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# Group $S_3$ cordial prime labeling of graphs

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**Abstract**
Let $G=(V(G),E(G))$ be a graph. Consider the group $S_3$. If $u \in S_3$, we denote by $o(u)$ the order of $u$ in $S_3$. Define a function $g: V(G) \rightarrow S_3$ in such a way that $xy \in E(G) \Leftrightarrow (o(g(x)),o(g(y)) = 1$.Let $n_j(g)$ denote the number of vertices of $G$ having label $j$ under $g$. Now $g$ is called a group $S_3$ cordial prime labeling if $|n_i(g)-n_j(g)| \leq 1$ for every $i,j \in S_3, i \neq j$. A graph which admits a group $S_3$ cordial prime labeling is called a group $S_3$ cordial prime graph. In this paper, we prove that all paths, cycles, Gear graphs, Ladder and fan are group $S_3$ cordial prime. We further characterize wheels that are group $S_3$ cordial prime.

**Keywords**
Cordial labeling, prime labeling, group $S_3$ cordial prime labeling.

**AMS Subject Classification**
05C78.

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## 1. Introduction

Graphs considered here are finite, undirected and simple. Let $A$ be a group. The order of $a \in A$ is the least positive integer $n$ such that $a^n = e$. We denote the order of $a$ by $o(a)$.

Cahit [1] introduced the concept of cordial labeling.

## 2. Preliminaries

**Definition 2.1.** *Consider any function $f: V(G) \rightarrow \{0,1\}$. Assign the label $|f(x)-f(y)|$ for each edge $xy$. $f$ is called a cordial labeling if the difference between the number of vertices labeled 0 and the number of vertices labeled 1 is at most 1. Also the difference between the number of edges labeled 0 and the number of edges labeled 1 is at most 1.*

The concept of prime labeling was introduced by Entringer. This was later studied by Tout et al.[4]

**Definition 2.2.** *A prime labeling of a graph G of order n is an injective function $f: V \rightarrow \{1,2,....n\}$ such that for every pair of adjacent vertices u and v $gcd\{f(u),f(v)\} = 1$.*

Motivated by these two definitions, we introduce group $S_3$ cordial prime labeling of graphs. Terms not defined here are used in the sense of Harary [3] and Gallian [2].

The greatest common divisor of two integers $m$ and $n$ is denoted by $(m,n)$ and $m$ and $n$ are said to be relatively prime if $(m,n)=1$. For any real number $x$, we denoted by $\lfloor x \rfloor$, the greatest integer smaller than or equal to $x$ and by $\lceil x \rceil$, we mean the smallest integer greater than or equal to $x$.

A path is an alternating sequence of vertices and edges, $v_1,e_1,v_2,e_2,...,v_{n-1},e_{n-1},v_n$ which are distinct, such that $e_i$ is an edge joining $v_i$ and $v_{i+1}$ for $1 \leq i \leq n-1$. A path on $n$ vertices is denoted by $P_n$. A path $v_1,e_1,v_2,e_2,...,e_{n-1},v_n,e_n,v_1$ is called a cycle and a cycle on $n$ vertices is denoted by $C_n$.

Given two graphs $G$ and $H$, $G+H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv | u \in V(G), v \in V(H)\}$. A wheel $W_n$ is defined as $C_n + K_1$. The graph $P_n + K_1$ is called a fan graph $F_n$. The Gear graph $G_n$ is obtained from the wheel $W_n$ by adding a vertex between every pair of adjacent vertices of the cycle $C_n$.

Given two graphs $G$ and $H$ , $G \times H$ is the graph with vertex set $V(G) \times V(H)$ and edge set $E(G) \times E(H)$. Two vertices $(u_1,v_1)$ and $(u_2,v_2)$ are adjacent in $G \times H$ if $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $u_1u_2 \in E(G)$ and $v_1 = v_2$. $P_n \times P_2$ is called the ladder graph $L_n$.

