The topology Ω^* on $\text{Spec}(\sigma, C)$ of L-component (σ, C)

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Abstract

The action of a locale L on a join semilattice J gives us the newly introduced notion of L-slices (σ, J) . We have tried to extend the idea of Zariski topology on modules to L-slices. Given a locale L and a L-slice (σ, J) , for $m \in (\sigma, J)$ and $r \in L$, we have constructed (σ, J) ideals $[r \rightarrow m] = \{n \in (\sigma, J) : \sigma(r, n) \leq m\}$. Their properties and characteristics are studied. In this paper we introduce the ideal elements of (σ, J) and develop a zariski topology on L-slices.

Keywords

L-slice, L-component, Zariski topology.

AMS Subject Classification

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1. Introduction

Locales are complete lattices which satisfy the infinite distributivity law. Algebraically, locales can be thought of as a commutative idempotent ring. With this idea, we have extended the notion of modules to that of L-slices. Through, L-slices, we study the action of a locale on the join semilattice. In literature, we can find many different generalisation of Zariski topology for modules over commutative rings. Here, we have tried to develop a Zariski topology on a more generalised concept of modules. Through the action of the locale, we expect the join semilattice to adopt some topological properties of the locale. We define the ideal elements of L-slice and some of their properties are studied.

2. Preliminaries

Definition 2.1. [2] A join semilattice is a poset $(J, \leq)$ in which every finite subset has a join.

Example 2.2. The set of all natural numbers with the partial order less than or equal to.

Definition 2.3. [2] A lattice is a poset in which every finite subset has both a join and meet.

Definition 2.4. [2] A subset I of a lattice J is said to be an ideal if

i) I is a sub-join semilattice of J ; i.e. $0 \in I$, and $a, b \in I$ imply $a \vee b \in I$ and ii) I is a lower set; i.e. $a \in I$ and $b \leq a$ imply $b \in I$

Definition 2.5. [2] A subset F of a lattice A is a set which satisfies the axioms dual to those defining an ideal, is called filter of A .

Proposition 2.6. [2] Let I be an ideal of a lattice A . The following conditions are equivalent

- i. The complement of I in A is a filter
- ii. $1 \notin I$, and $a \wedge b \in I$ implies either $a \in I$ or $b \in I$
- iii. I is the kernel of a lattice homomorphism $f : A \rightarrow 2$ where 2 denotes the two element lattice $\{0, 1\}$.

Definition 2.7. [2] An ideal satisfying the equivalent conditions of the above proposition is called a prime ideal; its complement is called a prime filter.

Definition 2.8. [2] A poset is a complete lattice if every subset has a join and meet.

Definition 2.9. [5] A frame is a complete lattice L satisfying the infinite distributivity law $a \sqcap \bigsqcup B = \bigsqcup \{a \sqcap b; b \in B\}$ for all $a \in L$ and $B \subseteq L$.

Definition 2.10. [5] A map $f : L \rightarrow M$ between frames L, M preserving all finite meets (including the top 1) and all joins (including the bottom 0) is called a frame homomorphism. A bijective frame homomorphism is called a frame isomorphism.

Remark 2.11. The category of frames is denoted by **Frm**. The opposite of category **Frm** is the category **Loc** of locales. We can represent the morphism in **Loc** as the infima-preserving $f : L \rightarrow M$ such that the corresponding left adjoint $f^* : M \rightarrow L$ preserves finite meet. If we do not refer to the morphisms in the category **Loc** of locales and the category **Frm** of frames, then the objects frames and locales are same.

Example 2.12. [5] i. The lattice of open subsets of topological space.

ii. The Boolean algebra B of all regularly open subsets of Real line R .

Definition 2.13. In a locale L an element p is said to be meet irreducible if whenever $a \wedge b \leq p$ implies either $a \leq p$ or $b \leq p$.

Definition 2.14. [6] Let L be a locale and $(J, \vee)$ be join semilattice with bottom element 0_J . By the “action of L on J ” we mean a function $\sigma : L \times J \rightarrow J$ such that the following conditions are satisfied.

