x}^T,\overline{x}^I,\overline{x}^F\rangle$ be given. If $\mathbf{X}^<$ is a monotone strong tolerance FNSEv of $\mathbf{A}$, then $\mathbf{X}^<$ is a monotone tolerance FNSEv of $\mathbf{A}$.
Proof. The implication follows directly from Definition 3.1.

The next example shows that the converse implication is not true.

Example 4.7. $(T5 \nRightarrow T4)$ Let $\underline{A}, \overline{A}$ and $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle$ have the form

$$\underline{A} = \begin{bmatrix} \langle 0.2, 0.1, 0.8 \rangle & \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.3, 0.2, 0.7 \rangle & \langle 0.4, 0.3, 0.6 \rangle, \end{bmatrix},$$
$$\overline{A} = \begin{bmatrix} \langle 0.2, 0.1, 0.8 \rangle & \langle 0.4, 0.3, 0.6 \rangle \\ \langle 0.3, 0.2, 0.7 \rangle & \langle 0.4, 0.3, 0.6 \rangle \end{bmatrix}$$
$$\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle = \begin{bmatrix} \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.4, 0.3, 0.6 \rangle \end{bmatrix},$$
$$\langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle = \begin{bmatrix} \langle 0.4, 0.3, 0.6 \rangle \\ \langle 0.4, 0.3, 0.6 \rangle \end{bmatrix}.$$
with $m^*(\underline{A}) = \begin{bmatrix} \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.3, 0.2, 0.7 \rangle \end{bmatrix}$, $M^*(\overline{A}) = \begin{bmatrix} \langle 0.4, 0.3, 0.6 \rangle \\ \langle 0.4, 0.3, 0.6 \rangle \end{bmatrix}$.

The following inequalities
$$m^*(\underline{A}) = \begin{bmatrix} \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.3, 0.2, 0.7 \rangle \end{bmatrix} \leq \langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle = \begin{bmatrix} \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.4, 0.3, 0.6 \rangle \end{bmatrix},$$
$$\langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle = \begin{bmatrix} \langle 0.4, 0.3, 0.6 \rangle \\ \langle 0.4, 0.3, 0.6 \rangle \end{bmatrix} \leq M^*(\overline{A}) = \begin{bmatrix} \langle 0.4, 0.3, 0.6 \rangle \\ \langle 0.4, 0.3, 0.6 \rangle \end{bmatrix}$$

that means $\mathbf{X}^<$ is a monotone tolerance FNSEv of $\mathbf{A}$ but $\mathbf{X}^<$ is not a monotone strong tolerance FNSEv of $\mathbf{A}$ because

$$\tilde{A} = \begin{bmatrix} \langle 0.2, 0.1, 0.8 \rangle & \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.3, 0.2, 0.7 \rangle & \langle 0.4, 0.3, 0.6 \rangle, \end{bmatrix}, \text{ and}$$
$$\langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle = \begin{bmatrix} \langle 0.4, 0.3, 0.6 \rangle \\ \langle 0.4, 0.3, 0.6 \rangle \end{bmatrix} \nleq M^*(\tilde{A}) = \begin{bmatrix} \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.4, 0.3, 0.6 \rangle \end{bmatrix}.$$

Next two examples indicate further non-implication between types of monotone IFNSEvs.

Example 4.8. $(T4 \nRightarrow T3)$ Let $\underline{A}, \overline{A}$ and $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle$ have the form

$$\underline{A} = \begin{bmatrix} \langle 0.3, 0.2, 0.7 \rangle & \langle 0.2, 0.1, 0.8 \rangle \\ \langle 0.3, 0.2, 0.7 \rangle & \langle 0.3, 0.2, 0.7 \rangle \end{bmatrix},$$
$$\overline{A} = \begin{bmatrix} \langle 0.3, 0.2, 0.7 \rangle & \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.3, 0.2, 0.7 \rangle & \langle 0.5, 0.4, 0.6 \rangle \end{bmatrix},$$
$$\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle = \begin{bmatrix} \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.4, 0.3, 0.6 \rangle \end{bmatrix},$$
$$\langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle = \begin{bmatrix} \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.5, 0.4, 0.6 \rangle \end{bmatrix}.$$

Then we have

$$m^*(\underline{A}) = \begin{bmatrix} \langle 0.2, 0.1, 0.8 \rangle \\ \langle 0.2, 0.1, 0.8 \rangle \end{bmatrix} \leq \langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle,$$
$$\tilde{A} = \begin{bmatrix} \langle 0.3, 0.2, 0.7 \rangle & \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.3, 0.2, 0.7 \rangle & \langle 0.5, 0.4, 0.6 \rangle \end{bmatrix},$$
$$\langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle = \begin{bmatrix} \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.5, 0.4, 0.6 \rangle \end{bmatrix} \leq M^*(\tilde{A}) = \begin{bmatrix} \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.5, 0.4, 0.6 \rangle \end{bmatrix}$$

which means that $\mathbf{X}^<$ is a monotone strong tolerance FNSEv of $\mathbf{A}$, but $\mathbf{X}^<$ is not a monotone universal FNSEv of $\mathbf{A}$ because

$$\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle = \begin{bmatrix} \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.4, 0.3, 0.6 \rangle \end{bmatrix} \nleq M^*(\underline{A}) = \begin{bmatrix} \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.3, 0.2, 0.7 \rangle \end{bmatrix}.$$

