Locating number of a tree

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Abstract
In this paper we study about the properties of locating number of a graph. In [1] the locating number of a complete graph, a path graph and bound of locating number of a graph is carried out. Based on this we established the locating number of a tree, cycle and some of its properties.

Keywords
Locating code, locating set, minimum locating set, locating number, locating vertex.

AMS Subject Classification
04A72, 05CXX.

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Article History: Received 24 January 2019; Accepted 24 May 2019

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1. Introduction
Consider a certain facility consists of 4 rooms $R_1, R_2, R_3, R_4$ as shown in Fig 1.

![Figure 1](image)

Figure 1. A facility consisting of 4 rooms

Now if we place a certain (red) sensor in one of the rooms. Then the sensor is able to detect the distance from the room with the red sensor to the room containing the fire when the fire took place.

For example, if the sensor is placed in $R_1$ when fire occurs in $R_3$, then the sensor alerts us that a fire has occurred in a room at distance two from $R_3$. ie, it informs us that the fire is in $R_3$ since $R_3$ is the only room at distance 2 from $R_1$. If the fire is in $R_1$ then the sensor indicates that the fire has occurred in room at distance zero from $R_1$, ie, the fire is in $R_1$. If the fire is in any of the other 2 rooms, then the sensor tells us that there is a fire in a room at distances one from $R_1$. But this information is not enough to determine the precise room in which the fire has occurred. Since there is no room in which the (red) sensor can be placed to identify the exact location of a fire in every instance, we forced to use another (blue) sensor.

ie, if we place the red sensor in $R_1$ and a blue sensor in $R_2$, and a fire occurs in $R_4$, say, then the red sensor in $R_1$ tells us that there is a fire in a room at distance one from $R_1$, while the blue sensor tells us that the fire is in a room at distance one from $R_2$; ie, the ordered pair (1, 1) is produced for $R_4$. Since these ordered pairs are distinct for all rooms, the minimum number of sensors required to detect the exact location of any fire is two. We must take care that where the 2 sensors to be placed.

For example, we can’t place sensors in $R_1$ and $R_3$, since in this case the ordered pairs of $R_2$ and $R_4$ are all (1, 1) and so we can’t determine the precise of the fire.
This facility can be modeled by a graph $G$, whose vertices are the rooms and two vertices of this graph are adjacent if the corresponding 2 rooms are adjacent.

Therefore the graph becomes $G(V,E)$ with $V = \{R_1, R_2, R_3, R_4\}$ and $E = \{R_1R_2, R_2R_3, R_3R_4, R_4R_1, R_2R_4\}$. This gives rise to a problem involving graph.

### 2. Locating number of a graph

**Definition 2.1.** [1] Let $H$ be a connected graph. For an ordered set $X = \{x_1, x_2, \ldots, x_r\}$ of vertices of $H$ and a vertex $v$ of $H$, the locating code of $v$ w.r.t $X$ is the $r-$vector, $C_X(v) = (d(v,x_1), d(v,x_2), \ldots, d(v,x_r))$.

**Definition 2.2.** [1] The set $X$ is a locating set for $H$ if distinct vertices have distinct codes.

A locating set containing a minimum number of vertices is a minimum locating set for $H$. The locating number is the number of vertices in a minimum locating set for $H$. It is denoted as $\text{loc}(H)$.

**Theorem 2.3.** [1] A connected graph $G$ of order $n$ is isomorphic to $P_n$ if and only if $G$ has locating number one.

**Theorem 2.4.** [1] A connected graph $G$ of order $n \geq 2$ is isomorphic to $K_n$ if and only if $G$ has locating number $n - 1$.

**Definition 2.5.** [3] The degree of a vertex $v$ is the number of vertices at a distance one.

Let $G$ be a connected graph of order $n$ and let $V(G) = \{v_1, v_2, \ldots, v_n\}$.

Then the maximum degree of $G$ is denoted as $\Delta$ and it is defined as $\Delta = \max\{\deg(v_1), \deg(v_2), \ldots, \deg(v_n)\}$.

**Theorem 2.6.** [1] If $G$ be a non-trivial connected graph of order $n \geq 2$ then $\lceil \log_3(\Delta + 1) \rceil \leq \text{loc}(G) \leq n - d$, where $d$ be the diameter of $G$ and $\Delta$ be the maximum degree of $G$.

### 3. Locating Number of a Tree

**Proposition 3.1.** Let $G$ be a tree with $n \geq 3$ then $2 \leq d \leq n - 1$, where $d$ be the diameter of $G$ and $n$ be the number of vertices of $G$.

