Fuzzy $M^*$-open sets in Šostak’s fuzzy topological spaces

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Abstract
The aim of this paper is to define $t$-fuzzy $M^*$-open and $t$-fuzzy $M^*$-closed sets in Šostak’s fuzzy topological spaces. Also, $t$-fuzzy $M^*$-interior, $t$-fuzzy $M^*$-closure are introduced and their properties are investigated. Moreover, we investigate the connections between $t$-fuzzy open, $t$-fuzzy $\theta$-semiopen, $t$-fuzzy $\theta$-open, $t$-fuzzy $\delta$-semiopen, $t$-fuzzy $\delta$-preopen, $t$-fuzzy $\alpha$-open, $t$-fuzzy $\epsilon$-open and $t$-fuzzy $\epsilon^*$-open in fuzzy topological spaces in the sense of Šostak.

Keywords
$t$-fuzzy $M^*$-open, $t$-fuzzy $M^*$-closed, $t$-fuzzy $M^*$-interior, $t$-fuzzy $M^*$-closure, $t$-fuzzy $M$-open, $t$-fuzzy $M$-closed, $t$-fuzzy $M$-interior, $t$-fuzzy $M$-closure, $t$-fuzzy $\theta$-interior, $t$-fuzzy $\theta$-closure, $t$-fuzzy $\delta$-semiopen, $t$-fuzzy $\delta$-preopen, $t$-fuzzy $\alpha$-open, $t$-fuzzy $\epsilon$-open and $t$-fuzzy $\epsilon^*$-open.

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1. Introduction

Chang’s fuzzy topology [4] has been extended by Šostak [28] in fuzzy topology and different level of growths have been made in [13, 14, 26]. Various authors [2, 3, 5, 10, 12, 21, 23] have developed fuzzy continuity between fuzzy topological space in weaker forms also using the idea of fuzzy semi-open sets [2], fuzzy regular open sets [2], fuzzy preopen sets, fuzzy strongly semiopen sets [3], fuzzy $\gamma$-open sets [12], fuzzy $\delta$-semiopen sets [1], fuzzy $\delta$-preopen sets [1], fuzzy semi $\delta$-preopen sets [33] and fuzzy $\epsilon$-open sets [27]. In the sense of Chang [4] fits, Ganguly and Saha [11] developed the idea of fuzzy $\delta$-cluster points in fits. In the sense of Šostak’s fuzzy topological space, Kim and Park [15] developed $t$-$\delta$-cluster points and $\delta$-closure operators. The weaker forms of fuzzy semi-preopen sets was developed by Park et al. [19] than any other from of fuzzy semi-open or fuzzy preopen sets. In 2008, the formations of $e$-open sets, $e^*$-open sets and $a$-open sets in topological spaces are due to Erdal Ekici [8], [9]. Sobana et al. [29] defined $t$-fuzzy $e$-open and $t$-fuzzy $e^*$-closed sets in a fuzzy topological space in the sense of Šostak. Velicko [30] in 1968 has developed and analyzed a specific variety of $\theta$-open sets and $\delta$-open sets in characerizations of $H$-closed topological spaces.


This article is a classified analysis of a different class of open set namely $t$-fuzzy $M^*$-open set. In this paper, we iden-
tify the idea of \( t \)-fuzzy \( M^* \)-open (resp. \( t \)-fuzzy \( M^* \)-closed) sets in fuzzy topological spaces in the sense of Šostak’s. Also, we defined \( t \)-fuzzy \( M^* \)-interior (resp. \( t \)-fuzzy \( M^* \)-closure) and analysed some of their properties. Also the relationships of \( t \)-fuzzy open, \( t \)-fuzzy \( \theta \)-semiopen, \( t \)-fuzzy \( \theta \)-open, \( t \)-fuzzy \( \delta \)-semiopen, \( t \)-fuzzy \( \delta \)-preopen, \( t \)-fuzzy \( a \)-open, \( t \)-fuzzy \( e \)-open and \( t \)-fuzzy \( e^* \)-open in Šostak’s fuzzy topological spaces are analysed.

2. Preliminaries

Throughout this article, we denote nonempty sets by \( X, Y \) etc., \( I = [0, 1] \) and \( I_0 = (0, 1] \). For \( \alpha \in I, \, \varpi(x) = \alpha, \, \forall x \in X \). A fuzzy point \( x_t \) for \( t \in I_0 \) is an element of \( I^X \) such that
\[
x_t(y) = \begin{cases} t & \text{if } y \text{ is equal to } x \\ 0 & \text{if } y \text{ is not equal to } x. \end{cases}
\]

Let \( Pr(X) \) denote the set of all fuzzy points in \( X \). A fuzzy point \( x_t \in x \) iff \( t < \mu(x) \). \( \mu \in I^X \) is quasi-coincident with \( v \), denoted by \( \mu_q \), \( v \), \( \exists x \in X \) such that \( \mu(x) + v(x) > 0 \). If \( \mu \) is not quasi-coincident with \( v \), we denoted \( \mu qv \). If \( A \) is a subset of \( X \), we define the characteristic function \( \chi_a \) on \( X \) by
\[
\chi_a(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}
\]
All notations and definitions will be standard in the fuzzy set theory.

Lemma 2.1. [28] Consider \( X \) be a nonempty set and \( \mu, v \in I^X \). Then
(i) \( \mu_qv \) iff there exists \( x_t \in \mu \) such that \( x_tqv \).
(ii) \( \mu_qv \), then \( \mu \wedge v \neq 0 \).
(iii) \( \mu qv \) iff \( \mu \leq \tau - v \).
(iv) \( \mu_qv \) iff \( x_t \in \mu \) implies \( x_t \in v \) iff \( x_tq\mu \) implies \( x_tq\mu \).
(v) \( x_tq \bigvee_{\mu qv} v_t \) iff there exists \( i_0 \in \mu \) such that \( x_tq\nu_{i_0} \).

