Tree domination polynomial of some graphs

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Abstract

The tree domination polynomial of a connected graph G of order n is the polynomial $D_{td}(G, x) = \sum_{i=1}^n d_{td}(G, i) x^i$, where $d_{td}(G, i)$ is the number of tree dominating sets of G of cardinality i and $\gamma_{td}(G)$ is the tree domination number of G . In this paper the td - polynomial of some graphs are studied.

Keywords

Domination, Tree domination, domination polynomial, Tree domination polynomial(td - polynomial).

AMS Subject Classification

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1. Introduction

The domination polynomial of a graph is introduced by Saeid Alikhani and Yee-hock Peng in [5]. While extending the concept of domination polynomial in view of tree dominating set, we came across with many interesting relations with the coefficients of the polynomial and the graph parameters. Also, the coefficients of the polynomial of some important class of graphs have attractive patterns and the roots of the polynomial have interesting nature. This paper includes the td - polynomial of some class of graphs.

Let $G = (V, E)$ be a simple graph. For any vertex $v \in V$, the *open neighbourhood* of v is the set $N(v) = \{u \in V : uv \in E\}$ and the *closed neighbourhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the *open neighbourhood* of S is $N(S) = \cup_{v \in S} N(v)$ and the *closed neighbourhood* of S is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a *dominating set* of G , if $N[S] = V$, or equivalently every vertex in $V \setminus S$ is adjacent to atleast one vertex in S . The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G . A dominat-

ing set with cardinality $\gamma(G)$ is called a γ -set. Let G be a simple connected graph of order n . A *tree dominating set* of G is a set S of vertices of G such that every vertex in $V \setminus S$ is adjacent to some vertex in S and the induced subgraph $\langle S \rangle$ is tree. The *tree domination number* $\gamma_{td}(G)$ is the minimum cardinality of a tree dominating set in G .

Saeid Alikhani and Yee-hock Peng introduced the concept of domination polynomial of a graph as the poly-

nomial $D(G, x) = \sum_{i=\gamma(G)}^n d(G, i) x^i$, where $d(G, i)$ denotes

the number of dominating sets of cardinality i .

Considering the polynomial idea of Alikhani et.al., we introduced the td - polynomial of a connected graph and study the information about the graph that we can obtain from the polynomial. For the basic concepts in graph theory we refer mainly Bondy and Murthy[1]. The graphs considered here are all connected and simple of order n .

2. Tree Domination Polynomial

As the elements of a td - set form a tree, the maximum cardinality of td - set may be n , and the minimum cardinality is $\gamma_{td}(G)$.

Definition 2.1. Let G be a connected graph of order n . The *tree domination polynomial* of G is the polynomial $D_{td}(G, x) =$

$\sum_{i=\gamma_{td}(G)}^n d_{td}(G, i)x^i$, where $d_{td}(G, i)$ is the number of tree dominating sets of G of cardinality i .

Example 2.2. Consider the path $P_3 = v_1 v_2 v_3$. Then the tree domination number is one and the tree dominating set of size one is $\{v_2\}$, the tree dominating set of size two are $\{v_1, v_2\}$, $\{v_2, v_3\}$, the tree dominating sets of size three is $\{v_1, v_2, v_3\}$. So that the polynomial is

$$x + 2x^2 + x^3.$$

A dominating set S is called a tree dominating set if the induced subgraph $\langle S \rangle$ is a tree.

Observations

Let G be a connected graph with n vertices then

1. If G contains a cycle then $d_{td}(G, n) = 0$. Because if we take all the vertices then the induced graph won't form a tree.
2. For any tree $d_{td}(G, n) = \text{The number of pendant(end) vertices(end)}$.
3. For any tree $d_{td}(G, n-1) = n-1$.
4. $D_{td}(G, x)$ has no constant term.
5. Zero is a root of $D_{td}(G, x)$ with multiplicity $\gamma_{td}(G)$.

Theorem 2.3. For any path P_n , $n \geq 3$ vertices, $D_{td}(P_n, x) = x^{n-2} + 2x^{n-1} + x^n$.