**Proof.** $d = 1$ is only possible if $n = 2 \Rightarrow 2 \leq d \leq n - 1$.(1) We know that a tree with maximum diameter is a path and diameter of a path, $d(P_n) = n - 1$.

Therefore $d \leq n - 1$..........(2)

Hence from (1) and (2), we get $2 \leq d \leq n - 1$.

**Definition 3.2.** Let $X$ be a minimum locating set of a connected graph $G$ and let $v \in V(G)$. If $v \in X$ then $v$ is called a locating vertex of $G$ w.r.t $X$.

**Theorem 3.3.** Let $G$ be a tree with $n \geq 3$ vertices and let $d$ be the diameter of $G$. If $v$ be a vertex of $G$ such that $d(u,v) = d$ for $u \in V(G)$ then $n(P_{d+1}) \leq \text{loc}(G) \leq n - 2$, where $P_{d+1}$ be the path of length $d$ from a fixed vertex $v$ passing through exactly one $w$ such that $(w,u)$ is an arc in $G$ with $\text{deg}(w)$ is maximum and $n(P_{d+1})$ be the number of $P_{d+1}$ path .

**Proof.** Let $G$ be a tree with $n \geq 3$ vertices. Let $v$ be a vertex of $G$ such that $d(u,v) = d$ for $u \in V(G)$, i.e., take $u$ and $v$ as eccentric nodes of $G$. And let $d$ be the diameter of $G$. Since $2 \leq d \leq n - 1$ (by(3.1)), it follows that $u_1, u_2, \ldots, u_{n-2}$ be the maximum possible end points of the tree except $v$. Then there exist paths from $v$ to each $u_i; 1 \leq i \leq n - 2$.

Next let $u_1, u_2, \ldots, u_r; 1 \leq r \leq n - 2$ are the only vertices of $G$ at a distance $d$ from $v$ to each $u_i$ passing through exactly one $w$ such that $(w,u)$ is an arc in $G$ with $\text{deg}(w)$ is maximum (ie, each $u_i$’s are eccentric nodes w.r.t $v$). Now fix $v_i$. Then there exists $r$ vertices in $N(v_i)$ at a distance $d$ from $v$ where there is an arc $(w,u)$ in $G$ with $\text{deg}(w)$ is maximum.

$\Rightarrow vu_1, vu_2, \ldots, vu_r$ are paths with $d + 1$ vertices passing through exactly one $w$ with maximum $\text{deg}(w)$ . ie, $n(P'_{d+1}) = r$..........(1)

And also we have $\text{loc}(P_n) = 1$ (by (2.3)). By assumption we have $w$ is an adjacent vertex of $u_1$ and let $w_j; \alpha < j \leq (n - 3)$ be the vertices between $v$ and $u_i (i \leq r)$ other than $w$ in each path $vu_i (0 < j \leq n - 3)$. Let $X_1$ be a minimum locating set for $G$.

Now if we let $v \in X_1$ then we can distinguish $v$ and at least one $w_j; \alpha < j \leq n - 3$ and if let $u_1 \in X_1$ then we can distinguish at least $w$ and $u_1$ ($w$, since $2 \leq d$). Now if there exist $w_j$ in $G$ and if we let both $v$ and $u_1$ are in $X_1$ then we can distinguish all $w_j; 1 \leq j \leq n - 3$ with $v, w$ and $u_1$. And if we let $u_2$ is also in $X_1$ then we can distinguish $v, w, u_1, u_2$ and all $w_j (0 < j \leq n - 3)$.

Similarly if we let $v, u_1, u_2, \ldots, u_{r-1} \in X_1$ then we can distinguish $v, w, u_1, u_2, \ldots, u_r$ and all $w_j (\alpha < j \leq n - 3)$. Since $r \leq n - 2$, if we take $r = n - 2$ then $X_1 = \{v, u_1, u_2, \ldots, u_{n-3}\}$, ie, we can distinguish all vertices with this $r$ vertices belongs to $X_1$. Also if $r = 1$ then we can distinguish $v, w, u_1$ and all $w_j (\alpha < j \leq n - 3)$ and $X_1 = \{v\}$ or $\{u_1\}$, ie, we can distinguish all.
vertices with 1 vertex.

And in all other cases $X_1$ may contains more than $r$ vertices w.r.t their intersections with different paths. For, if $r = n - 4$ and if one of $w_j$ has 2 more adjacent vertices $u_{r+1}$ and $u_{r+2}$ then $X_1$ becomes \{v, $u_1, u_2, \ldots, u_{r-1}, u_{r+1}$\} or \{v, $u_1, u_2, \ldots, u_{r-1}, u_{r+2}$\}. Therefore in this case $loc(G) = r+1$ and hence $r < loc(G)$.