## 3. Main Results

**Definition 3.1.** *Let $g : V(G) \to S_3$ be a function defined in such a way that $xy \in E(G) \Leftrightarrow (o(g(x)), o(g(y)) = 1$.Let $n_j(g)$ denote the number of vertices of G having label j under g. Now g is called a group $S_3$ cordial prime labeling if $|n_i(g) - n_j(g)| \leq 1$ for every $i, j \in S_3, i \neq j$. A graph which admits a group $S_3$ cordial prime labeling is called a group $S_3$ cordial prime graph.*

**Definition 3.2.** Consider the symmetric group $S_3$.
Let the elements of $S_3$ be $\{e, a, b, c, d, f\}$ where

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}; a = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}; b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix};$$
$$c = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}; d = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}; f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

Now $o(e) = 1$
$o(a) = o(b) = o(c) = 2$
$o(d) = o(f) = 3$

**Example 3.3.** A group $S_3$ cordial prime labeling of two graphs is given in Figure 1 and Figure 2.

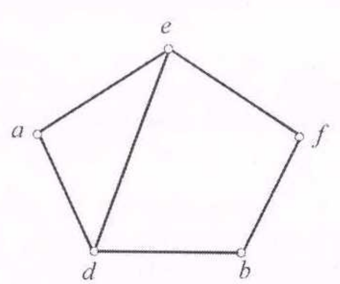


Figure 1

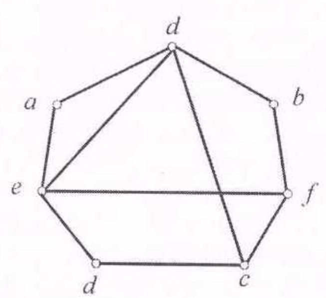


Figure 2

**Theorem 3.4.** *All paths $P_n (n \geq 1)$ are group $S_3$ cordial prime.*

*Proof.* Let $v_1, v_2, ..., v_n$ denote the vertices of $P_n$.
Define a function $g : V(P_n) \to S_3$ as follows.
For $k \geq 0$, $g(v_{6k+1}) = a$
$g(v_{6k+2}) = e$
$g(v_{6k+3}) = b$
$g(v_{6k+4}) = d$
$g(v_{6k+5}) = c$
For $k \geq 1$, $g(v_{6k}) = f$
Clearly $g$ is a group $S_3$ cordial prime labeling,
Hence all paths $P_n$ are group $S_3$ cordial prime. □

Illustration of the labeling for $P_5$ is given in Figure 3.

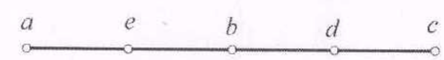


Figure 3

**Theorem 3.5.** *All cycles $C_n$ are group $S_3$ cordial prime.*

*Proof.* Let the vertices of $C_n$ be denoted by $v_1, v_2, ..., v_n$.
Define a function $g : V(C_n) \to S_3$ as follows.

$$g(v_i) = \begin{cases} \text{e,} & \text{if k} \equiv 1 (mod\ 6) \\ \text{a,} & \text{if k} \equiv 2 (mod\ 6) \\ \text{d,} & \text{if k} \equiv 3 (mod\ 6) \\ \text{b,} & \text{if k} \equiv 4 (mod\ 6) \\ \text{f,} & \text{if k} \equiv 5 (mod\ 6) \\ \text{c,} & \text{if k} \equiv 0 (mod\ 6) \end{cases}$$

Clearly $g$ is a group $S_3$ cordial prime labeling. □

Illustration of the labeling for $C_8$ is given in Figure 4.

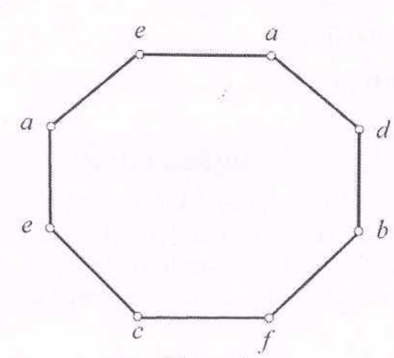


Figure 4.