Proof. Let P_n be the path and $\{v_1, v_2, \dots, v_n\}$, $n \geq 3$. The tree domination number is $n-2$ and there is only one tree dominating set of order $n-2$. That means $d_{td}(P_n, n-2) = 1$. Also there are only two tree dominating sets of order $n-1$ namely $\{v_1, v_2, \dots, v_{n-1}\}$ and $\{v_2, v_3, \dots, v_n\}$. Therefore $d_{td}(P_n, n-1) = 2$ and there is only one tree dominating sets of order n .

This completes the proof. $\square$

Theorem 2.4. For any cycle C_n , $n \geq 3$ vertices, $D_{td}(C_n, x) = nx^{n-1} + nx^{n-2}$.

Proof. Let $C_n = \{v_1, v_2, \dots, v_n, v_1\}$, $n \geq 3$. The tree domination number is $n-2$. Removal of any one vertex from V will yield a tree dominating set of cardinality $n-1$ and removal of two adjacent vertices will give a tree dominating set of cardinality $n-2$ so that $d_{td}(C_n, n-1) = n$ and $d_{td}(C_n, n-2) = n$.

This completes the proof. $\square$

Theorem 2.5. For any complete graph K_n , $n \geq 3$, $D_{td}(K_n, x) = \binom{n}{1}x + \binom{n}{2}x^2$.

Proof. Let G be a complete graph with n vertices $\{v_1, v_2, \dots, v_n\}$ and $\gamma_{td}(K_n) = 1$. There are n tree dominating sets of order 1. Any set of three or more vertices induce a cycle i.e. $\gamma_{td}(K_n, m) = 0$, $m \geq 3$. Also as any two vertices induce a tree there are $\binom{n}{2}$ tree domination set of cardinality 2.

This completes the proof. $\square$

Theorem 2.6. For any star graph $K_{1,t}$, $t \geq 2$, $D_{td}(K_{1,t}, x) = x + \binom{t}{1}x^2 + \binom{t}{2}x^3 + \binom{t}{3}x^4 + \dots + \binom{t}{t-1}x^t + \binom{t}{t}x^{t+1} = x(1+x)^t$.

Proof. Let $K_{1,t}$ be a star graph with vertices $\{v, v_1, v_2, \dots, v_t\}$, where v is the centre vertex of the graph. Since $t \geq 2$ any tree dominating set must contain the centre vertex v , so that there is only one tree dominating set of order 1 namely $\{v\}$, there are $\binom{t}{1}$ tree dominating sets of cardinality 2, and so on. Therefore $D_{td}(K_{1,t}, x) = x + \binom{t}{1}x^2 + \binom{t}{2}x^3 + \binom{t}{3}x^4 + \dots + \binom{t}{t-1}x^t + \binom{t}{t}x^{t+1} = x(1+x)^t$. This completes the proof. $\square$

Theorem 2.7. The tree domination polynomial of $K_{m,n}$ for $1 < m \leq n$ is

$$D_{td}(K_{m,n}, x) = \binom{m}{1}\binom{n}{1}x^2 + \sum_{k=3}^{m+1} [\binom{m}{k-1}\binom{n}{k-1} + \binom{n}{k-1}\binom{m}{k-1}]x^k + \sum_{k=m+2}^{n+1} \binom{m}{1}\binom{n}{k-1}x^k.$$

Proof. $\gamma_{td}(K_{m,n}) = 2$. There are $\binom{m}{1}\binom{n}{1}$ tree dominating sets of cardinality 2, because we cannot choose both these vertices from the same bipartite set of vertices as it won't dominate the remaining vertices of the same bipartite set. There are $[\binom{m}{1}\binom{n}{2} + \binom{m}{2}\binom{n}{1}]$ choices for tree dominating sets of cardinality 3. If we choose two or more vertices from both bipartite sets then the induced subgraph of these vertices will form a cycle. So that a tree dominating set cannot contain two vertices of each bipartite set. Therefore at a time we can choose one vertex from either of the two bipartite set and the remaining ones from the other set of vertices(*). So there are $[\binom{m}{1}\binom{n}{k-1} + \binom{m}{k-1}\binom{n}{1}]$ choices of tree dominating sets of order k elements, for $k \leq m+1$. Now for $k \geq m+2$, we have to choose $k-1 \geq m+1$ vertices from the bipartite set of n elements and one vertex from the other set(because of *). That gives $\binom{n}{k-1}\binom{m}{1}$ tree dominating sets. Again by *, the number of vertices in a tree dominating set cannot exceed $n+1$. Note that for $1 < m \leq n$, the polynomial of $K_{m,n}$ can be expressed as $D_{td}(K_{m,n}, x) = \binom{m}{1}\binom{n}{1}x^2 + \sum_{k=3}^{m+1} [\binom{m}{k-1}\binom{n}{k-1} + \binom{n}{k-1}\binom{m}{k-1}]x^k + \sum_{k=m+2}^{n+1} \binom{m}{1}\binom{n}{k-1}x^k$ with an additional term x in equation (1).

