A new concept of energy from eccentricity matrix of graphs

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Abstract

In this paper, the eccentricity energy, which is calculated from the eccentricity-matrix of the graph is defined. Eccentricity-matrix is constructed by retaining from the distance matrix, only the largest distances both row-wise and column-wise. Complete Graphs, Cocktail-Party Graphs, and Crown graphs were considered while finding the eccentricity energy.

Keywords

Distance matrix, eccentricity matrix, eccentric Eigen values, eccentricity energy of a graph.

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1. Introduction

The energy of a graph popularized by Gutman [4] is an important concept of spectral graph theory which links organic chemistry to linear algebra of mathematics. Generally, a graph's energy is summation of absolute values of eigenvalues of the adjacency matrix. Similar energies got from the eigenvalues of various graphs are considered in recent times. For more details see [1], [2] and the reference cited there in. Irreducibility of the polynomial of eccentricity-matrix of a tree is spoken by JianfengWang[5] and energy of a graph is spoken in Gutman[4]. I combined these two concepts and put forth the concept of eccentricity energy of graphs.

2. Eccentricity Energy

In a graph G , the distance, $d(u, v)$, between any two vertices u and v , is the length of the minimum path connecting them. It is denoted by $d(u, v)$. For a vertex v of G , the eccentricity of a vertex v is $e(v) = \max\{d(u, v) : u \in V(G)\}$

Let $D(G) = (d_{uv})$ be the distance matrix of G , where $d_{uv} = d_G(u, v)$ Wang[1] renamed Randić's DMAX-matrix as eccentricity matrix $\varepsilon(G)$. It is the d_{uv} obtained from $D = D(G)$, the distance matrix, by retaining only the eccentricities row-wise

and column-wise. Remaining elements of the distance matrix are set to zero.

The elements of eccentricity matrix can precisely defined as:

$$[\varepsilon(G)]_{ij} = \begin{cases} (D)_{ij} & \text{if } (D)_{ij} = \min\{e_G(u_i), e_G(u_j)\} \\ 0 & \text{if } (D)_{ij} < \min\{e_G(u_i), e_G(u_j)\} \end{cases}$$

Eccentricity matrix and adjacency matrix can considered to be in exact opposition by the concepts of obtaining them. While adjacency matrix is obtained from the distance matrix by selecting only the smallest distance row wise and column wise, the eccentricity matrix takes the largest distance in a similar fashion. Thus the two matrices can be thought of as two extremes of distance-like matrix.

Normally, graph spectrum is formed by the eigenvalues of its adjacency matrix. Similarly, the ε -Spectrum of a graph consists of the ε -eigenvalues of its eccentricity matrix.

Definition 2.1 (Eccentricity energy).

The eccentricity energy of the graph defined by $E(G) = \sum_{i=1}^n |\lambda_i|$

where $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ are the eigenvalues of eccentricity matrix of the corresponding graph G .

Definition 2.2 (Complete graph).

A graph in which each pair of graph vertices is connected by an edge.

Definition 2.3 (Cocktail Party graph).

A graph consisting of two rows of paired vertices in which all the vertices except the paired ones are joined by an edge.

Denoted by CP_k , where $k = 2n$, for all $n \geq 2$. It is also called hyper octahedral graph or Roberts graph.

Definition 2.4 (Crown graph).

Crown graph on $2n$ vertices is an undirected graph with two set of vertices u_i and v_j and with an edge from u_i to v_j where $i \neq j$ the crown graph can be viewed as a complete bipartite graph from which the edges of a perfect matching have been removed.

3. Main Results

Derivation of Eccentricity energies of Complete, Cocktail-Party and Crown graphs based on the respective eccentricity matrices with an example for each type are given below.

Theorem 3.1.

Let K_n be a complete graph with n vertices and m edges. Then eccentricity energy of K_n is $2n - 2$, $n > 2$.

Proof.

Let K_n be a complete graph with n vertices $\{v_1, v_2, \dots, v_n\}$ and m edges. Each vertex is adjacent to every other vertex. Therefore, distance from one vertex to any other vertex is one. Eccentricity of each vertex is one. For a complete graph, eccentricity matrix is same as distance matrix.

$$\varepsilon(K_n) = \begin{bmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & \dots & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & \dots & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & \dots & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 0 \end{bmatrix}_{(n \times n)}$$

The characteristic equation is $(\lambda + 1)^{n-1}[\lambda - (n - 1)] = 0$
The eccentricity Eigenvalues are $\lambda = (-1)[(n - 1)\text{times}]$, $\lambda = (n - 1)(\text{onetime})$
Eccentricity energy $K_n = (n - 1)|-1| + (n - 1) = 2n - 2$ $\square$

Example 3.2.

An example of Complete graph with 5 vertices ($n = 5$) is given in Fig. 1.