**Theorem 3.6.** *All Gear graphs $G_n$ are group $S_3$ cordial prime.*

*Proof.* Lew $w$ be the center vertex and $u_1, u_2, ..., u_{2n}$ be the vertices on the cycle $C_n$.
Then $w$ should be labelled as $e$. Suppose if, $w$ is labelled with $a$ (or $b$ or $c$).Then alternate vertices of the cycle should be labelled as $e$ which is not possible by definition.
Similar case arises if $w$ is labelled as $d$(or $f$).
Define $g : V(G_n) \to S_3$ as follows.
$g(w) = e$

$$g(u_i) = \begin{cases} \text{a}, & \text{i} \equiv 1(mod\ 6) \\ \text{d}, & \text{i} \equiv 2(mod\ 6) \\ \text{b}, & \text{i} \equiv 3(mod\ 6) \\ \text{f}, & \text{i} \equiv 4(mod\ 6) \\ \text{c}, & \text{i} \equiv 5(mod\ 6) \\ \text{e}, & \text{i} \equiv 0(mod\ 6) \end{cases}$$

Clearly $g$ is group $S_3$ cordial prime. □

Illustration of the labelings for the Gear graph $G_8$ is given in Figue 5.

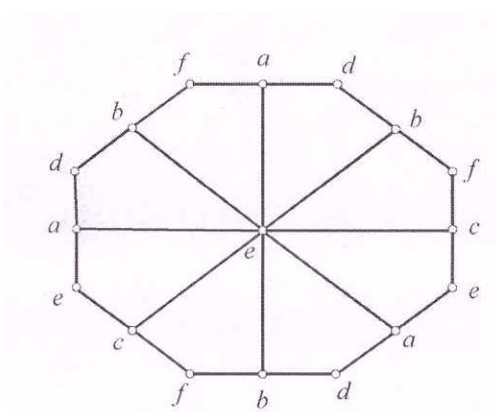

Figure 5

**Theorem 3.7.** *The ladder $L_n$ is group $S_3$ cordial prime.*

*Proof.* Let the vertices of $L_n$ be $\{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and

$$E(L_n) = \begin{cases} u_i v_i, & i = 1 \text{ to } n \\ u_i u_{i+1}, & i = 1 \text{ to } n-1 \\ v_i v_{i+1}, & i = 1 \text{ to } n-1 \end{cases}$$

Then define $g : V(L_n) \to S_3$ as follows.

$$g(u_i) = \begin{cases} e, & \text{if } i \equiv 1(mod\ 6) \\ a, & \text{if } i \equiv 2(mod\ 6) \\ d, & \text{if } i \equiv 3(mod\ 6) \\ b, & \text{if } i \equiv 4(mod\ 6) \\ f, & \text{if } i \equiv 5(mod\ 6) \\ c, & \text{if } i \equiv 0(mod\ 6) \end{cases}$$

The labels of vertices $v_i (1 \le i \le n)$ depend on the label of $u_n$.

If $g(u_n) = a$, define

$$g(v_i) = \begin{cases} d, & \text{if } i \equiv 2(mod\ 6) \\ b, & \text{if } i \equiv 3(mod\ 6) \\ f, & \text{if } i \equiv 4(mod\ 6) \\ c, & \text{if } i \equiv 5(mod\ 6) \\ e, & \text{if } i \equiv 0(mod\ 6) \\ a, & \text{if } i \equiv 1(mod\ 6) \end{cases}$$

If $g(u_n) = b$, define

$$g(v_i) = \begin{cases} f, & \text{if } i \equiv 2(mod\ 6) \\ c, & \text{if } i \equiv 3(mod\ 6) \\ e, & \text{if } i \equiv 4(mod\ 6) \\ a, & \text{if } i \equiv 5(mod\ 6) \\ d, & \text{if } i \equiv 0(mod\ 6) \\ b, & \text{if } i \equiv 1(mod\ 6) \end{cases}$$

If $g(u_n) = c$, define

$$g(v_i) = \begin{cases} e, & \text{if } i \equiv 2(mod\ 6) \\ a, & \text{if } i \equiv 3(mod\ 6) \\ d, & \text{if } i \equiv 4(mod\ 6) \\ b, & \text{if } i \equiv 5(mod\ 6) \\ f, & \text{if } i \equiv 0(mod\ 6) \\ c, & \text{if } i \equiv 1(mod\ 6) \end{cases}$$