- i. $\sigma(a, x_1 \vee x_2) = \sigma(a, x_1) \vee \sigma(a, x_2)$ for all $a \in L, x_1, x_2 \in J$.
- ii. $\sigma(a, 0_J) = 0_J$ for all $a \in L$.
- iii. $\sigma(a \sqcap b, x) = \sigma(a, \sigma(b, x)) = \sigma(b, \sigma(a, x))$ for all $a, b \in L, x \in J$.
- iv. $\sigma(1_L, x) = x$ and $\sigma(0_L, x) = 0_J$ for all $x \in J$.
- v. $\sigma(a \sqcup b, x) = \sigma(a, x) \vee \sigma(b, x)$ for $a, b \in L, x \in J$.

If σ is an action of the locale L on a join semilattice J , then we call (σ, J) as L -slice.

Example 2.15. 1. Let L be a locale and I be any ideal of L . For each $x \in I$ define $\sigma : L \times I \rightarrow I$ as $\sigma(a, x) = a \wedge x$. Then (σ, I) is an L -slice.

2. Let the locale L be a chain with Top and Bottom elements and J be any join semilattice with bottom element. Define $\sigma : L \times J \rightarrow J$ by $\sigma(a, j) = j \quad \forall a \neq 0$ and $\sigma(0_L, j) = 0_J$. Then σ is an action of L on J and (σ, J) is an L -slice.

3. Any locale L can be viewed as the meet L -slice $(\wedge, L)$ where the action σ is defined as $\sigma(a, x) = a \wedge x$.

Here we give some properties of the action $\sigma : L \times J \rightarrow J$.

Proposition 2.16. Let (σ, J) be an L -slice and $a \in L$

- i. $\sigma_a(x) \leq x$ for all $x \in (\sigma, J)$.
- ii. For $n, l \in (\sigma, J)$, if $l \leq n$, then $\sigma(a, l) \leq \sigma(a, n)$, for every $a \in L$.
- iii. If $a \leq b$, for $a, b \in L$, then $\sigma(a, x) \leq \sigma(b, x)$ for every $x \in (\sigma, J)$.

In [4] we have tried to develop a Zariski topology on L-components. Now we give some definition and propositions which were developed in [4]. One may refer to [4] for proofs of propositions and theorems.

Definition 2.17. [4] Let $n, l \in (\sigma, J)$, then we define a set $[l \rightarrow n]_L = \{r \in L : \sigma(r, l) \leq n\}$.

Proposition 2.18. [4] For $n, l \in (\sigma, J)$, then $[l \rightarrow n]_L$ is an ideal of L .

Definition 2.19. [4] For $n, l \in (\sigma, J)$, $[l \rightarrow n]_L$ is called the implicative ideal of the locale L .

Proposition 2.20. [4] i) For $n, l \in (\sigma, J)$ and $n \leq l$, $[x \rightarrow n]_L \subseteq [x \rightarrow l]_L \forall x \in (\sigma, J)$

ii) For $n, l, k \in (\sigma, J)$ and $n \leq l$, $[l \rightarrow k]_L \subseteq [n \rightarrow k]_L$.

Proposition 2.21. [4] For $n, l \in (\sigma, J)$, $[l \rightarrow k]_L \cap [n \rightarrow k]_L = [l \vee n \rightarrow k]_L$.

Proposition 2.22. [4] For $n, l, k \in (\sigma, J)$, $[k \rightarrow l]_L \cap [k \rightarrow n]_L = [k \rightarrow l \vee n]_L$.

Definition 2.23. [4] Let L be a locale and C be a sup lattice with bottom element 0_C and top element 1_C . The L -component (σ, C) , is the ‘action of L on C ’ which is defined as a map $\sigma : L \times C \rightarrow C$ such that it satisfies the following conditions in addition to the definition of L -slice

- i) $\sigma(a, \bigvee x_i) = \bigvee \sigma(a, x_i)$ for $\{x_i\}_{i \in I} \in (\sigma, C)$, for some indexed set I
- ii) $\sigma(\bigsqcup_i a_i, x) = \bigvee_i \sigma(a_i, x)$ for $\{a_i\}_{i \in I} \in L$, for some indexed set I .