Definition 2.2. [28] A function \( \tau : I^X \rightarrow I \) is called a fuzzy topology on \( X \) if it satisfies the following conditions:

1. \( \tau(\emptyset) = \tau(\emptyset) = 1 \).
2. \( \tau(\bigvee_{\mu} v) \geq \bigwedge_{\mu} \tau(v) \), for any \( \{v\}_{\mu} \in I^X \).
3. \( \tau(v_1 \wedge v_2) \geq \tau(v_1) \cap \tau(v_2) \), for any \( v_1, v_2 \in I^X \).

The pair \((X, \tau)\) is called a fuzzy topological space or Šostak’s fuzzy topological space or smooth topological space (for short, fts, sfts, sts).

Remark 2.3. [24] Let \((X, \tau)\) be a sfts. Then, for every \( t \in I_0 \), \( \tau = \{v \in I^X : \tau(v) \geq 1 \} \) is a Change’s fuzzy topology on \( X \).

Theorem 2.4. [26] Let \((X, \tau)\) be a sfts. Then for each \( \mu \in I^X \), \( t \in I_0 \), we define an operator \( C_t : I^X \times I_0 \rightarrow I^X \) as follows:
\[
C_t(\mu, t) = \bigwedge \{ v \in I^X : \mu \leq v, \tau(v - v) \geq 1 \}.
\]

Theorem 2.5. [26] Let \((X, \tau)\) be a sfts. Then for each \( t \in I_0 \), we define an operator \( I_t : I^X \times I_0 \rightarrow I^X \) as follows:

1. \( I_t(\emptyset, t) = \emptyset \).
2. \( \mu \leq I_t(\mu, t) \).
3. \( C_t(\mu, t) \wedge C_t(\nu, t) = C_t(\nu \vee t, t) \).
4. \( C_t(\mu, s) \leq C_t(\mu, t) \) if \( t \leq s \).
5. \( C_t(\mu, t) = C_t(\mu, t) \).

Theorem 2.6. [16] Let \((X, \tau)\) be a sfts. Then for each \( \mu \in I^X \), \( x_t \in P_1(X) \) and \( t \in I_0 \), \( v \) is called

1. \( t \)-open \( Q_\tau \)-neighbourhood of \( x_t \) if \( x_tqv \) with \( \tau(v) \geq 1 \).
2. \( t \)-open \( R_\tau \)-neighbourhood of \( x_t \) if \( x_tqv \) with \( v = I_t(\mu, t) \).

We denote \( Q_\tau(x_t, t) = \{ v \in I^X : x_tqv \wedge \tau(v) \geq 1 \}, R_\tau(x_t, t) = \{ v \in I^X : x_tqv = I_t(\mu, t) \} \).

Definition 2.7. [16] Let \((X, \tau)\) be a sfts. Then for each \( \mu \in I^X \), \( x_t \in P_1(X) \) and \( t \in I_0 \), \( x_t \) is called

1. \( t \)-\( \tau \)-cluster point of \( \mu \) if for every \( v \in Q_\tau(x_t, 1) \), we have \( v \mu qv \).
2. \( t \)-\( \delta \)-cluster point of \( \mu \) if for every \( v \in R_\tau(x_t, 1) \), we have \( v \mu qv \).
3. \( \delta \)-closure operator is a mapping \( D_\tau : I^X \times I_0 \rightarrow I^X \) defined as follows: \( D_\tau(\mu, t) \) or \( D_\tau(\mu, t) \) = \( \{ x_t : x_tqv \wedge \tau(v) \geq 1 \} \).

Definition 2.8. Let \((X, \tau)\) be a sfts. For \( \mu, v \in I^X \) and \( t \in I_0 \), \( \mu \) is called an

1. \( t \)-fuzzy \( \delta \)-semiopen (resp. \( t \)-fuzzy \( \delta \)-semiclosed) [29] set if \( \mu \leq C_t(\delta C_t(\delta C_t(\mu, t), t)) \).
2. \( t \)-fuzzy \( \delta \)-preopen (resp. \( t \)-fuzzy \( \delta \)-preclosed) [29] set if \( \mu \leq I_t(\delta C_t(\delta C_t(\mu, t), t)) \).


(iii) The intersection of all t-fuzzy θ-closed, (resp. t-fuzzy θ-semiclosed, t-fuzzy θ-preclosed) sets containing μ is called the t-fuzzy θ-closure (resp. t-fuzzy θ-semiclosure, t-fuzzy θ-preclosure) of μ and is denoted by θC(μ, t) (resp. θSC(μ, t), θPC(μ, t)).

Lemma 2.12. [32] Let (X, τ) be a sfts, μ, ν ∈ τX and t ∈ I0, then

(i) μ is t-fuzzy θ-open if and only if μ = θI(μ, t).

Lemma 2.13. [32] Let (X, τ) be a sfts, μ, ν ∈ τX and t ∈ I0, then

(i) θI(θI(μ, t)) < θI(μ, t).

Lemma 2.14. [32] Let (X, τ) be a sfts. For μ ∈ τX and t ∈ I0, then the following statements hold:

(i) θSC(μ, t) = μ ∨ I(θSC(μ, t), t), θPC(μ, t) = μ ∨ C(θPC(μ, t), t).

Lemma 2.15. [32] Let (X, τ) be a sfts, μ, ν ∈ τX and t ∈ I0, then

(i) pC(δC(μ, t), t) = δC(δC(μ, t), t) and pC(δC(μ, t), t) = δC(δC(μ, t), t).