Therefore as number of new vertices increases $loc(G)$ is either equal to $r$ or greater than $r$. 

$\Rightarrow r \leq loc(G)$. 

From (1) we get, $n(P_{d+1}) = r \leq loc(G)$......(2)

Also we have $loc(G) \leq n - d$ for any connected graph $G$ (by (2.6)) and for $n \geq 3$ we have $2 \leq d \leq n - 1$(by (3.1)). 

$\Rightarrow 1 \leq n - d \leq n - 2$. 

$\Rightarrow loc(G) \leq n - 2$ for $n \geq 3$.....(3)

Therefore from (2) and (3) we get $n(P_{d+1}) \leq loc(G) \leq n - 2$ for $n \geq 3$.

**Corollary 3.4.** Let $G$ be a tree of order $n \geq 3$. And let $P_{d+1}'$ be the path of length $d$ from a fixed vertex $v$ passing through exactly one $w$ such that $(w, u)$ is an arc in $G$ with deg($w$) is maximum. Then every path $P_{d+1}'$ of $G$ has at least one locating vertex w.r.t any minimum locating set $X$, where $d$ be the diameter of $G$.

This is also true in the case of $P_n, C_n$ and $K_n$ (it is clear from (2.3), (2.4), (3.3) and (3.9)).

**Proposition 3.5.** Let $G$ be a tree with $n$ vertices and let $d$ be the diameter of $G$, then $d = 2 \Leftrightarrow loc(G) = n - 2$.

**Proof.** If $d = 2$ then $n(P_3') = n - 2$. .... We have $n(P_{d+1}') \leq loc(G) \leq n - d$ (from (3.3)). 

$\Rightarrow n(P_{d+1}') \leq loc(G) \leq n - 2$. 

$\Rightarrow n - 2 \leq loc(G) \leq n - 2$. 

$\Rightarrow loc(G) = n - 2$.

Conversely, let $loc(G) = n - 2$. Since $2 \leq d \leq n - 1$ (by(3.1)), it follows that $1 \leq n - d \leq n - 2$.....(1)

From (3.3) we have $n(P_{d+1}') \leq loc(G) \leq n - d$.....(2)

Therefore from (1) and (2), we get $n - 2 \leq n - d \leq n - 1$. 

$\Rightarrow 1 \leq d \leq 2$. 

And we have $d = 1 \Leftrightarrow n = 2$ and $loc(G) = 1$.

Therefore $d = 1$ is not possible. Hence the only possibility is $d = 2$.

**Theorem 3.9.** For every $C_n$ with $n \geq 3$ has locating number 2.

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of $C_n$.

If we take $X_1 = \{v_1\}$ then the locating codes for the vertices of $C_n$ w.r.t $X_1$ are 

$0, 1, 2, \ldots, n/2, (n/2) - 1, \ldots, 2, 1$ respectively if $n$ is even and if $n$ is odd

$\Rightarrow loc(G) \leq n$. 

We know that $loc(G) = 0$ is not possible. Therefore $loc(G) = 1$.

Conversely suppose that $loc(G) = 1$. We have $loc(G) = 1 \Leftrightarrow G = P_n$ (by(2.3)) and diameter of $P_n$ is $n - 1$, so that $d = n - 1$.

Hence the proof.

**Proposition 3.7.** Let $G$ be a tree with $n$ vertices and let $d$ be the diameter of $G$. If $d = n - 2$ then $loc(G) = 2$.

**Proof.** Let $d = n - 2$ and we have $loc(G) \leq n - d$ (by(3.3)). 

$\Rightarrow loc(G) \leq n - (n - 2)$. 

$\Rightarrow loc(G) \leq 2$. 

$\Rightarrow loc(G) = 1$ or $loc(G) = 2$.

If $loc(G) = 1$ then $d = n - 1$ (by(3.6)), which is a contradiction to our assumption $d = n - 2$. Therefore $loc(G) = 2$ is the only possibility.

Hence $d = n - 2 \Rightarrow loc(G) = 2$.

**Proposition 3.8.** Let $G$ be a tree with $n$ vertices and $n < 6$ then $loc(G) = n - d$, where $d$ be the diameter of $G$.

**Proof.** If $n = 2$ then $d = 1$ is possible.

$\Rightarrow G = P_2$. 

