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Locating number of a tree

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Abstract
In this paper we study about the properties of locating number of a graph. In [1] the locating number of a complete graph, a path graph and bound of locating number of a graph is carried out. Based on this we established the locating number of a tree, cycle and some of its properties.

Keywords
Locating code, locating set, minimum locating set, locating number, locating vertex.

AMS Subject Classification
04A72, 05CXX.

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1. Introduction

Consider a certain facility consists of 4 rooms R_1, R_2, R_3, R_4 as shown in Fig 1.

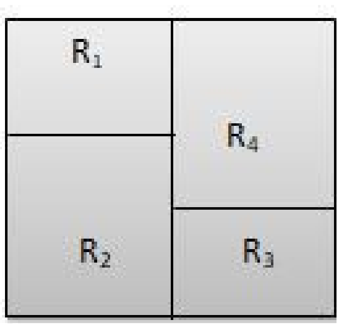

Figure 1. A facility consisting of 4 rooms

The distance between rooms R_1 and R_3 is 2 and the distance between all other pairs of distinct rooms is 1.The distance between a room and itself is zero.

Now if we place a certain (red) sensor in one of the rooms. Then the sensor is able to detect the distance from the room with the red sensor to the room containing the fire when the fire took place.

For example, if the sensor is placed in R_1 when fire occurs in R_3, then the sensor alerts us that a fire has occurred in a room at distance two from R_3 . ie, it informs us that the fire is in R_3 since R_3 is the only room at distance 2 from R_1. If the fire is in R_1 then the sensor indicates that the fire has occurred in room at distance zero from R_1. ie, the fire is in R_1. If the fire is in any of the other 2 rooms, then the sensor tells us that there is a fire in a room at distances one from R_1. But this information is not enough to determine the precise room in which the fire has occurred. Since there is no room in which the (red) sensor can be placed to identify the exact location of a fire in every instance, we forced to use another (blue) sensor.

ie, if we place the red sensor in R_1 and a blue sensor in R_2, and a fire occurs in R_4, say, then the red sensor in R_1 tells us that there is a fire in a room at distance one from R_1, while the blue sensor tells us that the fire is in a room at distance one from R_2; ie, the ordered pair $(1,1)$ is produced for R_4. Since these ordered pairs are distinct for all rooms, the minimum number of sensors required to detect the exact location of any fire is two. We must take care that where the 2 sensors to be placed.

For example, we can't place sensors in R_1 and R_3, since in this case the ordered pairs of R_2 and R_4 are all (1,1) and so we can't determine the precise of the fire.

This facility can be modeled by a graph G, whose vertices are the rooms and two vertices of this graph are adjacent if the corresponding 2 rooms are adjacent.

Therefore the graph becomes $G(V,E)$ with $V = \{R_1,R_2,R_3,R_4\}$ and $E = \{R_1R_2,R_2R_3,R_3R_4,R_4R_1,R_2R_4\}$. This gives rise to a problem involving graph.

2. Locating number of a graph

Definition 2.1. *[1] Let H be a connected graph. For an ordered set $X = \{x_1,x_2,.....x_r\}$ of vertices of H and a vertex v of H, the locating code of v w.r.t X is the $r-$ vector, $C_X(v) = (d(v,x_1),d(v,x_2),......d(v,x_r))$.*

Definition 2.2. *[1] The set X is a locating set for H if distinct vertices have distinct codes.*
A locating set containing a minimum number of vertices is a minimum locating set for H.
The locating number is the number of vertices in a minimum locating set for H. It is denoted as $loc(H)$.

Theorem 2.3. *[1] A connected graph G of order n is isomorphic to P_n if and only if G has locating number one.*

Theorem 2.4. *[1] A connected graph G of order $n \geq 2$ is isomorphic to K_n if and only if G has locating number $n-1$.*

Definition 2.5. *[3] The degree of a vertex v is the number of vertices at a distance one.*
Let G be a connected graph of order n and let $V(G) = \{v_1,v_2,......v_n\}$. Then the maximum degree of G is denoted as Δ and it is defined as
$\Delta = \max\{deg(v_1),deg(v_2),......,deg(v_n)\}$.