Definition 2.9. [32] Let (X, τ) be a sfts, μ, ν ∈ τX and t ∈ I0, then

(i) The t-fuzzy θ-interior (resp. θ-closure) of μ is
\[ \thetaI(μ, t) = \bigvee \{I(μ, t) : μ ≥ ν, τ(ν - τ) ≥ t \} \]
\[ \thetaC(μ, t) = \bigwedge \{C(μ, t) : μ ≥ ν, τ(ν - τ) ≥ t \}. \]

(ii) The t-fuzzy θ-semi-interior (resp. θ-semi-closure) of μ is
\[ θS(μ, t) = \bigvee \{S(μ, t) : μ ≥ ν, ν is t-fsc \} \]
\[ θC(μ, t) = \bigwedge \{C(μ, t) : μ ≥ ν, ν is t-fsc \}. \]

(iii) The t-fuzzy θ-pre-interior (resp. θ-pre-closure) of μ is
\[ θP(μ, t) = \bigvee \{P(μ, t) : μ ≥ ν, ν is t-fpc \} \]
\[ θC(μ, t) = \bigwedge \{C(μ, t) : μ ≥ ν, ν is t-fpc \}. \]

Definition 2.10. [32] Let (X, τ) be a sfts. For μ, ν ∈ τX and t ∈ I0, μ is called an

(i) t-fuzzy θ-open (resp. θ-closed) set if μ = θI(μ, t)
\[ \text{(resp. μ = θC(μ, t)).} \]

(ii) t-fuzzy θ-semiopen (resp. θ-semiclosed)
set if μ ≤ C(θI(μ, t), t)
\[ \text{(resp. } I(θC(μ, t), t) \text{).} \]

(iii) t-fuzzy θ-preopen (resp. θ-preclosed) set if μ ≤ I(θC(μ, t), t)
\[ \text{(resp. } C(θI(μ, t), t) \text{).} \]

Definition 2.11. [32] Let (X, τ) be a sfts. Then

(i) The union of all t-fuzzy θ-open, (resp. t-fuzzy θ-semiopen, t-fuzzy θ-preopen) sets contained in μ is called the t-fuzzy θ-interior (resp. t-fuzzy θ-semiinterior, t-fuzzy θ-preinterior) of μ and is denoted by θI(μ, t) (resp. θS(μ, t), θP(μ, t)).
Proof. of a sfts

Proposition 3.3. Hence

(ii) $MC_\tau(\mu, t) = \bigwedge \{ v \in I^X : \mu \leq v, v \text{ is a } t\text{-fM set} \}$ is called the $t$-fuzzy $M$-closure of $\mu$.

Proposition 2.16. [32] Let $(X, \tau)$ be a sfts, $\mu \in I^X$ and $t \in I_0$, then

(i) Every $t$-fuzzy $\theta$-semiopen (resp. $t$-fuzzy $\delta$-preopen) set is $t$-fuzzy $M$-open.

(ii) Every $t$-fuzzy $M$-open set is $t$-fuzzy e-open.

3. $t$-fuzzy $M^*$-open sets and $t$-fuzzy $M^*$-closed sets

Definition 3.1. Let $(X, \tau)$ be a sfts. For $\mu \in I^X$ and $t \in I_0$, $\mu$ is called an $t$-fuzzy

(i) $M^*$-open set if $\mu \leq I_t(C_\tau(\theta I_t(\mu, t), t), t)$.

(ii) $M^*$-closed set if $\mu \geq C_\tau(I_t(\theta C_\tau(\mu, t), t), t)$.

The collection of all $t$-fuzzy $M^*$-open (resp. $t$-fuzzy $M^*$-closed) sets will be denoted by $t\text{-fM}_0^*$ (resp. $t\text{-fM}_c^*$) respectively.

Definition 3.2. Let $(X, \tau)$ be a sfts, $\mu, v \in I^X$ and $t \in I_0$,

(i) $M^*_I(\mu, t) = \bigvee \{ v \in I^X : \mu \geq v, v \text{ is a } t\text{-fM}_0^* \text{ set} \}$ is called the $t$-fuzzy $M^*$-interior of $\mu$.

(ii) $M^*_C(\mu, t) = \bigwedge \{ v \in I^X : \mu \leq v, v \text{ is a } t\text{-fM}_c^* \text{ set} \}$ is called the $t$-fuzzy $M^*$-closure of $\mu$.

Remarks. $M^*_C(\mu, t)$ is the smallest $t$-fM$^*$ set which contains $\mu$ and $M^*_I(\mu, t)$ is the largest $t$-fM$^*$ set which is contained in $\mu$. Also $M^*_C(\mu, t) = (\mu, t)$ for any $t$-fM$^*$ set $\mu$ and $M^*_I(\mu, t) = (\mu, t)$ for any $t$-fM$^*$ set $\mu$.

Proposition 3.3. The following are equivalent for a subset $\mu$ of a sfts $(X, \tau)$.

(i) Every $t$-fuzzy $\theta$-open set is an $t$-fuzzy $M^*$-open set.

(ii) Every $t$-fuzzy $M^*$-open set is an $t$-fuzzy $\theta$-semiopen set.

(iii) Every $t$-fuzzy $M^*$-open set is an $t$-fuzzy $M$-open set.

Proof. (i) Let $\mu$ be a $t$-fuzzy $\theta$-open set. Then $\mu = \theta I_t(\mu, t)$ and by Lemma 2.12(iv), $\theta I_t(\mu, t) \leq I_t(\mu, t) \leq \mu$. Hence $\mu = I_t(\mu, t)$. Since $\mu = \theta I_t(\mu, t) \leq C_\tau(\theta I_t(\mu, t), t)$, then $\mu = I_t(\mu, t) \leq I_t(C_\tau(\theta I_t(\mu, t), t), t)$. Thus $\mu$ is a $t$-fM$^*$ set.

(ii) Obvious from the definition.

(iii) Let $\mu$ be a $t$-fM$^*$ set. Then

\[
\begin{align*}
\mu & \leq I_t(C_\tau(\theta I_t(\mu, t), t), t) \\
& \leq C_\tau(\theta I_t(\mu, t), t) \\
& \leq C_\tau(\theta I_t(\mu, t), t) \vee I_t(\delta C_\tau(\mu, t), t).
\end{align*}
\]

Hence $\mu$ is an $t$-fM$^*$ set.