Figure 1. Complete graph with 5 vertices

The eccentricity matrix of the graph shown in fig. 1 is

$$\varepsilon(K_5) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Characteristic equation of K_5 is $(\lambda + 1)^4(\lambda - 4) = 0$

$\lambda = (-1)4$ times, $\lambda = 4$ one time

Eccentricity eigenvalues of $K_5 = 4, -1, -1, -1, -1$

Eccentricity energy of $K_5 = |4| + |(-1)4| = 8$

Theorem 3.3.

The eccentricity energy of a Cocktail Party graph CP_k where $k = 2n$ for $n \geq 2$ is $4n$.

Proof.

Let CP_k , $n \geq 2$ be the Cocktail Party graph with vertex set $V = \bigcup_{i=1}^n \{u_i, v_i\}$. Each vertex u_i is adjacent to every $(n - 1)$ vertices in the vertex set $\{v_i\}$.

The distance matrix contains $d(u_i, u_i) = 0$ and $d(v_i, v_i) = 0$.

$$d(u_i, v_i) = \begin{cases} 1 & \text{if } i \neq j \\ 2 & \text{if } i = j \end{cases}$$

Eccentricity of every vertex is two. Hence, by the definition of eccentricity matrix, it contains values 0 and 2 only.

$$\varepsilon(CP_k) = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}_{(n \times n)}$$

Characteristic equation of CP_k is $(\lambda - 2)^n(\lambda + 2)^n = 0$
Eigen values of CP_k : $\lambda = 2(n \text{ times})$, $\lambda = -2(n \text{ times})$
Eccentricity energy of $CP_k = |2|n + |(-2)|n = 4n$ $\square$

Example 3.4.

An example of Cocktail Party graph with 4 vertices is given in Fig. 2. In this case, $k = 4$ and $n = 2$.

The eccentricity matrix of Cocktail Party graph of fig. 2 is

$$\varepsilon(CP_4) = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

Figure 2. Cocktail Party graph of 4 vertices

Characteristic equation of CP_4 is $(\lambda - 2)^2(\lambda + 2)^2 = 0$

Eccentricity eigenvalues of CP_4 :

$\lambda = 2$ (2 times), $\lambda = -2$ (2 times)

Eccentricity energy of $CP_4 = |2|2 + |(-2)|2 = 8$

Theorem 3.5.

The eccentricity energy of a Crown graph S_k , where $k = 2n$ for $n > 2$ is $6n$.

Proof.

Suppose S_k is the crown graph with k vertices. So by definition of Crown graph, the vertex set of S_k is partitioned into two subsets V_1 and V_2 such that $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$

Let $V_1 = \{v_1, v_2, v_3, \dots, v_n\}$ and $V_2 = \{u_1, u_2, u_3, \dots, u_n\}$ be the two subsets of $V(S_k)$ and all the vertices of V_1 are connected to each vertex of V_2 except the paired ones. The distance matrix contains the values:

$$d(u_i, v_j) = \begin{cases} 3 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$$

$$d(u_i, u_j) = \begin{cases} 0 & \text{if } i = j \\ 2 & \text{if } i \neq j \end{cases}$$

$$d(v_i, v_j) = \begin{cases} 0 & \text{if } i = j \\ 2 & \text{if } i \neq j \end{cases}$$

Eccentricity of every vertex is three. Hence, by definition of eccentricity matrix, it contains values zero and three only.

$$\varepsilon(S_k) = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 3 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}_{(2n \times 2n)}$$

Characteristic equation of S_k is $(\lambda - 3)^n(\lambda + 3)^n = 0$

Eccentricity eigenvalues of S_k :

$\lambda = 3$ (n times), $\lambda = -3$ (n times)

Eccentricity energy of $S_k = |3|n + |(-3)|n = 6n$

□

Example 3.6.

An example of Crown graph with 8 vertices is given in Fig. 3. In this case, $k = 8$ and $n = 4$.

Figure 3. Crown Graph with $n = 4$, $k = 8$

The eccentricity matrix of Crown graph is

$$\varepsilon(S_8) = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & v_1 & v_2 & v_3 & v_4 \\ u_1 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ u_2 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ u_3 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ u_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_2 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ v_3 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ v_4 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \end{bmatrix}$$

Characteristic equation of S_8 is $(\lambda - 3)^4(\lambda + 3)^4 = 0$

Eccentricity eigenvalues of S_8 :

$\lambda = 3$ (4 times), $\lambda = -3$ (4 times). Eccentricity energy of $S_8 = |3|4 + |(-3)|4 = 24$.

Conclusion

The eccentricity energy based on the eccentricity matrix were found for a Complete graph, Cocktail Party graph and Crown graph. It can be proved that the results obtained for every complete graph, the eccentricity energy is $2n - 2$ always. The eccentricity energy of any Cocktail Party graph is $4n$ and for every Crown graph is $6n$.

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