Definition 2.24. [4] An element $p \neq 1_C$, of (σ, C) is said to be L -prime element if for every $r \in L$ and $n \in (\sigma, C)$, $\sigma(r, n) \leq p$, implies that either $r \in [1_C \rightarrow p]_L$ or $n \leq p$.

Theorem 2.25. [4] Let p be a L -prime element and $x \in (\sigma, C)$ then $[x \rightarrow p]_L$ is a prime ideal of L .

Definition 2.26. [4] The set of all L -prime elements of (σ, C) is called the spectrum of (σ, C) and is denoted by $\text{Spec}(\sigma, C)$.

3. Main Results

In this section, we introduce the concept of L -component (σ, C) . We construct a L -slice over a sup lattice. A complete L -slice is called a L -component, that is, it is the action of a locale on a sup lattice. Further, we define ‘the action of an ideal on $x \in (\sigma, C)$, and ideal elements of (σ, C) ’.

Definition 3.1. Let L be a locale and C be a sup lattice with bottom element 0_C and top element 1_C . The L -component (σ, C) , is the ‘action of L on C ’ which is defined as a map $\sigma : L \times C \rightarrow C$ such that it satisfies the following conditions in addition to the definition of L -slice

- i) $\sigma(a, \bigvee x_i) = \bigvee \sigma(a, x_i)$ for $\{x_i\}_{i \in I} \in (\sigma, C)$, for some indexed set I
- ii) $\sigma(\bigsqcup_i a_i, x) = \bigvee_i \sigma(a_i, x)$ for $\{a_i\}_{i \in I} \in L$, for some indexed set I .

Example 3.2. The trivial L -slice is an example.

Definition 3.3. Let I be an ideal of the locale L , then for $x \in (\sigma, C)$ we define the ‘action of I on x ’, $\sigma(I, x) = \bigvee \{\bigvee_{i=1}^k \sigma(a_i, x) : k \in \mathbb{N}, a_i \in I\}$ and is denoted as Ix .

Definition 3.4. Let i be an ideal of the locale L and $n \in (\sigma, C)$ then $[n : I] = \bigvee \{x \in (\sigma, C) : Ix \leq n\}$ and is called the ideal element of (σ, C) .

Now we discuss some properties of ideal elements.

Proposition 3.5. Let (σ, C) be a L-component and I be an ideal of a locale L , then for every $n \in (\sigma, C)$

- i. $n \leq [n : I]$
- ii. $I[n : I] \leq n$

Proof. i. We have

$$\begin{aligned} \sigma(a_i, n) \leq n \forall a_i \in I &\Rightarrow \bigvee_{i=1}^k \sigma(a_i, n) \leq n \\ &\Rightarrow \bigvee \{ \bigvee_{i=1}^k \sigma(a_i, x) : k \in N, a_i \in I \} \leq n \\ &\Rightarrow In \leq n \end{aligned}$$

Therefore, $n \leq [n : I]$

- ii. $[n : I] = \bigvee \{x \in (\sigma, C) : Ix \leq n\}$

Let $[n : I] = m$ and $\{x \in (\sigma, C) : Ix \leq n\} = X$, so that we have $\bigvee x \in Xx = m$.

For any $a \in I$, $\sigma(a, m) = \sigma(a, \bigvee x \in Xx) = \bigvee x \in X \sigma(a, x)$.

Therefore, $\sigma(a, x) \leq Ix \leq n$, implies $\bigvee x \in X \sigma(a, x) \leq n$ so that $\sigma(a, m) \leq n$. Thus, for every $a_i \in I$,

$$\begin{aligned} \sigma(a_i, m) \leq n &\Rightarrow \bigvee_{i=1}^k \sigma(a_i, m) \leq n \\ &\Rightarrow \bigvee \{ \bigvee_{i=1}^k \sigma(a_i, m) : k \in N, \} \leq n \\ &\Rightarrow Im \leq n \\ &\Rightarrow I[n : I] \leq n \end{aligned}$$

□

Proposition 3.6. If $I \subseteq J$, then $Ix \leq Jx$, for I, J being ideals of locale L and $x \in (\sigma, C)$.