Remark 3.4. From the above Definitions and Proposition 3.3, it is clear that the following implications are true for a subset $\mu$ of a sfts $X$ and $t \in I_0$.

Example 3.5. Let $\mu$ and $v$ be fuzzy subsets of $X = \{a, b, c\}$ defined as follows:

$\mu(a) = 0.3, \mu(b) = 0.5, \mu(c) = 0.5$;
$v(a) = 0.5, v(b) = 0.5, v(c) = 0.5$.

Then $\tau : I^X \to I_0$ defined as

\[
\tau(\mu) = \begin{cases} 
1, & \text{if } \mu = \emptyset \text{ or } \overline{X}, \\
\frac{1}{2}, & \text{if } \mu = \mu, v, \\
0, & \text{otherwise,} 
\end{cases}
\]

is a sfts on $X$. For $t = \frac{1}{2}$, then $v$ is a $\frac{1}{2}$-fM$^*$ set and $\frac{1}{2}$-f$\emptyset$ set.

Example 3.6. Let $\mu$, $v$ and $\omega$ be fuzzy subsets of $X = \{a, b, c\}$ defined as follows:

$\mu(a) = 0.3, \mu(b) = 0.4, \mu(c) = 0.5$;
$v(a) = 0.6, v(b) = 0.5, v(c) = 0.5$;
$\omega(a) = 0.7, \omega(b) = 0.6, \omega(c) = 0.5$.

Then $\tau : I^X \to I_0$ defined as

\[
\tau(\mu) = \begin{cases} 
1, & \text{if } \mu = \emptyset \text{ or } \overline{X}, \\
\frac{1}{2}, & \text{if } \mu = \mu, v, \\
0, & \text{otherwise,} 
\end{cases}
\]

is a sfts on $X$. For $t = \frac{1}{2}$, then $\omega$ is a $\frac{1}{2}$-f$\emptyset$ set but $\omega$ is not a $\frac{1}{2}$-fM$^*$ set.
Example 3.7. Let \( \mu \) and \( \nu \) be fuzzy subsets of \( X = \{a, b, c\} \) as follows:
\[
\begin{align*}
\mu(a) &= 0.1, \mu(b) = 0.1, \mu(c) = 0.1; \\
\nu(a) &= 0.9, \nu(b) = 0.9, \nu(c) = 0.9.
\end{align*}
\]
Then \( \tau : I^X \rightarrow I \) defined as
\[
\tau(\mu) = \begin{cases} 
1, & \text{if } \mu = 0 \text{ or } 1, \\
\frac{1}{2}, & \text{if } \mu = \mu, \\
0, & \text{otherwise},
\end{cases}
\]
is a smooth fuzzy topology on \( X \). For \( t = \frac{1}{2} \), then \( \nu \) is \( \frac{1}{2} \)-fMoset but \( \nu \) is not \( \frac{1}{2} \)-fM* o set.

**Theorem 3.8.** Let \( (X, \tau) \) be a sfts and \( t \in I_\sigma \).

(i) Arbitrary union of \( t \)-fM* o sets is an \( t \)-fM* o set.

(ii) Arbitrary intersection of \( t \)-fM* c sets is an \( t \)-fM* c set.

**Proof.** (i) Let \( \{\mu_\alpha : \alpha \in \Gamma\} \) be a family of \( t \)-fM* o sets. For each \( \alpha \in \Gamma \),
\[
\mu_\alpha \leq I_\tau(C_\tau(\theta I_\tau(\mu_\alpha, t), t), t).
\]
\[
\bigvee_{\alpha \in \Gamma} \mu_\alpha \leq \bigvee_{\alpha \in \Gamma} I_\tau(C_\tau(\theta I_\tau(\mu_\alpha, t), t), t).
\]
\[
\leq I_\tau(C_\tau(\theta I_\tau(\bigvee_{\alpha \in \Gamma} \mu_\alpha, t), t), t).
\]
(ii) Similar to the proof of (i).

**Lemma 3.9.** For a sfts \( (X, \tau) \), the family of all \( t \)-fuzzy \( M^* \)-open sets of \( X \) forms a smooth topology denoted by \( \tau_{M^*} \) for \( X \).

**Proof.** It is obvious that \( \emptyset \) and \( \top \) are \( fM^* \) sets of \( X \) and from Theorem 3.8, we have arbitrary union of \( t \)-fM* o sets is an \( t \)-fM* o set.

Let \( \mu \) and \( \nu \) be \( t \)-fM* o sets. Then
\[
\mu \leq I_\tau(C_\tau(\theta I_\tau(\mu, t), t), t)
\]
and
\[
\nu \leq I_\tau(C_\tau(\theta I_\tau(\nu, t), t), t).
\]
Hence \( \mu \wedge \nu \)
\[
\leq I_\tau(C_\tau(\theta I_\tau(\mu, t), t) \wedge I_\tau(C_\tau(\theta I_\tau(\nu, t), t), t), t)
\]
\[
\leq I_\tau(C_\tau(\theta I_\tau(\mu, t) \wedge \theta I_\tau(\nu, t), t), t), t)
\]
\[
\leq I_\tau(C_\tau(\theta I_\tau(\nu, t), t), t)
\]
Hence the finite intersection of \( t \)-fM* o sets is \( t \)-fM* o and hence \( \tau_{M^*} \) is a smooth topology for \( X \).

**Theorem 3.10.** The following hold for a subset \( \mu \) of a sfts \( X \).

(i) \( \mu \) is \( t \)-fM* o \( \iff \) \( \mu = \mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu, t), t), t) \).

(ii) \( \mu \) is \( t \)-fM* c \( \iff \) \( \mu = \mu \vee I_\tau(C_\tau(\theta I_\tau(\mu, t), t), t) \).

(iii) \( M^* I_\tau(\mu, t) = \mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu, t), t), t) \).

