Independent tracking polynomial of a graph

Jalsiya M P¹* and Raji Pilakkat²

Abstract

Let $G(V, E)$ be a graph. For $v \in V(G)$, the *cycle trace* of v is defined as the set of all vertices $u \in V$ such that u and v belong to same cycle and is denoted by $T_G(v)$.

Two vertices u, v of a graph G are said to be trace independent if $u \notin T_G(v)$. A set S of vertices in a graph $G(V, E)$ is called a *trace independent set* if any two vertices of S are trace independent in G . A cycle tracing set of G which is also trace independent is called an independent cycle tracking set of G . The cardinality of such set with minimum number of vertices is called independent cycle tracking number and is denoted by $\tau_i(G)$. Let $T_i(G, j)$ be the family of independent cycle tracking sets of a graph G with cardinality j and let $t_i(G, j) = |T_i(G, j)|$. Then the independent tracking polynomial $T_i(G, x)$ of G is defined as

$$T_i(G, x) = \sum_{j=\tau_i(G)}^{|V(G)|} t_i(G, j)x^j$$

where $\tau_i(G)$ is the independent cycle tracking number of G . The roots of the polynomial $T_i(G, x)$ are called the independent tracking roots of G . We investigate in this paper properties of the independent cycle tracking polynomial and determine independent cycle tracking polynomial for some special graphs.

Keywords

Cycle tracking set, cycle tracking number $\tau(G)$, independent tracking polynomial $T_i(G, x)$.

AMS Subject Classification

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¹ Department of Mathematics, T M Government College, Tirur, Kerala-676502, India.

² Department of Mathematics, University of Calicut, Malappuram, Kerala-673635, India.

*Corresponding author: ¹ jalsiyajamal@gmail.com; ² rajiunical@rediffmail.com

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1. Introduction

The concept of cycle tracking set is introduced in [5] by Jalsiya and Raji Pilakkat. Let $G(V, E)$ be a graph. For $v \in V(G)$, the *cycle trace (simply trace)* of v is defined as the set of all vertices $u \in V$ such that u and v belong to same cycle of G and is denoted by $T_G(v)$. For $u, v \in V$, v is said to be *cycle traced (traced)* by u if $v \in T_G(u)$. A set S of vertices in a graph $G(V, E)$ is called a *cycle tracking set* if for every vertex $v \in V \setminus S$, there exists a vertex $u \in S$ such that $v \in T_G(u)$. A

cycle tracking set is a *minimal cycle tracking set* if no proper subset $S' \subset S$ is a cycle tracking set. The *cycle tracking number* $\tau(G)$ of a graph G is the minimum cardinality of a minimal cycle tracking set of G . The *upper cycle tracking number* $T(G)$ of a graph G is the maximum cardinality of minimal cycle tracking set of G . A cycle tracking set with minimum cardinality is called a τ -set of G .

A graph $G(V, E)$ is said to be *transitive tracking graph* if for every $u, v, w \in V(G)$, $w \in T_G(u)$ and $u \in T_G(v)$ imply $w \in T_G(v)$. A graph $G(V, E)$ is said to be *track connected* if for every pair of vertices $u, v \in V(G)$ there exist two internally disjoint paths connecting u and v . A vertex is said to be *trace free vertex* if it belongs to no cycle.

Through out this paper the letter G denotes a simple, undirected graph of order n .

2. Independent cycle tracking polynomial of a graph

In the paper [8] Jalsiya and Raji Pilakkat introduced the concept of independent cycle tracking set. Two vertices u, v of a graph G are said to be trace independent if $u \notin T_G(v)$. A set S of vertices in a graph $G(V, E)$ is called a *trace independent set* if any two vertices of S are trace independent in G . A trace independent set is a *maximal trace independent set* if no proper superset $S' \supset S$ is a trace independent set. If a cycle tracking set S is trace independent then S is called independent cycle tracking set. The minimum cardinality of an independent cycle tracking set of G is the *independent cycle tracking number* and is denoted by $\tau_i(G)$. The maximum cardinality of independent cycle tracking set of G is called the *upper independent cycle tracking number* and is denoted by $T_i(G)$.

Definition 2.1. Let $T_i(G, j)$ be the family of independent cycle tracking sets of a graph G with cardinality j and let $t_i(G, j) = |T_i(G, j)|$. Then the independent tracking polynomial $T_i(G, x)$ of G is defined as

$$T_i(G, x) = \sum_{j=\tau_i(G)}^{|V(G)|} t_i(G, j)x^j$$

where $\tau_i(G)$ is the independent cycle tracking number of G . The roots of the polynomial $T_i(G, x)$ are called the independent tracking roots of G .