Therefore $loc(G) = 1 = n - d$.

If $n = 3$ then $d = 2$ is only possible.

$\Rightarrow loc(G) = n - 2$ (by(3.5)) = $n - d$.

If $n = 4$ then $d = 3$. 

So if $d = 3 = n - 1$, $\Leftrightarrow loc(G) = 1$ (by(3.6)) = $n - d$.

And if $d = 2 = n - 2$ $\Rightarrow loc(G) = 2$ (by(3.7)) = $n - (n - 2) = n - d$.

If $n = 5$ then $d = 4, 3, 2$.

So if $d = 4 = n - 1$ $\Leftrightarrow loc(G) = 1$ (by(3.6)) = $n - (n - 1) = n - d$.

And if $d = 3 = n - 2$ $\Rightarrow loc(G) = 2$ (by(3.7)) = $n - (n - 2) = n - d$.

Also if $d = 2$ then $loc(G) = n - 2$ (by(3.5)) = $n - d$.

Hence the proof.

**Theorem 3.9.** For every $C_n$ with $n \geq 3$ has locating number 2.
0, 1, 2, ... , (n - 1)/2, (n - 1)/2, (n - 1)/2 - 1, ... , 2, 1 respectively.

Therefore it is not a locating set (since distinct vertices have same locating code).

Now consider \( X_2 = \{v_1, v_2\} \) where \( v_1 \) and \( v_2 \) are adjacent vertices then the locating codes for the vertices of \( C_n \) w.r.t \( X_2 \) becomes
\[
(0, 1), (1, 2), \ldots, (n/2 - 1, n/2), (n/2, n/2 - 1), \ldots, (2, 1), (1, 0)
\]
if \( n \) is even and
\[
(0, 1), (1, 2), \ldots, (n - 1)/2 - 1, (n - 1)/2, (n - 1)/2, \ldots, (2, 1), (1, 0)
\]
if \( n \) is odd respectively. Since distinct vertices have distinct locating code, \( X_2 \) becomes a locating set and it follows that \( X_2 \) is a minimum locating set.

Therefore \( \text{loc}(C_n) = 2 \) for every \( n \geq 3 \).

\[ \square \]

**Corollary 3.10.** Every adjacent vertices of \( C_n (n \geq 3) \) are locating vertices of \( C_n \) w.r.t some minimum locating set.

**Theorem 3.11.** [2] Let \( G \) be a connected graph of order \( n \geq 3 \). If \( \text{loc}(G) = 2 \), then \( 3 \leq n \leq 6 \), where \( n = |V(G)| \).

**Remark 3.12.** But the above theorem fails in the case of trees and cycles. Counter example for this:

1) Consider the graph \( G(V, E) \) with
\[
V = \{v_1, v_2, v_3, v_4, v_5, v_6\}
\]
and
\[
E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1\}.
\]
This is a tree with \( \text{loc}(G) = 2 \).

2) Next consider a graph \( G(V, E) \) with
\[
V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}
\]
and
\[
E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_7v_1\}.
\]
This is a cycle, \( C_7 \) with \( \text{loc}(G) = 2 \).

Therefore if \( G \) be a connected graph of order \( n \geq 3 \) and \( \text{loc}(G) = 2 \) then \( 3 \leq n \leq 6 \) is not true.

**Proposition 3.13.** Let \( G \) be a tree or \( K_n \) or \( C_n \) and if \( n = d + 2 \) then \( \text{loc}(G) = 2 \), where \( n \) be the vertices of \( G \) and \( d \) be the diameter of \( G \).

**Proof.** In case of a tree we have, if \( d = n - 2 \) then \( \text{loc}(G) = 2 \) (by (3.7)).

\[ \Rightarrow \] If \( n = d + 2 \) then \( \text{loc}(G) = 2 \).

In case of \( K_n \), \( K_3 \) is the one and only complete graph which satisfies \( n = d + 2 \). And we know that \( \text{loc}(K_n) = n - 1 \) (by (2.4)).

Therefore \( \text{loc}(K_3) = 3 - 1 = 2 \).

In case of \( C_n \), \( C_3 \) and \( C_4 \) are the only two cycles which satisfies \( n = d + 2 \). And since \( \text{loc}(C_n) = 2 \forall n \geq 3 \) (by (3.9)), \( \text{loc}(C_3) = \text{loc}(C_4) = 2 \).

Hence the proof.

\[ \square \]

**Acknowledgment**

This work was supported by Kerala State Council for Science, Technology and Environment with ref.no: 489/SPS63/2018.

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