(iv) \( M^* C_\tau(\mu, t) = \mu \vee C_\tau(I_\tau(\theta C_\tau(\mu, t), t), t), t) \).

**Proof.** (i) Let \( \mu \) be \( t \)-fM* o. Then
\[
\mu \leq I_\tau(C_\tau(\theta I_\tau(\mu, t), t), t)
\]
We obtain \( \mu = \mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu, t), t), t) \).
Conversely, let \( \mu = \mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu, t), t), t) \).
We have
\[
\mu = \mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu, t), t), t)
\]
\[
\leq I_\tau(C_\tau(\theta I_\tau(\mu, t), t), t).
\]
Hence \( \mu \) is \( t \)-fM* o.

(ii) Let \( \mu \) be \( t \)-fM* c. Then \( \mu \geq C_\tau(I_\tau(\theta C_\tau(\mu, t), t), t) \).
We obtain \( \mu = \mu \vee C_\tau(I_\tau(\theta C_\tau(\mu, t), t), t) \).
Conversely, let \( \mu = \mu \vee C_\tau(I_\tau(\theta C_\tau(\mu, t), t), t) \).
We have
\[
\mu = \mu \vee C_\tau(I_\tau(\theta C_\tau(\mu, t), t), t)
\]
\[
\geq C_\tau(I_\tau(\theta C_\tau(\mu, t), t), t).
\]
Hence \( \mu \) is \( t \)-fM* c.

(iii) Since \( M^* I_\tau(\mu, t) \) is \( t \)-fM* o, we have,
\[
I_\tau(C_\tau(\theta I_\tau(\mu, t), t), t) \geq I_\tau(C_\tau(\theta I_\tau(M^* I_\tau(\mu, t), t), t), t)
\]
\[
\geq M^* I_\tau(\mu, t).
\]
Hence,
\[
\mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu, t), t), t) \geq M^* I_\tau(\mu, t).
\]
On the other way, since
\[
I_\tau(C_\tau(\theta I_\tau(\mu, t), t), t)
\]
\[
\geq I_\tau(C_\tau(\theta I_\tau(\mu, t) \wedge \theta I_\tau(\theta I_\tau(\mu, t), t), t), t), t)
\]
\[
\geq I_\tau(C_\tau(\theta I_\tau(\mu, t) \wedge \theta I_\tau(\theta I_\tau(\mu, t), t), t), t)
\]
\[
= I_\tau(C_\tau(\theta I_\tau(\mu, t) \wedge \theta I_\tau(\theta I_\tau(\mu, t), t), t), t)
\]
\[
\geq \mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu, t), t), t), t)
\]
Hence, \( \mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu, t), t), t) \) is \( t \)-fM* o contained in \( \mu \). Hence
\[
M^* I_\tau(\mu, t) \geq \mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu, t), t), t).
\]
Thus, we obtain
\[
M^* I_\tau(\mu, t) = \mu \wedge I_\tau(C_\tau(\theta I_\tau(\mu, t), t), t).
\]
(iv) Since \( M^* C_\tau(\mu, t) \) is \( t \)-fM* c, we have, \( C_\tau(I_\tau(\theta C_\tau(\mu, t), t), t) \).
\[
= C_\tau(I_\tau(\theta C_\tau(M^* C_\tau(\mu, t), t), t), t)
\]
\[
\leq M^* C_\tau(\mu, t).
\]
Hence
\[
\mu \vee C_\tau(I_\tau(\theta C_\tau(\mu, t), t), t), t) \leq M^* C_\tau(\mu, t).
\]
On the other way, since
\[
C_\tau(I_\tau(\theta C_\tau(\mu, t) \vee C_\tau(I_\tau(\theta C_\tau(\mu, t), t), t), t), t)
\]
\begin{align*}
\leq C_2(I_2(\theta C_2(\mu, t) \lor \theta C_2(\theta I_2(\theta C_2(\mu, t), t), t), t)) \\
= C_2(I_2(\theta C_2(\mu, t) \lor \theta C_2(\theta I_2(\theta C_2(\mu, t), t), t), t)) \\
= C_2(I_2(\theta C_2(\theta I_2(\theta C_2(\mu, t), t), t), t), t) \\
= C_2(I_2(\theta C_2(\mu, t), t), t) \\
\leq \mu \lor C_2(I_2(\theta C_2(\mu, t), t), t).
\end{align*}

Then

\[ \mu \lor C_2(I_2(\theta C_2(\mu, t), t), t) \text{ is } t\text{-fM}^c\text{ containing } \mu. \]

Hence

\[ M^c C_2(\mu, t) \leq \mu \lor C_2(I_2(\theta C_2(\mu, t), t), t). \]

Thus, we obtain

\[ M^c C_2(\mu, t) = \mu \lor C_2(I_2(\theta C_2(\mu, t), t), t). \]

\[ \Box \]

**Theorem 3.11.** Let \((X, \tau)\) be a fts. Let \(\mu \in \mathcal{P}^X\) and \(t \in I_0.\)

(i) \(\mu\) is t-fM\(^c\) o if \(\mu = M^t I_2(\mu, t).\)

(ii) \(\mu\) is t-fM\(^c\) o if \(\mu = M^c C_2(\mu, t).\)

**Proof.** (i) Let \(\mu\) be an t-fM\(^c\) o, then

\[ M^t I_2(\mu, t) = \{ v : \mu \geq v, v \text{ is a t-fM}^c\} \]

Conversely, let \(\mu = M^t I_2(\mu, t),\) since \(M^t I_2(\mu, t)\) is the arbitrary union of t-fM\(^c\) o then \(\mu\) is t-fM\(^c\) o.