Remark 2.2. As $t_i(G, j) = 0$ for $j > T_i(G)$, the upper independent cycle tracking number of G , the independent tracking polynomial $T_i(G, x)$ of G , in fact is,

$$T_i(G, x) = \sum_{j=\tau_i(G)}^{T_i(G)} t_i(G, j)x^j$$

where $T_i(G)$ is the independent cycle tracking number of G as $t_i(G, j) = 0$ for $j > T_i(G)$.

The path P_3 on three vertices has only one independent cycle tracking set with cardinality 3 ($\tau_i(G) = 3$) its independent tracking polynomial is then $T_i(P_3, x) = x^3$. Similarly every independent cycle tracking set of C_n contains one and only one vertex and for every vertex $v \in V$, $\{v\}$ forms an independent cycle tracking set. So $T_i(C_n, x) = nx$.

Theorem 2.3. Let the graph $G = G_1 \cup G_2$ be the union of two graphs with disjoint vertex set. Then $T_i(G, x) = T_i(G_1, x)T_i(G_2, x)$.

Proof. For $k \geq \tau_i(G)$, an independent cycle tracking set of k vertices in G arises by choosing an independent cycle tracking set of p vertices in G_1 (for some p such that $\tau_i(G_1) \leq p \leq |V(G_1)|$) and an independent cycle tracking set of $k - p$ vertices in G_2 . The number of ways of doing this over all $j = \tau_i(G_1), \dots, |V(G_1)|$ is exactly the coefficient of x^k in $T_i(G_1, x)T_i(G_2, x)$. So $T_i(G, x) = T_i(G_1, x)T_i(G_2, x)$. $\square$

Corollary 2.4. If a graph G consist of m components $G_1, G_2, \dots, G_m$ then $T_i(G, x) = T_i(G_1, x)T_i(G_2, x) \dots T_i(G_m, x)$.

Proof. We prove this result by mathematical induction on m . For $m = 1$ the result is trivial. The case $m = 2$ holds by theorem 2.3. Suppose that the result is true for $m = k$. Now we have to prove that the result is true for $m = k + 1$,

So let G consist of $k + 1$ components $G_1, G_2, \dots, G_{k+1}$. Then $T_i(G, x) = T_i(G_1 \cup G_2 \cup \dots \cup G_k, x)T_i(G_{k+1}, x)$ by theorem 2.4.
 $= T_i(G_1, x)T_i(G_2, x) \dots T_i(G_{k+1}, x)$, by induction hypothesis. $\square$

Theorem 2.5. Let G be a graph of order n then

1. if $i < \tau_i(G)$ or $i > T_i(G)$, then $t(G, i) = 0$
2. $T_i(G, x)$ has no constant term.
3. $T_i(G, x)$ is a strictly increasing function on $(0, \infty)$
4. zero is a root of $T_i(G, x)$ with multiplicity $\tau_i(G)$

Theorem 2.6. [8] Let G be a graph of order n . Then $\tau_i(G) = n$ if and only if G is a forest.

Theorem 2.7. Let G be a graph of order n . Then $T(G, x) = x^n$ if and only if G is a forest.

Proof. The independent cycle tracking polynomial of G is x^n if and only if $V(G)$ is the only cycle tracking set for G . That is if and only if $\tau_i(G) = n$. That is if and only if G is a forest (by Theorem 2.6) $\square$

Theorem 2.8. For a graph G of order n , $T_i(G, x) = nx$ if and only if G is track connected.

Proof. suppose $T_i(G, x) = nx$. Then every independent cycle tracking set contains only one vertex. That is every vertex in G traces all other vertices and hence the graph is track connected.

Conversely suppose that G is track connected. Then $T_i(G) = 1$ and every vertex trace all other vertices. So $T_i(G, x) = nx$ $\square$

Definition 2.9. [5]

Let G be a graph with exactly one cut vertex. Let v be the cut vertex of G and $G_1, G_2, \dots, G_k$ be the components of $G \setminus \{v\}$. If the order of G_i is greater than or equal to three and the graphs induced by $V(G_i) \cup \{v\}$, $i = 1, 2, \dots, k$ are track connected then G is called a track connected floral graph. And the graph induced by $V(G_i) \cup \{v\}$, $i = 1, 2, \dots, k$ are called petals of G .