But the expression is not true. However if $m = 1$, and $n > 1$ This completes the proof. $\square$

This completes the proof. $\square$

A complete m -partite graph $K_{n_1, n_2, \dots, n_m}$ is a graph whose vertex set can be partitioned into m non-empty sets V_i , $i = 1, 2, \dots, m$ with $|V_i| = n_i$ such that every vertex in V_i is adjacent to every vertex in V_j for every $i \neq j$ and $i, j \in \{1, 2, \dots, m\}$.

Theorem 2.8. For a complete m -partite graph $K_{n_1, n_2, \dots, n_m}$, with $1 < n_1 \leq n_2 \leq \dots \leq n_m$; $D_{td}(K_{n_1, n_2, \dots, n_m}, x) = \sum_{k=2}^{n_m+1} a_k x^k$, where $a_2 = n_1 \sum_{j>1} n_j + n_2 \sum_{j>2} n_j + \dots + n_{m-1} n_m = \sum_{i=1}^{m-1} (n_i \sum_{j>i} n_j)$ and $a_k = \sum_{i=1}^m [(\binom{n_i}{c_i}) (\sum_{j \neq i} (\binom{n_j}{c_{k-1}}))] \rightarrow (1)$, for $k > 2$; with the convention that $\binom{n_j}{c_k} = 0$ for $k > n_j$. If $n_i = 1$ for k vertices of i , then $D_{td}(G, x) = kx + \sum_{k=2}^{n_m+1} a_k x^k$, where $a_k, k > 2$ gives as in (1).

Proof. Since $m > 1$, any two vertices from two different partite sets form a tree domination set and $\gamma_{td}(K_{n_1, n_2, \dots, n_m}) = 2$. Also at a time we can't choose three or more vertices from three or more different partite sets because the induced subgraph of these vertices will form a cycle so it won't form a td -set. So that for a td -set of cardinality 2, we have $[\binom{n_1}{c_1} (\binom{n_2+n_3+\dots+n_m}{c_1}) + \binom{n_2}{c_1} (\binom{n_3+\dots+n_m}{c_1}) + \dots + \binom{n_{m-1}}{c_1} (\binom{n_m}{c_1})]$ choices. For td -sets of cardinality 3, we can choose one vertex from any of the partite sets namely V_j and the other two vertices both from one partite set other than V_j . Therefore there are $\binom{n_1}{c_1} [\binom{n_2}{c_2} + \binom{n_3}{c_2} + \dots + \binom{n_m}{c_2}] + \binom{n_2}{c_1} [\binom{n_1}{c_2} + \binom{n_3}{c_2} + \dots + \binom{n_m}{c_2}] + \dots + \binom{n_m}{c_1} [\binom{n_1}{c_2} + \binom{n_2}{c_2} + \dots + \binom{n_{m-1}}{c_2}]$ td -sets of cardinality 3. Similarly there are $\binom{n_1}{c_1} [\binom{n_2}{c_3} + \binom{n_3}{c_3} + \dots + \binom{n_m}{c_3}] + \binom{n_2}{c_1} [\binom{n_1}{c_3} + \binom{n_3}{c_3} + \dots + \binom{n_m}{c_3}] + \dots + \binom{n_m}{c_1} [\binom{n_1}{c_3} + \binom{n_2}{c_3} + \binom{n_3}{c_3} + \dots + \binom{n_{m-1}}{c_3}]$ td -sets of cardinality 4.