(ii) It is similar to part (i).

\[ \Box \]

**Theorem 3.12.** Let \((X, \tau)\) be a fts. For \(\mu \in \mathcal{P}^X\) and \(t \in I_0\) we have

(i) \(M^t I_2(T - \mu, t) = \overline{T} - M^c C_2(\mu, t).\)

(ii) \(M^c C_2(T - \mu, t) = \overline{T} - M^t I_2(\mu, t).\)

**Proof.** By Theorem 3.10 and 3.11, we have for all \(\mu \in \mathcal{P}^X\) and \(t \in I_0, (i) M^t I_2(T - \mu, t)

\[ = (T - \mu) \lor I_2(\theta C_2(\mu, t), t), t). \]

\[ = (T - \mu) \lor (T - C_2(I_2(\theta C_2(\mu, t), t), t)), t). \]

\[ = (T - \mu) \lor (T - C_2(I_2(\theta C_2(\mu, t), t)), t)). \]

\[ = (T - \mu) \lor (T - C_2(I_2(\theta C_2(\mu, t), t), t)). \]

\[ = T - M^c C_2(\mu, t). \]

(ii) \(M^c C_2(T - \mu, t)

\[ = (T - \mu) \lor C_2(I_2(\theta C_2(\mu, t), t), t), t). \]

\[ = (T - \mu) \lor (T - I_2(\theta C_2(\mu, t), t), t)). \]

\[ = (T - \mu) \lor (T - I_2(\theta C_2(\mu, t), t), t)). \]

\[ = (T - \mu) \lor (T - I_2(\theta C_2(\mu, t), t), t)). \]

\[ = T - M^t I_2(\mu, t). \]

\[ \Box \]

**Theorem 3.13.** Let \((X, \tau)\) be a fts. Let \(\mu \in \mathcal{P}^X\) and \(t \in I_0,\) the following statements hold:

(i) \(M^c C_2(\overline{0}, t) = \overline{0}\) and \(M^t I_2(\overline{T}, t) = \overline{T}.\)

(ii) \(I_2(\mu, t) \leq M^t I_2(\mu, t) \leq \mu \leq M^c C_2(\mu, t) \leq C_2(\mu, t).\)

(iii) \(\mu \leq v \Rightarrow M^t I_2(\mu, t) \leq M^t I_2(v, t) \text{ and } M^c C_2(\mu, t) \leq M^c C_2(v, t).\)

(iv) \(M^t C_2(M^t C_2(\mu, t), t) = M^t C_2(\mu, t) \text{ and } M^t I_2(M^t I_2(\mu, t), t) = M^t I_2(\mu, t).\)

(v) \(M^t C_2(\mu, t) \lor M^t C_2(v, t) \leq M^t C_2(\mu \lor v, t).\)

\[ M^t I_2(\mu, t) \lor M^t I_2(v, t) \leq M^t I_2(\mu \lor v, t). \]

(vi) \(M^t C_2(\mu, t) \land M^t C_2(v, t) \geq M^t C_2(\mu \land v, t) \text{ and } M^t I_2(\mu, t) \land M^t I_2(v, t) \geq M^t I_2(\mu \land v, t).\)

**Proof.** (i), (ii) and (iii) are trivial from the Definitions of \(M^t C_2\) and \(M^t I_2.\)

(iv) By Theorem 3.10 and 3.11, \(M^t C_2(M^t C_2(\mu, t), t) = C_2(I_2(\theta C_2(\mu, t), t), t). \)

But

\[ M^t C_2(\mu, t) \leq M^t C_2(M^t C_2(\mu, t), t). \]

Hence

\[ M^t C_2(\mu, t) = M^t C_2(M^t C_2(\mu, t), t). \]

\[ M^t I_2(\mu, t) = M^t I_2(M^t I_2(\mu, t), t). \]

\[ \leq I_2(\theta C_2(\mu, t) \lor I_2(\theta C_2(\mu, t), t), t), t). \]

\[ \leq I_2(\theta C_2(\mu, t) \lor I_2(\theta C_2(\mu, t), t), t), t). \]

\[ \leq I_2(\theta C_2(\mu, t), t), t). \]

\[ \leq M^t I_2(\mu, t). \]

But \(M^t I_2(\mu, t) \geq M^t I_2(M^t I_2(\mu, t), t).\) Hence \(M^t I_2(\mu, t) = M^t I_2(M^t I_2(\mu, t), t).\)

(v) By Theorem 3.10 and 3.11, we have,

\[ M^t C_2(\mu, t) \lor M^t C_2(v, t) = [\mu \lor I_2(\theta C_2(\mu, t), t), t) \lor [v \lor I_2(\theta C_2(\mu, t), t), t)] \]

\[ = [\mu \lor v) \lor (I_2(\theta C_2(\mu, t), t), t) \lor (I_2(\theta C_2(\mu, t), t), t)] \]

\[ \leq M^t C_2(\mu \lor v, t). \]

Hence \(M^t C_2(\mu, t) \lor M^t C_2(v, t) \leq M^t C_2(\mu \lor v, t).\)

\[ M^t I_2(\mu, t) \lor M^t I_2(v, t) = [\mu \lor I_2(\theta C_2(\mu, t), t), t) \lor [v \lor I_2(\theta C_2(\mu, t), t), t)] \]

\[ = [\mu \lor v) \lor (I_2(\theta C_2(\mu, t), t), t) \lor (I_2(\theta C_2(\mu, t), t), t)] \]

\[ \leq M^t I_2(\mu \lor v, t). \]

Hence \(M^t I_2(\mu, t) \lor M^t I_2(v, t) \leq M^t I_2(\mu \lor v, t).\)

(vi) \(M^t C_2(\mu \lor v, t) = [\mu \lor v) \lor C_2(I_2(\theta C_2(\mu \lor v, t), t), t)] \)

\[ = [\mu \lor v) \lor C_2(I_2(\theta C_2(\mu \lor v, t), t), t) \]

\[ \leq [\mu \lor v) \lor C_2(I_2(\theta C_2(\mu \lor v, t), t), t)] \]

\[ = M^t I_2(\mu \lor v, t). \]

Hence \(M^t C_2(\mu, t) \land M^t C_2(v, t) \geq M^t C_2(\mu \land v, t).\)
Theorem 3.15. The following are equivalent for a subset $\mu$ of a sfts $(X, \tau)$.