Theorem 2.10. Let G be a track connected floral graph with k petals having $m_1, m_2, \dots, m_k$ vertices respectively then $T_i(G, x) = x + m_1 m_2 \dots m_k x^n$

Proof. Let G be a track connected floral graph with k petals with each petal having $m_1, m_2, \dots, m_k$ vertices respectively. Then every maximal trace independent set contains exactly 1 or n vertices. Here the central vertex is the only vertex that can trace all other vertices and, a set with one vertex from each petal form an independent cycle tracking set. So there are $m_1 m_2 \dots m_k$ trace independent sets which contain n vertices. Therefore $T_i(G, x) = x + m_1 m_2 \dots m_k x^n$. $\square$

Theorem 2.11. Let G be a transitively tracked and let $V(G)$ be partitioned into $V_1, V_2, \dots, V_k$ of cardinality $m_1, m_2, \dots, m_k$ respectively such that the graph $\langle V_i \rangle$ induced by each V_i is maximal track connected subgraph of G . Then $T_i(G, x) = m_1 m_2 \dots m_k x^k$.

Proof. G be a transitively tracked graph then its vertex set can be partitioned into $V_1, V_2, \dots, V_k$ of cardinality, say $m_1, m_2, \dots, m_k$ respectively such that the graph $\langle V_i \rangle$ induced by each V_i is maximal track connected subgraph of G . Then the independent cycle tracking set of G are precisely the sets of vertices which contains exactly one element from each V_i . So an independent cycle tracking set of G with cardinality k can be chosen in $m_1 m_2 \dots m_k$ ways. Therefore $T_i(G, x) = m_1 m_2 \dots m_k x^k$. $\square$

Theorem 2.12. For a graph G , $t_i(G, 1) = 1$ if and only if G is a track connected floral graph.

Proof. Let G be any graph with $t_i(G, 1) = 1$. Then there exists one and only one independent cycle tracking set S with $|S| = 1$. That is there exist a vertex $v \in V$ such that $T_G(v) = V$ and no other vertex can trace G . And since $\tau_i(G) = 1$, G must be a track connected floral graph [5]. $\square$

3. Independent cycle tracking polynomial of some graphs

Definition 3.1. A firefly graph $F_{s,t,n-2s-2t-1}$ ($s \geq 0, t \geq 0$ and $n-2s-2t-1 \geq 0$) is a graph of order n that consists of s triangles, t pendent paths of length 2 and $n-2s-2t-1$ pendant edges sharing a common vertex.

Theorem 3.2. $\tau_i(F_{s,t,n-2s-2t-1}) = n - 2s$.

Proof. The graph $F_{s,t,n-2s-2t-1}$ has $n - 2s - 1$ trace free vertices and the common vertex traces all s triangles. So the $n - 2s - 1$ trace free vertices together with the common vertex form a τ_i -set. Hence $\tau(F_{s,t,n-2s-2t-1}) = n - 2s$. $\square$

Theorem 3.3. $t(F_{s,t,n-2s-2t-1}, p) = \begin{cases} 1 & \text{if } p = n - 2s \\ 2^s & \text{if } p = n - s - 1 \\ 0 & \text{otherwise} \end{cases}$

Proof. Let S be an independent cycle tracking set of $F_{s,t,n-2s-2t-1}$. Then S contain all the $n - 2s - 1$ trace free vertices. Suppose v be the common vertex shared by the triangles, the pendant paths of length 2 and the pendant edges.

Case(i) $v \in S$. Then the other vertices in the s triangles does not belong to S . So $|S| = n - 2s$ and S has only one choice.

Case(ii) $v \notin S$. Then S contains exactly one vertex from each triangle other than v . So $|S| = n - s - 1$ and there are 2^s choices for S

Hence $t(F_{s,t,n-2s-2t-1}, p) = \begin{cases} 1 & \text{if } p = n - 2s \\ 2^s & \text{if } p = n - s - 1 \\ 0 & \text{otherwise} \end{cases}$ $\square$

Theorem 3.4. $T(F_{s,t,n-2s-2t-1}, x) = x^{n-2s} + 2^s x^{n-s-1}$

Definition 3.5. A Lollipop graph $L_{n,m}$ is obtained by joining K_n to a path P_m of length m with a bridge.

Theorem 3.6. $\tau_i(L_{n,m}) = m + 1$

Proof. Since a vertex in K_n can trace all vertices in it and all vertices of P_m are trace free vertices we need atleast $m + 1$ vertices to trace $L_{n,m}$ and hence $\tau_i(L_{n,m}) = m + 1$. $\square$

Theorem 3.7. $T_i(L_{n,m}, x) = nx^{m+1}$

Proof. Let S be an independent cycle tracking set of $L_{n,m}$. Then S contain all the m trace free vertices and exactly one vertex from K_n . So $|S| = m + 1$ and S can be chosen in n ways.