if $k > n_j + 1$ for $j = 1, 2, \dots, p$ we have to choose one vertex from a partite set and the remaining $k-1$ vertices all from another single partite set. As $k-1$ vertices cannot be chosen together from partite sets $V_1, \dots, V_p$; the coefficient of x^k is

$$\begin{aligned} & \binom{n_1}{c_1} \left[\sum_{j>p} \binom{n_j}{c_{k-1}} \right] + \dots + \binom{n_p}{c_1} \left[\sum_{j>p} \binom{n_j}{c_{k-1}} \right] \\ & + \binom{n_{p+1}}{c_1} \left[\sum_{j>p+1} \binom{n_j}{c_{k-1}} \right] + \binom{n_{p+2}}{c_1} \left[\sum_{j>p, j \neq p+2} \binom{n_j}{c_{k-1}} \right] + \dots + \\ & \binom{n_m}{c_1} \left[\sum_{p < j < m} \binom{n_j}{c_{k-1}} \right] = \sum_{i=1}^m \left[\binom{n_i}{c_1} \left(\sum_{j \neq i} \binom{n_j}{c_{k-1}} \right) \right], \text{ where } \binom{n_j}{c_{k-1}} = 0 \text{ for } n_j < k-1. \end{aligned}$$

Thus we can conclude that $a_k = \sum_i \left[\binom{n_i}{c_1} \left(\sum_{j \neq i} \binom{n_j}{c_{k-1}} \right) \right]$, for $k \geq 3$. Since $n_1 \leq n_2 \leq \dots \leq n_m$, the maximum cardinality of a td -set is $n_m + 1$; which completes the arguments. $\square$

Theorem 2.9. For a complete m partite graph $K_{n, n, \dots, n}$, where n repeats m times,

$$D_{td}(K_{n, n, \dots, n}, x) = m \{ [(nC_1)((m-1)nC_1)x^2 + [(nC_1)(nC_2)]x^3 + [(nC_1)(nC_3)]x^4 + \dots + [(nC_1)(nC_n)]x^{n+1}] \}$$

A friendship graph F_n is the one point union of n copies of the cycle C_3 . Equivalently $F_n = nP_2 \circ K_1$. When $n=1$, $F_n = K_3 = C_3$.

Theorem 2.10. For $n > 1$, $D_{td}(F_n, x) = \sum_{k=1}^{n+1} a_k x^k$, where $a_k = 2^{k-1} \binom{n}{k-1}$.

Proof. $\gamma_{td}(F_n) = 1$.

Let v be the apex vertex of F_n common to all C_n . Since $n > 1$, every td -set must contain the vertex v and no td -set can contain all the three vertices from same block of F_n . There is only one td -set of one element, $\{v\}$; so that $a_1 = 1$. Now, v together with any other vertex will form a td -set of 2 elements, and hence $a_n = 2n$.

In general, a td -set of k elements is obtained by taking v along with $k-1$ vertices from $k-1$ blocks of F_n . The $k-1$ blocks can be selected in $\binom{n}{k-1}$ ways, and for selecting one vertex from each block we have 2^{k-1} choices. So that $a_k = 2^{k-1} \binom{n}{k-1}$; which completes the proof. $\square$

A tadpole $T_{(n,l)}$, $n \geq 3, l \geq 1$ is a graph obtained by joining one of the end vertices of the path P_l to any one of the vertices of the cycle C_n by a bridge. The graph $T_{(n,l)}$ has $n+l$ vertices and $n+l$ edges.

Theorem 2.11. $D_{td}(T_{(n,l)}, x) = (n-2)x^{n+l-3} + (2n-3)x^{n+l-2} + (n-1)x^{n+l-1}$

Proof. Let v_1 be the vertex of the cycle $C_n = v_1 v_2 \dots v_n v_1$ to which the path $P_l = u_1 u_2 \dots u_l$ is attached. Since v_1 is the only vertex connecting C_n and P_l , v_1 should be a member of every td -set of a tadpole graph. At least $n-2$ vertices are required from $\{v_1, \dots, v_n\}$ to form a tree dominating set and all the n vertices cannot be included. So that from $\{v_1, \dots, v_n\}$, we have to choose either $n-2$ vertices including v_1 or $n-1$ vertices including v_1 . There are $n-2$ choices for choosing $n-2$ vertices and $n-1$ choices for $n-1$ vertices.

Similarly if the vertex in P_l adjacent to v_1 is u_1 , then a td -set of $T_{(n,l)}$ must include either all the vertices $u_1, \dots, u_l$ or the $l-1$ vertices $u_1, \dots, u_{l-1}$.