If $g(u_n) = d$, define

$$g(v_i) = \begin{cases} b, & \text{if } i \equiv 1(mod\ 6) \\ f, & \text{if } i \equiv 2(mod\ 6) \\ c, & \text{if } i \equiv 3(mod\ 6) \\ e, & \text{if } i \equiv 4(mod\ 6) \\ a, & \text{if } i \equiv 5(mod\ 6) \\ d, & \text{if } i \equiv 0(mod\ 6) \end{cases}$$

If $g(u_n) = e$, define

$$g(v_i) = \begin{cases} a, & \text{if } i \equiv 1(mod\ 6) \\ d, & \text{if } i \equiv 2(mod\ 6) \\ b, & \text{if } i \equiv 3(mod\ 6) \\ f, & \text{if } i \equiv 4(mod\ 6) \\ c, & \text{if } i \equiv 5(mod\ 6) \\ e, & \text{if } i \equiv 0(mod\ 6) \end{cases}$$

If $g(u_n) = f$, define

$$g(v_i) = \begin{cases} c, & \text{if } i \equiv 1(mod\ 6) \\ e, & \text{if } i \equiv 2(mod\ 6) \\ a, & \text{if } i \equiv 3(mod\ 6) \\ d, & \text{if } i \equiv 4(mod\ 6) \\ b, & \text{if } i \equiv 5(mod\ 6) \\ f, & \text{if } i \equiv 0(mod\ 6) \end{cases}$$

Clearly $g$ is a group $S_3$ cordial prime labeling. □

Illustration of the labeling for the Ladder graph $L_5$ is given in Figue 6.

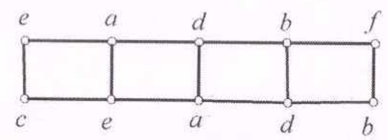


Figure 6

**Theorem 3.8.** *Fan graph $F_n$ is group $S_3$ cordial prime.*

*Proof.* Let $F_n = P_n + K_1$. Let the $n$ vertices of $P_n$ be denoted by $v_1, v_2, ..., v_n$ and the single vertex of $K_1$ is denoted by $w$.
Define $g : V(F_n) \to S_3$ as follows.
Note that $w$ should be labelled as $e$. Suppose if $g(w) = a$ (or $b$ or $c$), then no label from $\{a, b, c,\}$ can be used to label $v_i$. If $g(w) = d$ (or $f$) similar case arises which is not possible by definition. Let $g(w) = e$. Then the $n$ vertices of $P_n$ can be labelled as $a, d, b, f, c, e$ in order. Clearly $g$ is group $S_3$ cordial prime. □

Illustration of the labeling for the fan graph $F_4$ is given in Figure 7.

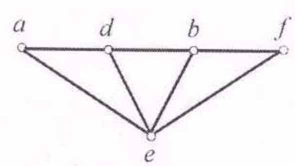


Figure 7

**Theorem 3.9.** *Wheel graphs $W_n (n \geq 4)$ are group $S_3$ cordial prime iff $n \not\equiv 5(mod 6)$.*

*Proof.* Let $W_n$ be the wheel $C_n + K_1$. Let $v_1, v_2, ..., v_n$ be the vertices on the cycle $C_n$ and $w$ be the center of the wheel.
Suppose $n = 3$. Suppose the center vertex $w$ is labelled as $e$. Now only one of $v_1, v_2, v_3$ can be labelled with a label of $\{a, b, c\}$. Another one can be labelled with a label of $\{d, f\}$. Hence it is not possible to label the third vertex. Similarly if $w$ is labelled with a label of $\{a, b, c\}$, it is not possible to label the other three vertices. Same is the case if $w$ is labelled with a label of $\{d, f\}$. Thus $n \geq 4$.
The center vertex $w$ is labelled as $e$. Otherwise, if we label $w$ as $a$ (or $b$ or $c$) then we have to label alternate vertices of $C_n$ by $d, e, f, ...$ Then $e$ appears $\lceil n/2 \rceil$ times which is not possible by definition.
Similar contradiction arises if we label $w$ by $d$ (or $f$).

**Case 1**: $n$ is even.
Now $n \equiv 0$ or 2 or $4(mod\ 6)$.
Define $g : V(w_n) \to S_3$ as follows.
$g(w) = e$

$$\text{For } k \geq 0,\ g(v_i) = \begin{cases} a, & i = 6k+1 \\ d, & i = 6k+2 \\ b, & i = 6k+3 \\ f, & i = 6k+4 \\ c, & i = 6k+5 \end{cases}$$

For $k \geq 1$, $g(v_i) = e$ for $i = 6k$
From Table 1, $g$ is group $S_3$ cordial prime labeling.