Example 4.9. $(T2 \nRightarrow T5)$ Let $\underline{A}, \overline{A}$ and $\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle, \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle$ have the form

$$\underline{A} = \begin{bmatrix} \langle 0.2, 0.1, 0.8 \rangle & \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.3, 0.2, 0.7 \rangle & \langle 0.4, 0.3, 0.6 \rangle \end{bmatrix},$$
$$\overline{A} = \begin{bmatrix} \langle 0.2, 0.1, 0.8 \rangle & \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.3, 0.2, 0.7 \rangle & \langle 0.5, 0.4, 0.6 \rangle \end{bmatrix},$$

$$\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle = \begin{bmatrix} \langle 0.2, 0.1, 0.8 \rangle \\ \langle 0.4, 0.3, 0.6 \rangle \end{bmatrix},$$
$$\langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle = \begin{bmatrix} \langle 0.5, 0.4, 0.6 \rangle \\ \langle 0.6, 0.5, 0.3 \rangle \end{bmatrix}.$$

The following inequalities

$$m^*(\overline{A}) = \begin{bmatrix} \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.3, 0.2, 0.7 \rangle \end{bmatrix} \leq \langle \overline{x}^T, \overline{x}^I, \overline{x}^F \rangle = \begin{bmatrix} \langle 0.5, 0.4, 0.6 \rangle \\ \langle 0.6, 0.5, 0.3 \rangle \end{bmatrix}.$$

$$\langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle = \begin{bmatrix} \langle 0.2, 0.1, 0.8 \rangle \\ \langle 0.4, 0.3, 0.6 \rangle \end{bmatrix} \leq M^*(\underline{A}) = \begin{bmatrix} \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.4, 0.3, 0.6 \rangle \end{bmatrix},$$

$$\langle m^*(\overline{A}), M^*(\underline{A}) \rangle \cap \mathcal{N}^<_{(n)} =$$
$$\left\langle \begin{bmatrix} \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.3, 0.2, 0.7 \rangle \end{bmatrix}, \begin{bmatrix} \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.4, 0.3, 0.6 \rangle \end{bmatrix} \right\rangle \cap \mathcal{N}^<_{(n)} \neq \phi,$$

that mean $\mathbf{X}^<$ is a monotone strong universal FNSEv of $\mathbf{A}$, but $\mathbf{X}^<$ is not a monotone tolerance FNSEv of $\mathbf{A}$ because

$$m^*(\underline{A}) = \begin{bmatrix} \langle 0.3, 0.2, 0.7 \rangle \\ \langle 0.3, 0.2, 0.7 \rangle \end{bmatrix} \nleq \langle \underline{x}^T, \underline{x}^I, \underline{x}^F \rangle = \begin{bmatrix} \langle 0.2, 0.1, 0.8 \rangle \\ \langle 0.4, 0.3, 0.6 \rangle \end{bmatrix}.$$

Remark 4.10. The previous two examples shown that implications $T4 \Rightarrow T3$ and $T2 \Rightarrow T5$ are not fulfilled. It can be easily seen that if $T4 \nRightarrow T3$ and $T2 \nRightarrow T5$, then also $T2 \nRightarrow T4$, $T3 \nRightarrow T4$, $T3 \nRightarrow T5$, $T4 \nRightarrow T2$, $T5 \nRightarrow T2$, $T5 \nRightarrow T3$.

E.g. if the implication $T2 \Rightarrow T4$ holds true, then by $T4 \Rightarrow T5$ (Theorem 4.6) we get $T2 \Rightarrow T5$, a contradiction. The proofs of the remaining five non-implications are analogous.

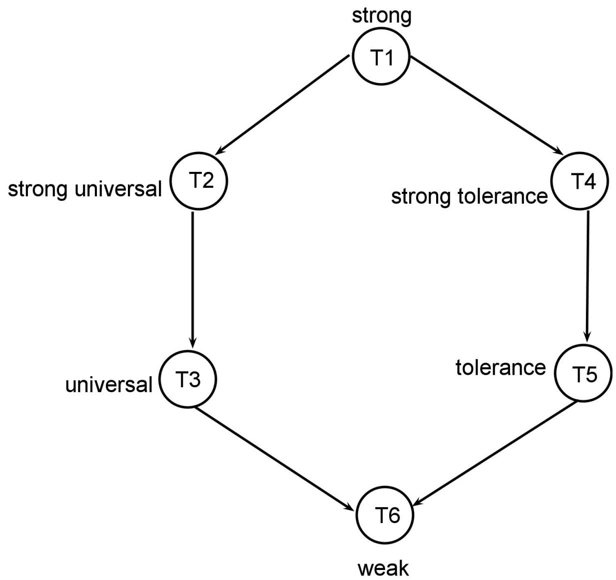

Fig. Hasse diagram of the relations between different types of monotone IFNSEvs.

The results from this section are summarized by the Hasse diagram in Figure where every arrow indicates an implication between corresponding types, and a non-existing arrow indicates that there is no implication between corresponding types of monotone IFNEvs.

5. Conclusion

In this paper we proposed six types of IFNSEvs of IFNSMs and the necessary and sufficient conditions are described. Also, we discussed the relations between these types of monotone FNSEvs and this results are summarized by the hasse diagram.

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