Thus $T_{(n,l)}$ has td -sets of cardinalities

1. $n+l-3$ (when $n-l$ v_i 's and $l-1$ u_j 's),
2. $n+l-2$ (when $n-2$ v_i 's and l u_j 's or $n-1$ v_i 's and $l-1$ u_j 's and
3. $n+l-1$ (when $n-1$ v_i 's and l u_j 's)

There are $n-2$ choices in category (1), $n-2+n-1 = 2n-3$ choices in category (2) and $n-1$ choices in category (3). So the polynomial is $(n-2)x^{n+l-3} + (2n-3)x^{n+l-2} + (n-1)x^{n+l-1}$. This completes the proof. $\square$

The n -barbell graph $B_{n,1}$ is a graph obtained by connecting two copies of complete graph K_n by a bridge. $B_{1,1} = K_2 = P_2$ and $B_{2,1} = P_4$.

Theorem 2.12. For $n \geq 2$ $D_{td}(B_{n,1}, x) = x^2 + 2(n-1)x^3 + (n-1)^2 x^4$.

Proof. Since $n \geq 2$, $\gamma_{td}(B_{n,1}) = 2$. Let uv be the bridge connecting two K_n 's. Any td -set of a n -barbell graph contains the two vertices u and v . The only minimal td -set is $\{u, v\}$; so that the coefficient of x^2 in $D_{td}(B_{n,1}, x) = 1$. There are td -sets of 3 elements consisting of u, v and any other vertex; so

that the coefficient of x^3 is $2(n-1)$. The td -sets of cardinality 4 consisting of u, v and two other vertices from two different blocks. So the coefficient of x^4 is $(n-1)^2$. As more than two vertices from a block constitute a cycle, these are the only td -sets.

□

In the n -barbel graph $B_{n,1}$, two K'_n 's are joined by an edge. If we consider the graph K_n and K_m by an edge, the arguments similar to the above, show the following.

Theorem 2.13. *If G is the graph obtained by connecting two complete graphs K_m and K_n by an edge, where $m, n > 1$, then $D_{td}(G, x) = x^2 + (m+n-2)x^3 + (m-1)(n-1)x^4$.*

A bistar graph $B_{m,n}$ is obtained by connecting the center vertices of two star graphs $K_{1,m}$ and $K_{1,n}$ by an edge.

Theorem 2.14. *For $m, n \geq 1$, $D_{td}(B_{m,n}, x) = \sum_{k=2}^{m+n+2} a_k x^k$, where $a_k = \binom{m+n}{k-2}$.*

Proof. Let the center vertex of the two star graphs $K_{1,m}$ and $K_{1,n}$ be u and v respectively. Also any td -set of $B_{m,n}$ must contain the two vertices u and v otherwise the induced subgraph won't form a tree and $\gamma_d(B_{m,n}) = 2$. There is only one td -set of order 2 namely $\{u, v\}$. For td -sets of $k > 2$ elements, we can consider any k -subset of the vertex set containing u and v . As there are $m+n$ vertices other than u and v , we have $\binom{m+n}{k-2}$ td -sets of k elements; hence proved. This completes the proof. □

A bistar graph $B_{n,n}$ is the union of two star graphs $K_{1,n}$ with centers u and v together with a new edge uv .

In the bistar graph $B_{m,n}$, two star graphs $K_{1,n}$ and $K_{1,m}$ are joined by an edge. If we consider the graph obtained by connecting two star graphs $K_{1,n}$ by an edge, the arguments similar to the above show the following.

Theorem 2.15. *For a bistar graph $D_{td}(B_{n,n}, x) = \sum_{k=2}^{2n+2} a_k x^k$, where $a_k = \binom{2n}{k-2}$.*

A lollipop graph $L_{n,1}$ is obtained by joining K_n to a path of length 1 with a bridge.