So $t(L_{n,m}, p) = \begin{cases} n & \text{if } p = m + 1 \\ 0 & \text{otherwise} \end{cases}$.

Hence $T_i(L_{n,m}, x) = nx^{m+1}$. $\square$

Definition 3.8. A Tadpole $T_{(n,l)}$ is a graph obtained by attaching a path P_l to one of the vertices of the cycle C_n by a bridge.

Theorem 3.9. $\tau_i(T_{(n,l)}) = l + 1$

Proof. Since a vertex in C_n can trace all vertices in it and all vertices of P_l are trace free vertices we need at least $m + 1$ vertices to trace $T_{(n,l)}$ and these $m + 1$ vertices are independent. Hence $\tau_i(T_{(n,l)}) = l + 1$ $\square$

Theorem 3.10. $T_i(T_{(n,l)}, x) = nx^{l+1}$

Proof. Let S be an independent cycle tracking set of $T_{(n,l)}$. Then S contain all the l trace free vertices and exactly one vertex from C_n . So $|S| = l + 1$ and S can be chosen in n distinct ways. So,

$t(T_{(n,l)}, p) = \begin{cases} n & \text{if } p = l + 1 \\ 0 & \text{otherwise} \end{cases}$.

Hence $T_i(T_{(n,l)}, x) = nx^{l+1}$. $\square$

Definition 3.11. A helm graph, denoted by H_n is obtained from the Wheel W_n by joining a pendant vertex to each vertex in the outer circle of W_n by means of an edge.

Theorem 3.12. $\tau_i(H_n) = n$

Proof. Since H_n contains $n - 1$ pendant vertices, all these vertices belong to every independent cycle tracking set. Since W_n is track connected, a vertex of W_n together with the pendant vertices form an independent cycle tracking set. So $\tau(H_n) = n$. $\square$

Theorem 3.13. $T_i(H_n, x) = nx^n$

Proof. Let S be an independent cycle tracking set of H_n . Then S contain all the $n - 1$ trace free vertices and exactly one vertex from W_n . So $|S| = n$ and S can be chosen in n distinct ways. So,

$t(H_n, p) = \begin{cases} n & \text{if } p = n \\ 0 & \text{otherwise} \end{cases}$.

Hence $T_i(H_n) = nx^n$. $\square$

Definition 3.14. A web graph $WB_n, n > 3$ is obtained by joining the pendent vertices of a helm H_n to form a cycle and then adding a single pendent edge to each vertex of this outer cycle. WB_n has $3n - 2$ vertices and $3(n - 1)$ edges.

Theorem 3.15. $\tau_i(WB_n, x) = n$

Theorem 3.16. $T_i(WB_n) = (2n - 1)x^n$

Definition 3.17. A friendship graph F_n is the one point union of n copies of the cycle C_3 .

Since F_n is a track connected floral graph with n petals each having 3 vertices, we have:

Theorem 3.18. $\tau_i(F_n) = 1$

Theorem 3.19. $T(F_n, x) = x + 2^n x^n$

Definition 3.20. An armed crown $C_n \odot P_m$ is a graph obtained by attaching the path P_m to every vertex of the cycle C_n .

Theorem 3.21. $\tau_i(C_n \odot P_m) = mn + 1$

Proof. Since a vertex in C_n can trace all vertices in it and the remaining mn vertices are trace free vertices. So we need at least $mn + 1$ independent vertices to trace $C_n \odot P_m$ and hence $\tau(C_n \odot P_m) = mn + 1$ $\square$

Theorem 3.22. $T_i(C_n \odot P_m) = nx^{mn+1}$

Definition 3.23. [4] The corona of two graphs G_1 and G_2 is the graph $G = G_1 \odot G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 , where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2

Theorem 3.24. For any graph G of order n , $\tau_i(G \odot K_1) = n + \tau(G)$.

By Theorem 2.3 we have the following theorem

Theorem 3.25. $T_i(G \odot K_1, x) = T_i(G, x)x^n$

Theorem 3.26. $\tau_i(G_n \odot H_m) = n$ where G_n denote a graph with n vertices and H_m denote a connected graph with $m(m > 1)$ vertices.

Corollary 3.27. If G denote a connected graph with $m(m > 1)$ vertices. Then $\tau_i(K_1 \odot G) = 1$

Theorem 3.28. Let G be a connected graph with $m(m > 1)$ vertices. Then $T_i(K_1 \odot G, x) = (m + 1)x$.

Proof. Since $K_1 \odot G$ is a track connected graph with $m + 1$ vertices, $T(K_1 \odot G, x) = (m + 1)x$ $\square$

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