Theorem 2.6. *[1] If G be a non-trivial connected graph of order $n \geq 2$ then $[\log_3(\Delta+1)] \leq loc(G) \leq n-d$, where d be the diameter of G and Δ be the maximum degree of G.*

3. Locating Number of a Tree

Proposition 3.1. *Let G be a tree with $n \geq 3$ then $2 \leq d \leq n-1$., where d be the diameter of G and n be the number of vertices of G.*

Proof. $d = 1$ is only possible if $n = 2 \Rightarrow 2 \leq d$.........(1)
We know that a tree with maximum diameter is a path and diameter of a path, $d(P_n) = n-1$.
Therefore $d \leq n-1$.........(2)
Hence from (1) and (2), we get $2 \leq d \leq n-1$. □

Definition 3.2. *Let X be a minimum locating set of a connected graph G and let $v \in V(G)$. If $v \in X$ then v is called a locating vertex of G w.r.t X.*

Theorem 3.3. *Let G be a tree with $n \geq 3$ vertices and let d be the diameter of G. If v be a vertex of G such that $d(u,v) = d$ for $u \in V(G)$ then $n(P'_{d+1}) \leq loc(G) \leq n-2$, where P'_{d+1} be the path of length d from a fixed vertex v passing through exactly one w such that (w,u) is an arc in G with $deg(w)$ is maximum and $n(P'_{d+1})$ be the number of P'_{d+1} path .*

Proof. Let G be a tree with $n \leq 3$ vertices. Let v be a vertex of G such that $d(u,v) = d$ for $u \in V(G)$. ie,take u and v as eccentric nodes of G. And let d be the diameter of G. Since $2 \leq d \leq n-1$ (by(3.1)), it follows that $u_1,u_2,.....u_{n-2}$ be the maximum possible end points of the tree except v. Then there exist paths from v to each $u_i; 1 \leq i \leq n-2$.

Next let $u_1,u_2,......u_r; 1 \leq r \leq n-2$ are the only vertices of G at a distance d from v to each u_i passing through exactly one w such that (w,u) is an arc in G with $deg(w)$ is maximum (ie, each u_i's are eccentric nodes w.r.t v). Now fix v . Then there exists r vertices in $N(w)$ at a distance d from v where there is an arc (w,u) in G with $deg(w)$ is maximum.

$\Rightarrow vu_1,vu_2,......vu_r$ are paths with $d+1$ vertices passing through exactly one w with maximum $deg(w)$.
ie, $n(P'_{d+1}) = r$.......(1)

And also we have $loc(P_n) = 1$ (by (2.3)). By assumption we have w is an adjacent vertex of u_1 and let $w_j; o < j \leq (n-3)$ be the vertices between v and $u_i (i \leq i \leq r)$ other than w in each path $vu_i (0 < j \leq n-3)$. Let X_1 be a minimum locating set for G.

Now if we let $v \in X_1$ then we can distinguish v and at least one $wj(o < j \leq n-3)$ and if let $u_1 \in X_1$ then we can distinguish at least w and u_1 (w, since $2 \leq d$). Now if there exist w_j in G and if we let both v and u_1 are in X_1 then we can distinguish all $w_j; 1 \leq j \leq n-3$ with v,w and u_1. And if we let u_2 is also in X_1 then we can distinguish v,w,u_1,u_2 and all $wj(0 < j \leq n-3)$.

Similarly if we let $v,u_1,u_2,......u_{r-1} \in X_1$ then we can distinguish
$v,w,u_1,u_2,.....u_r$ and all $w_j(o < j \leq n-3)$. Since $r \leq n-2$, if we take $r = n-2$ then $X_1 = \{v,u_1,u_2,......u_{n-3}\}$. ie, we can distinguish all vertices with this r vertices belongs to X_1.
Also if $r = 1$ then we can distinguish v,w,u_1 and all $w_j(o < j \leq n-3)$ and $X_1 = \{v\}$ or $\{u_1\}$. ie, we can distinguish all

vertices with 1 vertex.

And in all other cases X_1 may contains more than r vertices w.r.t their intersections with different paths. For, if $r = n-4$ and if one of w_j has 2 more adjacent vertices u_{r+1} and u_{r+2} then X_1 becomes $\{v, u_1, u_2, ...u_{r-1}, u_{r+1}\}$ or $\{v, u_1, u_2, ...u_{r-1}, u_{r+2}\}$. Therefore in this case $loc(G) = r+1$ and hence $r < loc(G)$.