Theorem 2.16. *For a lollipop graph $L_{n,1}$ $D_{td}(L_{n,1}, x) = x + nx^2 + (n-1)x^3$.*

Proof. Let the vertices of $L_{n,1}$ be $v, v_1, \dots, v_n$, where $\{v_1, \dots, v_n\}$ induces K_n and v is adjacent to v_1 . Any td -set of $L_{n,1}$ must include the vertex v_1 . The set $\{v_1\}$ is the only singleton td -set. We can't choose more than two vertices from K_n because the induced subgraph will contain a cycle and they won't form a td -set. So that the only td -sets of two elements is $\{v, v_1\}$, $\{v_1, v_i\}$; $i = 1, 2, \dots, n$. Therefore there are n td -sets of 2 elements. The td -sets of 3 elements must contain v and v_1 , exactly one vertex from $\{v_2, \dots, v_n\}$. So that td -sets of cardinality 3 is $n-1$. This completes the proof. □

The lollipop graph $L_{m,n}$ is a graph obtained by joining one vertex of a complete graph K_m to the end vertex of a path P_n with an edge. When $m = 1$, $L_{m,n}$ is the path P_{n+1} . For $m \geq 2$ we have the following result.

Theorem 2.17. *The tree domination polynomial of the lollipop graph $L_{m,n}$ for $m \geq 2$ is $x^n + mx^{n+1} + (m-1)x^{n+2}$.*

Proof. $\gamma_d(L_{m,n}) = n$. Let $\{v_1, v_2, \dots, v_m\}$ be the vertices of the complete graph K_m and $\{u_1, u_2, \dots, u_n\}$ be the vertices of the path P_n and $v_1 u_1$ be the edge joining them. The vertices $u_1, \dots, u_{n-1}$ and v_1 must be a part of every td -set, otherwise the induced graph will not be a tree. There is only one td -set of n elements namely $S = \{v_1, u_1, \dots, u_{n-1}\}$. The td -sets of cardinality $n+1$ are $S \cup \{u_n\}$, $S \cup \{v_j\}$; $j = 2, \dots, m$. Now any three vertices of K_m form a cycle; so that no td -set can contain more than 2 vertices from $\{v_1, \dots, v_m\}$. Thus the only td -sets of $n+2$ elements are $S \cup \{u_n, v_j\}$, $j = 2, \dots, m$. So we conclude that $D_{td}(L_{m,n}, x) = x^n + mx^{n+1} + (m-1)x^{n+2}$. This completes the proof. □

Note 2.18. *Theorem 2.16 is a particular case of the above theorem; $n = 1$.*

The bipartite cocktail party graph B_n is the graph obtained by removing a perfect matching from the complete bipartite graph $K_{n,n}$. This graph B_n is connected iff $n \geq 3$.

Theorem 2.19. $D_{td}(B_n, x) = \left[\binom{n}{2} \binom{n}{2} \right] x^4 + \left[2 \binom{n}{2} \binom{n}{3} \right] x^5$, for $n \geq 3$.

Proof. $\gamma_d(B_n) = 4$. We have to take 2 vertices from one set of vertices and 2 from other set of vertices otherwise it won't form a td -set. $d_{td}(B_n, 4) = \binom{n}{2} \binom{n}{2}$. Also $d_{td}(B_n, 5) = \binom{n}{2} \binom{n}{3} + \binom{n}{2} \binom{n}{3}$. Any set of more than 5 vertices doesn't induce a tree. $\therefore$ It won't form a td -set. This completes the proof. □

The windmill graph $W_n^{(m)}$ is obtained by taking m copies of K_n with a vertex in common.

Theorem 2.20. $D_{td}(W_n^{(m)}, x) = x + m(n-1)x^2 + \frac{1}{2!}m(m-1)(n-1)^2x^3 + \frac{1}{3!}m(m-1)(m-2)(n-1)^4 + \dots + (n-1)^m x^m = x(1 + (n-1)x)^m$.

Proof. If v is the vertex in common, then v should be a member of every td -set of $W_n^{(m)}$ otherwise the induced graph won't be connected. Also we know that the maximum number of elements in a td -set from a complete graph is 2. Here ' v ' is common to all K_n . So that any td -set can include at most one more vertex other than v . As there are m copies of K_n and $m(n-1)$ vertices other than v , we have $m(n-1)$ choices for choosing one vertex, $\frac{m(n-1)(m-1)(n-1)}{2!}$ choices for choosing two vertices from two different copies of $K_n - v$, $\frac{m(n-1)(m-1)(m-2)(n-1)}{3!}$ choices for choosing three vertices from three different $K_n - v$, and so on. Thus for $2 \leq k \leq m$,

the coefficient of x^k is $\frac{m(m-1)\dots(m-(k-2))(n-1)^{k-1}}{(k-1)!}$. Hence the result. $\square$

Dutch windmill graph $D_n^{(m)}$ is a windmill graph $W_n^{(m)}$ with $n=3$.