(i) $\mu$ is an $t$-$M^*o$ set.

(ii) $\mu \leq \theta sC_\tau(\theta I_\tau(\mu, t), 1)$.

(iii) $\theta sC_\tau(\mu, t) = \theta sC_\tau(\theta I_\tau(\mu, t), t)$.

Proof. 
(i) $\Rightarrow$ (ii): Let $\mu$ be an $t$-$M^*o$ set.
Then by Theorem 3.13, $\mu = M^*I_\tau(\mu, 1)$.
By Lemma 3.14,

$$\mu = \mu \land \theta sC_\tau(\theta I_\tau(\mu, t), t) \leq \theta sC_\tau(\theta I_\tau(\mu, t), t).$$

Hence $\mu \leq \theta sC_\tau(\theta I_\tau(\mu, t), t)$.

(ii) $\Rightarrow$ (i): Let $\mu \leq \theta sC_\tau(\theta I_\tau(\mu, t), t)$. This implies that $\mu \leq \mu \land \theta sC_\tau(\theta I_\tau(\mu, t), t) = M^*I_\tau(\mu, 1)$.
Thus $\mu \leq M^*I_\tau(\mu, 1)$ and hence

$$\mu = M^*I_\tau(\mu, 1),$$

therefore $\mu$ is $t$-$M^*o$.

(ii) $\Rightarrow$ (iii): Let $\mu \leq \theta sC_\tau(\theta I_\tau(\mu, t), t)$. Then
$\theta sC_\tau(\mu, t) \leq \theta sC_\tau(\theta I_\tau(\mu, t), t)$. But $\theta I_\tau(\mu, t) \leq \mu$.
Hence $\theta sC_\tau(\theta I_\tau(\mu, t), t) \leq \theta sC_\tau(\mu, t)$.
Thus $\theta sC_\tau(\mu, t) = \theta sC_\tau(\theta I_\tau(\mu, t), t)$.

(iii) $\Rightarrow$ (ii): Let $\theta sC_\tau(\mu, t) = \theta sC_\tau(\theta I_\tau(\mu, t), t)$. Then $\theta sC_\tau(\mu, t) \leq \theta sC_\tau(\theta I_\tau(\mu, t), t)$.
But $\mu \leq \theta sC_\tau(\mu, t)$ and therefore

$$\mu \leq \theta sC_\tau(\theta I_\tau(\mu, t), t).$$

Theorem 3.16. The following are equivalent for a $\mu$ of a sfts $(X, \tau)$.

(i) $\mu$ is an $t$-$M^*c$ set.

(ii) $\mu \geq \theta sI_\tau(\theta C_\tau(\mu, t), t)$.

(iii) $\theta sI_\tau(\mu, t) = \theta sI_\tau(\theta C_\tau(\mu, t), t)$.

Proof. (i) $\Rightarrow$ (ii): Let $\mu$ be an $t$-$M^*c$ set. Then by Theorem 3.13, $\mu = M^*C_\tau(\mu, 1)$.
By Lemma 3.14,

$$\mu = \mu \lor \theta sI_\tau(\theta C_\tau(\mu, t), t) \geq \theta sI_\tau(\theta C_\tau(\mu, t), t).$$

Hence $\mu \geq \theta sI_\tau(\theta C_\tau(\mu, t), t)$.

(ii) $\Rightarrow$ (i): Let $\mu \geq \theta sI_\tau(\theta C_\tau(\mu, t), t)$. This implies that

$$\mu \geq \mu \lor \theta sI_\tau(\theta C_\tau(\mu, t), t) = M^*C_\tau(\mu, t).$$

Hence $\mu \geq M^*C_\tau(\mu, t)$ and thus $\mu = M^*C_\tau(\mu, t)$ and therefore $\mu$ is $t$-$M^*c$.

(iii) $\Rightarrow$ (ii): Let $\mu \geq \theta sI_\tau(\theta C_\tau(\mu, t), t)$.
Then $\theta sI_\tau(\mu, t) \geq \theta sI_\tau(\theta C_\tau(\mu, t), t)$.
But $\theta C_\tau(\mu, t) \geq \mu$.
Hence $\theta sI_\tau(\theta C_\tau(\mu, t), t) \geq \theta sI_\tau(\mu, t)$.
Thus

$$\theta sI_\tau(\mu, t) = \theta sI_\tau(\theta C_\tau(\mu, t), t).$$

(iii) $\Rightarrow$ (iii): Let $\mu \geq \theta sI_\tau(\theta C_\tau(\mu, t), t)$.
Then $\theta sI_\tau(\mu, t) \geq \theta sI_\tau(\theta C_\tau(\mu, t), t)$.
But $\mu \geq \theta sI_\tau(\mu, t)$ and therefore $\mu \geq \theta sI_\tau(\theta C_\tau(\mu, t), t)$.

Theorem 3.17. Let $(X, \tau)$ be a sfts. For $\mu, v \in I^X$ and $t \in I_0$, then,

(i) If $\tau(v) \geq t$ where $v$ is a crisp subset and $\mu$ is an $t$-$M^*o$ set, then $\mu \land v$ is an $t$-$M^*o$ set.

(ii) If $\tau(\overline{v}) \geq t$ where $v$ is a crisp subset and $\mu$ is an $t$-$M^*c$ set, then $\mu \lor v$ is an $t$-$M^*c$ set.

Proof. 
(i) Let $\mu$ be $t$-$M^*o$ and $v \in I^X$ with $\tau(v) \geq t$ which is a crisp subset.
Then

$$\mu \land v \leq I_\tau(C_\tau(\theta I_\tau(\mu, t), t), t) \land v \leq I_\tau(C_\tau(\theta I_\tau(\mu \land v, t), t), t).$$

Hence $\mu \land v$ is $t$-$M^*o$.