**Case 2**: $n$ is odd.

**Case (a)**: $n \equiv 1(mod\ 6)$
Let $n = 6k+1, k \geq 1$
Define $g : V(W_n) \to S_3$ as follows
$g(w) = e$

$$g(v_i) = \begin{cases} a, & \text{for } i = 1, 7, ..., 6k-5 \\ d, & \text{for } i = 2, 8, ..., 6k-4 \\ b, & \text{for } i = 3, 9, ..., 6k-3 \\ f, & \text{for } i = 4, 10, ..., 6k-2 \\ c, & \text{for } i = 5, 11, ..., 6k-1 \\ e, & \text{for } i = 6, 12, ..., 6k \\ d, & \text{for } i = 6k+1 \end{cases}$$

From Table 1, $g$is group $S_3$ cordial prime labeling.

**Case (b)**: Suppose $n \equiv 3mod(6)$
Let $n = 6k+3, k \geq 1$
Define $g : V(W_n) \to S_3$ as follows
$g(w) = e$

$$\text{For } k \geq 1,\ g(v_i) = \begin{cases} a, & \text{for } i = 1, 7, ..., 6k-5 \\ d, & \text{for } i = 2, 8, ..., 6k-4 \\ b, & \text{for } i = 3, 9, ..., 6k-3 \\ f, & \text{for } i = 4, 10, ..., 6k-2 \\ c, & \text{for } i = 5, 11, ..., 6k-1 \\ e, & \text{for } i = 6, 12, ..., 6k-6 \\ d, & \text{for } i = 6k \\ a, & \text{for } i = 6k+1 \\ f, & \text{for } i = 6k+2 \\ e, & \text{for } i = 6k+3 \end{cases}$$

**Case (c)**: $n \equiv 5(mod\ 6)$
Let $n = 6k+5, k \geq 0$. Let $w$ be labelled as $e$. To maintain the condition of the labeling the vertices $v_1, v_2, ..., v_{6k}$ have to be labelled using the labels $a, b, c, d, e, f$ equal number of times in an acceptable pattern. Now the vertices $v_{6k+1}, v_{6k+2}, ..., v_{6k+5}$ have to be labelled using $a, b, c, d, e, f$. If some $v_i (6k+1 \leq i \leq 6k+5)$ is labelled with $e$, then one of $\{a, b, c, d, f\}$ would be left out say $x$. Then $|n_x(v) - n_e(v)| = 2$ which is a contradiction. So no $v_i$ can be labelled with $e$. It can be easily observed that the vertices $v_i (6k+1 \leq i \leq 6k+5)$ cannot be labelled using $\{a, b, c, d, f\}$ and satisfying group cordial prime condition. Hence $W_n$ are group $S_3$ cordial prime if $n \not\equiv 5(mod\ 6)$.

**Table 1**

| nature of $n$ | $n_a(g)$ | $n_b(g)$ | $n_c(g)$ | $n_d(g)$ | $n_f(g)$ | $n_e(g)$ |
|---|---|---|---|---|---|---|
| $n=6k+1$ | $k$ | $k$ | $k$ | $k+1$ | $k$ | $k+1$ |
| $n=6k+2$ | $k+1$ | $k$ | $k$ | $k+1$ | $k$ | $k+1$ |
| $n=6k+3$ | $k+1$ | $k$ | $k$ | $k+1$ | $k+1$ | $k+1$ |
| $n=6k+4$ | $k+1$ | $k+1$ | $k$ | $k+1$ | $k+1$ | $k+1$ |
| $n=6k$ | $k$ | $k$ | $k$ | $k$ | $k$ | $k+1$ |

From Table 1, it is clear that $g$ is a group $S_3$ cordial prime labeling.

□

Illustration of the labelings for $W_4$ is given in Figure 8.

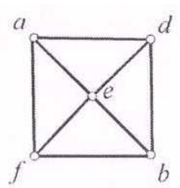


Figure 8

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