Theorem 2.21. $D_{td}(D_n^{(m)}, x) =$
 $x + [m \binom{2}{1}]x^2 + [m \binom{2}{1}][(m-1) \binom{2}{1}]x^3$
 $+ [m \binom{2}{1}][(m-1) \binom{2}{1}][(m-2) \binom{2}{1}]x^4 + \dots$
 $+ [m \binom{2}{1}][(m-1) \binom{2}{1}][(m-2) \binom{2}{1}] \dots [(m-(m-1)) \binom{2}{1}]x^{m+1}$

Proof. Let v be the common vertex of all K_3 . Now we consider all td -sets of order i , $1 \leq i \leq m+1$. **i=1:** There is only one td -set of order 1 namely $\{v\}$. **i=2:** One vertex should be ' v ' and the other one can be chosen from any of the K_3 ie; we have $m \binom{2}{1}$ choices exists in this category. So $\exists m(2C_1)$ td -sets of order 1. **i=3:** Here we have the freedom to choose 2 vertices from 2 different K_3 . $\therefore$ There are $[m \binom{2}{1}][(m-1) \binom{2}{1}]$ td -sets exists. When $i=4$ we have $[m \binom{2}{1}][(m-1) \binom{2}{1}][(m-2) \binom{2}{1}]$ td -sets exists. $\therefore$

Similarly continuing in this way we get, When $i = m+1$: There are $[m \binom{2}{1}][(m-1) \binom{2}{1}][(m-2) \binom{2}{1}] \dots [(m-(m-1)) \binom{2}{1}]$ td -sets exists.

This completes the proof. $\square$

There are certain class of graphs having no td - set. One such class is that of armed crown graphs. An armed crown $C_n \odot P_m$ For $n > 2$, $C_n \odot P_m$ cannot have a td - set because, all vertices in C_n are to be included for connection, but then the induced graph can never be a tree. For any graph G and for any $n \geq 3$, the corona graphs $K_n \circ G$ and $C_n \circ G$ and in general $H \circ G$, for a graph H containing a cycle, all belong to this category. In fact it is clear that the corona $H \circ G$ have a td - set if and only if H is a tree.

The td - polynomial of the corona of a tree G with a complete graph K_m can be find out by using the arguments used to prove theorem 2.20. Then 2.20 deals with the windmill graph $W_n^{(m)}$ which is m copies of K_n with a common vertex, the common vertex is in every td - set. Here all the vertices of G are in every td - set, and at each vertex we have a copy of K_{n+1} .

Theorem 2.22. For any tree G with n vertices, $D_{td}(G \circ K_1, x) = (x^2 + x)^n$

Proof. Let $H = t_n \circ K_1$. Then obviously $\gamma_{td}(H) = n$ and it is clear that there are $\binom{n}{i}$ possibilities to extend the tree dominating set into td -sets of size $n+1$. $\therefore D_{td}(H, x) = x^n + \binom{n}{1}x^{n+1} + \binom{n}{2}x^{n+2} + \dots + x^{2n} = x^n[1 + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + x^n] = x^n[\sum_{k=0}^n \binom{n}{k}x^k] = (x^2 + x)^n$.

This completes the proof. $\square$

Theorem 2.23. If G is any tree with n vertices then $D_{td}(G \circ K_m, x) = x^n + \binom{nm}{1}x^{n+1} + [\binom{nm}{1} \binom{(nm-(m+1))}{1}]x^{n+2} + [\binom{nm}{1} \binom{(nm-(m+1))}{1} \binom{(nm-2(m+1))}{1}]x^{n+3} + \dots + [\binom{nm}{1} \binom{(nm-(m+1))}{1} \binom{(nm-2(m+1))}{1} + \dots + \binom{(nm-(m-1)(n+1))}{1}]x^{2n}$.

Theorem 2.24. For a tree G with n vertices $D_{td}(G \circ \overline{K_m}, x) = x^n(1+x)^{mn}$.