Proof. $Ix = \bigvee \{ \bigvee_{i=1}^k \sigma(a_i, x) : k \in N, a_i \in I \} \leq \bigvee \{ \bigvee_{i=1}^k \sigma(b_i, x) : k \in N, b_i \in J \} = Jx$. Thus, $Ix \leq Jx$ for all $x \in (\sigma, C)$. □

Proposition 3.7. For, $n \in (\sigma, C)$, $[x \rightarrow n] \subseteq [x \rightarrow [x \rightarrow n]x]$.

Proof. Let $r \in [x \rightarrow n]$. Then $\sigma(r, x) \leq n$ implies $\sigma(r, x) \leq [x \rightarrow n]x$. Hence, $r \in [x \rightarrow [x \rightarrow n]x]$. Therefore, $[x \rightarrow n] \subseteq [x \rightarrow [x \rightarrow n]x]$. □

Proposition 3.8. For, $n \in (\sigma, C)$, $[x \rightarrow n]x \leq n$.

Proof. $[x \rightarrow n]x = \bigvee \{ \bigvee_{i=1}^k \sigma(a_i, x) : k \in N, a_i \in [x \rightarrow n] \} = \bigvee \{ \bigvee_{i=1}^k \sigma(a_i, x) : k \in N, \sigma(a_i, x) \leq n \}$. Hence the proof. □

Now we define a new type of sets $C^*(n)$ and have tried to define a Zariski topology on $\text{Spec}(\sigma, C)$.

Definition 3.9. For $n \in (\sigma, C)$, we define $C^*(n) = \{p \in \text{Spec}(\sigma, C) : [1_C \rightarrow n] \subseteq [1_C \rightarrow p]\}$.

Proposition 3.10. Let (σ, C) be a L-component. Then we have the following

- i. $C^*(0_C) = \text{Spec}(\sigma, C)$
- ii. $C^*(1_C) = \emptyset$
- iii. $\bigcap_{i \in I} C^*(n_i) = C^*(\bigvee_{i \in I} [1_C \rightarrow n_i] 1_C)$, for some indexed set I
- iv. $C^*(n) \cup C^*(l) = C^*(n \wedge l)$

Proof. i. $C^*(0_C) = \{p \in \text{Spec}(\sigma, C) : [1_C \rightarrow 0_C] \subseteq [1_C \rightarrow p]\}$. Let $r \in [1_C \rightarrow 0_C]$, then $\sigma(r, 1_C) = 0_C$ and $0_C \leq p, \forall p \in (\sigma, C)$. Therefore, $[1_C \rightarrow 0_C] \subseteq [1_C \rightarrow p], \forall p \in (\sigma, C)$. Hence, $C^*(0_C) = \text{Spec}(\sigma, C)$.

ii. $C^*(1_C) = \{p \in \text{Spec}(\sigma, C) : [1_C \rightarrow 1_C] \subseteq [1_C \rightarrow p]\}$. And, $[1_C \rightarrow 1_C] = \{r \in L : \sigma(r, 1_C) \leq 1_C\} = L$. Since, $p \neq 1_C, C^*(1_C) = \emptyset$.

iii. Let $p \in \bigcap_{i \in I} C^*(n_i)$, then $p \in C^*(n_i), \forall i \in I$. Then, $[1_C \rightarrow n_i] 1_C \leq p, \forall i \in I$ implies that $\bigvee_{i \in I} [1_C \rightarrow n_i] 1_C \leq p$. Thus, $p \in C^*(\bigvee_{i \in I} [1_C \rightarrow n_i] 1_C)$.

Now, consider $p \in C^*(\bigvee_{i \in I} [1_C \rightarrow n_i] 1_C)$, then for any $j \in I, [1_C \rightarrow n_j] \subseteq [1_C \rightarrow [1_C \rightarrow n_j] 1_C] \subseteq [1_C \rightarrow \bigvee_{j \in I} [1_C \rightarrow n_j] 1_C] \subseteq [1_C \rightarrow p]$. Therefore, $p \in C^*(n_j), \forall j \in I$.

iv. For $n, l \in (\sigma, C)$, first we prove that $C^*(n) \cup C^*(l) \subseteq C^*(n \wedge l)$.