(ii) Let $\mu$ be $t$-$M^*c$ and $v \in I^X$ with $\tau(\overline{v}) \geq t$ which is a crisp subset.
Then

$$\mu \lor v \geq C_\tau(I_\tau(\theta C_\tau(\mu, t), t), t) \lor v \geq C_\tau(I_\tau(\theta C_\tau(\mu \lor v, t), t), t).$$

Hence $\mu \lor v$ is $t$-$M^*c$.

Theorem 3.18. Let $(X, \tau)$ be a sfts. For $\mu, v \in I^X$ and $t \in I_0$, then,

$$M^*I_\tau(\mu \land v, t) = (\mu \land v) \land I_\tau(C_\tau(\theta I_\tau(\mu \land v, t), t), t) = (\mu \land v) \land I_\tau(C_\tau(I_\tau(\mu \land v, t), t), t) \leq [\mu \land I_\tau(C_\tau(\theta I_\tau(\mu \land v, t), t), t) \land [\nu \land I_\tau(C_\tau(I_\tau(\mu \land v, t), t), t)] = M^*I_\tau(\mu, t) \land M^*I_\tau(v, t).
Hence $M^*I_\tau(\mu, t) \land M^*I_\tau(v, t) \geq M^*I_\tau(\mu \land v, t)$.

Lemma 3.14. Let $(X, \tau)$ be a sfts. Let $\mu \in I^X$ and $t \in I_0$, then

(i) $M^*C_\tau(\mu, t) = \mu \land \theta sI_\tau(\theta C_\tau(\mu, t), t)$.

(ii) $M^*I_\tau(\mu, t) = \mu \land \theta sC_\tau(\theta I_\tau(\mu, t), t)$.

Proof. 
(i) From Lemma 2.14(4),

$$\mu \land \theta sI_\tau(\theta C_\tau(\mu, t), t) = \mu \land (I_\tau(C_\tau(\theta C_\tau(\mu, t), t), t)) = M^*C_\tau(\mu, t).$$

(ii) From Lemma 2.14(3),

$$\mu \land \theta sC_\tau(\theta I_\tau(\mu, t), t) = \mu \land (I_\tau(C_\tau(\theta I_\tau(\mu, t), t), t)) = M^*I_\tau(\mu, t).$$
(i) $\mu$ is $t$-$fM^o$ iff $1 - \mu$ is $t$-$fM^c$.

(ii) If $\tau(\mu) \geq t$, then $\mu$ is $t$-$fM^o$ set.

(iii) $I_C(\mu, t)$ is an $t$-$fM^o$ set.

(iv) $C_\tau(\mu, t)$ is an $t$-$fM^c$ set.

Proof. (i) and (ii) are trivial.

(iii) From the Definition of $I_C$ of Theorem 2.5 and Definition 2.2, since $\tau(I_C(\mu, t)) \geq t$, by (ii) $I_C(\mu, t)$ is an $t$-$fM^o$ set.

(iv) Since $\overline{1 - C_\tau(\mu, t)} = I_C(\overline{1 - \mu}, t)$ from Theorem 2.5, hence we have $\tau(\overline{1 - C_\tau(\mu, t)}) \geq t$. Hence by (ii), we have $\overline{1 - C_\tau(\mu, t)}$ is $t$-$fM^o$. By (i) $C_\tau(\mu, t)$ is an $t$-$fM^c$ set.

Theorem 3.19. Let $(X, \tau)$ be a sfts, $\mu \in P^X$ and $t \in I_0$. Then

(i) If $\mu$ is $t$-$fM^o$ with $\tau(\overline{1 - \mu}) \geq t$, then $\mu$ is $t$-$fM^o$ set.

(ii) If $\mu$ is $t$-$fM^c$ with $\tau(\mu) \geq t$, then $\mu$ is $t$-$fM^o$ set.

(iii) If $\mu$ is $t$-$f\theta$ set with $\tau(\mu) \geq t$, then $\mu$ is $t$-$fM^o$ set.

(iv) If $\mu$ is $t$-$f\theta$ set with $\tau(\overline{1 - \mu}) \geq t$, then $\mu$ is $t$-$fM^c$ set.

Proof. We prove (i) and (iii). Other results have similar proofs.

(i) Let $\mu$ be an $t$-$fM^o$ set and $\tau(\overline{1 - \mu}) \geq t$. Then $\delta C_\tau(\mu, t) = \mu$.

$\mu \leq C_\tau(\theta I_C(\mu, t), t) \lor I_C(\delta C_\tau(\mu, t), t)$

$= C_\tau(\theta I_C(\mu, t), t) \lor I_C(\mu, t)$

$\leq C_\tau(\theta I_C(\mu, t), t)$

$\leq I_C(\theta I_C(\mu, t), t)$.

Hence $\mu$ is a $t$-$fuzzy M^o$-open set.

(iii) Let $\mu$ be an $t$-$f\theta$ set and $\tau(\mu) \geq t$.

$\mu \leq C_\tau(\theta I_C(\mu, t), t)$

$\leq I_C(\theta I_C(\mu, t), t)$

Thus $\mu$ is a $t$-$fuzzy M^o$-open set.

4. Conclusion

In this paper, we introduce the idea of $t$-$fuzzy M^o$-open (resp. $t$-$fuzzy M^o$-closed) sets in fuzzy topological spaces in the sense of Šostak’s. Also, we introduce $t$-$fuzzy M^c$-interior (resp. $t$-$fuzzy M^c$-closure) and investigate some of their properties. Moreover, we investigate the relationships between $t$-$fuzzy open$, $t$-$fuzzy \theta$-semiopen, $t$-$fuzzy \theta$-open, $t$-$fuzzy \delta$-semiopen, $t$-$fuzzy \delta$-preopen, $t$-$fuzzy a$-open, $t$-$fuzzy e$-open and $t$-$fuzzy e^*$-open in Šostak’s fuzzy topological spaces.

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