Proof. If $\overline{K_m}$ denote the complement of K_m , then the graph $G \circ \overline{K_m}$ is a graph with a star $K_{1,m}$ at each vertex v in G ; with v as center. If G is a tree on n vertices, then $G \circ \overline{K_m}$ on $n + nm$ vertices. The only γ_{td} set is the vertex set of G and all its supersets will become a td - set. Therefore there are $\binom{nm}{i}$ td - sets of cardinality $n+i$. Thus we have proved the theorem. For a tree G with n vertices $D_{td}(G \circ \overline{K_m}, x) = x^n(1+x)^{mn}$.

This completes the proof. $\square$

The joint sum of two cycles C_m and C_n is a new graph obtained by joining a vertex of C_n to a vertex of C_m with new edge.

Theorem 2.25. Let G be the joint sum of two cycles C_m and C_n .

Then $\gamma_{td}(G) = m + n - 4$ and the number of minimum tree dominating set of G is $mn - 2(m+n) + 4$.

Proof. Assume that G be the joint sum of two cycles C_m and C_n . It is clear that the two joint vertices from C_m to C_n must include in any minimum tree dominating set of G . Therefore the minimum tree dominating set of G is $m + n - 4$. This follows that $\gamma_{td}(G) = m + n - 4$.

Let u and v be the joint vertices of C_m and C_n respectively. Let $N(u) = \{u_1, u_2, v\}$ and $N(v) = \{v_1, v_2, u\}$. To select $m-2$ vertices from C_m and $n-2$ vertices from C_n to be minimum tree dominating set of G , u and v must belong to any minimum tree dominating set of G . So from C_m number of ways selected vertices is equal to the number of edges in C_m excluding the edges uu_1 and uu_2 . So there are $m-2$ choices to select vertices from C_m . Similarly there are $n-2$ choices to select vertices from C_n . Hence the number of minimum td -sets in G is $mn - 2(m+n) + 4$.

This completes the proof. $\square$

The following results are easy to prove.

Theorem 2.26. Let G be the joint sum of two cycles C_m and C_n . Then

1. $d_{td}(G, m+n-3) = (m-2)(n-1) + (m-1)(n-2)$.
2. $d_{td}(G, m+n-2) = (m-1)(n-1)$.

Theorem 2.27. Let G be the joint sum of two cycles C_m and C_n . Then

$$D_{td}(G, x) = (m-2)(n-2)x^{m+n-4} + [(m-2)(n-1) + (m-1)(n-2)]x^{m+n-3} + (m-1)(n-1)x^{m+n-2}.$$

Proof. The proof is very clear by Theorems 23 and 24. $\square$

Theorem 2.28. Let G be the joint sum of two copies of C_n . Then
 $D_{td}(G, x) = (n-2)^2 x^{2n-4} + 2(n-2)(n-1)x^{2n-3} + (n-1)^2 x^{2n-2}$.

If we evaluate the graph polynomial at a particular point can give some interesting information about the graph. Also there are many studies doing in this area.

It is clear that $D_{td}(G, 1)$ is the number of tree dominating sets of G , and $D_{td}(G, -1)$ is the difference between the number of tree dominating sets of even size and the number of tree dominating sets of odd size. In the results given below we got the value of td -polynomial of some standard graphs at -1 .

Proposition 2.29. For any path P_n , $n \geq 3$ vertices, $D_{td}(P_n, -1) = 0$.

Proof. The proof is clear from Theorem 2.3. □

Proposition 2.30. For a cycle C_n , $D_{td}(C_n, -1) = 0$.

Proof. The proof is followed by Theorem 2.4. Also by using the Theorem 4 we get the following result. □

Proposition 2.31. For any star graph $K_{1,t}$, $t \geq 2$ with $t+1$ vertices, $D_{td}(K_{1,t}, -1) = 0$.

Similarly by Theorem 2.5, we get the following result.

Proposition 2.32. For a complete graph K_n , $D_{td}(K_n, -1) = \frac{n(n-3)}{2}$.

Proposition 2.33. 1. For any tree t_n , $D_{td}(t_n \circ K_1, -1) = 0$.

2. For any tree t_n , $D_{td}(t_n \circ \overline{K_m}, -1) = 0$.

3. Conclusion

We have studied the td - polynomial of many important classes of graphs, and continue for other classes of graphs, and derived graphs, using graphs operations like join, corona, etc. In this area our main task is to relate the td - polynomial of the derived graph with that of its underlying graphs. Also we are trying to find whether we can extract the important features of a graph from the parameters related to its td -polynomial.

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