Let $p \in C^*(n \wedge l)$ then, $[1_C \rightarrow n \wedge l] \subseteq [1_C \rightarrow p] \Rightarrow [1_C \rightarrow n] \cap [1_C \rightarrow l] \subseteq [1_C \rightarrow p]$. Since, $[1_C \rightarrow p]$ is a prime ideal, either $[1_C \rightarrow n] \subseteq [1_C \rightarrow p]$ or $[1_C \rightarrow l] \subseteq [1_C \rightarrow p]$.

Therefore $p \in C^*(n) \cup C^*(l)$.

Now, let $p \in C^*(n) \cup C^*(l)$, i.e., $[1_C \rightarrow n] \subseteq [1_C \rightarrow p]$ or $[1_C \rightarrow l] \subseteq [1_C \rightarrow p]$. Clearly, $[1_C \rightarrow n \wedge l] \subseteq [1_C \rightarrow p]$, which implies $p \in C^*(n \wedge l)$. Hence the proof. □

Theorem 3.11. The collection $\gamma^* = \{C^*(n) : n \in (\sigma, C)\}$ forms a collection of closed sets for some topology Ω^* on $\text{Spec}(\sigma, C)$.

Proof. Follows from the above proposition. □

Definition 3.12. The topology Ω^* on $\text{Spec}(\sigma, C)$ with γ^* as the collection of closed sets is called the Zariski topology on $\text{Spec}(\sigma, C)$.

Definition 3.13. An element $a \in \text{Spec}(\sigma, C)$ is said to be maximal if it satisfies the conditions

- i. $a \leq b$ implies $a = b$ for $b \in \text{Spec}(\sigma, C)$
- ii. $[1_C \rightarrow a]$ is a maximal ideal of the locale L .

Theorem 3.14. For $x \in \text{Spec}(\sigma, C)$, $\{x\}$ is closed if x is a maximal element of (σ, C) .

Proof. Since $\{x\}$ is closed, we have $\{x\} = C^*(n)$ for some $n \in \text{Spec}(\sigma, C)$. That is, $[1_C \rightarrow x]$ is the only prime ideal containing $[1_C \rightarrow n]$. Hence $[1_C \rightarrow x]$ is a maximal ideal. Thus, $\{x\}$ is closed if x is a maximal element of (σ, C) . □

Theorem 3.15. For $x \in \text{Spec}(\sigma, C)$, $Cl(\{q\}) = C^*(q)$.

Proof. We have to show that the smallest closed set containing $\{q\}$ is $C^*(q)$. Obviously, $q \in C^*(q)$. Now let $q \in C^*(n)$ for some $n \in (\sigma, C)$. We prove $C^*(q) \subseteq C^*(n)$. Let $y \in C^*(q)$, then $[1_C \rightarrow q] \subseteq [1_C \rightarrow y]$. Also, $[1_C \rightarrow n] \subseteq [1_C \rightarrow q]$. Therefore, $y \in C^*(n)$. Thus, $Cl(\{q\}) = C^*(q)$. □

Theorem 3.16. For $p, q \in \text{Spec}(\sigma, C)$, $p \leq q$ implies $q \in Cl(\{p\})$.

Proof. $p \leq q$, implies $[1_C \rightarrow p] \subseteq [1_C \rightarrow q]$. Thus, $q \in C^*(p)$ implying $q \in Cl(\{p\})$. □

Remark 3.17. If every element of $\text{Spec}(\sigma, C)$ is maximal element then the singleton sets will be closed in Ω^* and hence $(\text{Spec}(\sigma, C), \Omega^*)$ will be a T_1 space.

4. Conclusion

Further study on the topology Ψ on $\text{Spec}_\wedge(\sigma, C)$ can be done. As a sequel to this article we intend to investigate the relation between $\text{Spec}(\sigma, C)$ and the spectrum of the locale L .

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