Therefore as number of new vertices increases $loc(G)$ is either equal to r or greater than r.
$\Rightarrow r \leq loc(G)$.
From (1) we get, $n(P'_{d+1}) = r \leq loc(G)$......(2)

Also we have $loc(G) \leq n-d$ for any connected graph G (by (2.6)) and for $n \geq 3$ we have $2 \leq d \leq n-1$(by (3.1)).
$\Rightarrow 1 \leq n-d \leq n-2$.
$\Rightarrow loc(G) \leq n-2$ for $n \geq 3$......(3)
Therefore from (2) and (3) we get $n(P'_{d+1}) \leq loc(G) \leq n-2$ for $n \geq 3$.

□

Corollary 3.4. *Let G be a tree of order $n \geq 3$. And let P'_{d+1} be the path of length d from a fixed vertex v passing through exactly one w such that (w,u) is an arc in G with $deg(w)$ is maximum. Then every path P'_{d+1} of G has at least one locating vertex w.r.t any minimum locating set X, where d be the diameter of G.*
This is also true in the case of P_n, C_n and K_n (it is clear from (2.3), (2.4),(3.3) and (3.9).

Proposition 3.5. *Let G be a tree with n vertices and let d be the diameter of G, then $d = 2 \Leftrightarrow loc(G) = n-2$.*

Proof. If $d = 2$ then $n(P'_3) = n-2$.... We have $n(P'_{d+1}) \leq loc(G) \leq n-d$ (from (3.3)).
$\Rightarrow n(P'_3) \leq loc(G) \leq n-2$.
$\Rightarrow n-2 \leq loc(G) \leq n-2$.
$\Rightarrow loc(G) = n-2$.

Conversely, let $loc(G) = n-2$. Since $2 \leq d \leq n-1$ (by(3.1)), it follows that $1 \leq n-d \leq n-2$.........(1)
From (3.3) we have $n(P'_{d+1}) \leq loc(G) \leq n-d$..........(2)
Therefore from (1) and (2), we get $n-2 \leq n-d \leq n-1$.
$\Rightarrow 1 \leq d \leq 2$.
And we have $d = 1 \Leftrightarrow n = 2$ and $loc(G) = 1$.
Therefore $d = 1$ is not possible. Hence the only possibility is $d = 2$.

□

Proposition 3.6. *Let G be a tree with n vertices and let d be the diameter of G, then $d = n-1 \Leftrightarrow loc(G) = 1$.*

Proof. Let $d = n-1$ and we have $loc(G) \leq n-d$ (by(3.3)).
$\Rightarrow loc(G) \leq n-(n-1)$.
$\Rightarrow loc(G) \leq 1$.
We know that $loc(G) = 0$ is not possible. Therefore $loc(G) = 1$.

Conversely suppose that $loc(G) = 1$. We have $loc(G) = 1 \Leftrightarrow G = P_n$ (by(2.3)) and diameter of P_n is $n-1$, so that $d = n-1$.
Hence the proof.

□

Proposition 3.7. *Let G be a tree with n vertices and let d be the diameter of G. If $d = n-2$ then $loc(G) = 2$.*

Proof. Let $d = n-2$ and we have $loc(G) \leq n-d$ (by(3.3)).
$\Rightarrow loc(G) \leq n-(n-2)$.
$\Rightarrow loc(G) \leq 2$.
$\Rightarrow loc(G) = 1$ or $loc(G) = 2$.
If $loc(G) = 1$ then $d = n-1$ (by(3.6)), which is a contradiction to our assumption $d = n-2$. Therefore $loc(G) = 2$ is the only possibility.
Hence $d = n-2 \Rightarrow loc(G) = 2$.

□

Proposition 3.8. *Let G be a tree with n vertices and $n < 6$ then $loc(G) = n-d$, where d be the diameter of G.*

Proof. If $n = 2$ then $d = 1$ is possible.
$\Rightarrow G = P_2$.
Therefore $loc(G) = 1 = n-d$.

If $n = 3$ then $d = 2$ is only possible.
$\Rightarrow loc(G) = n-2$ (by(3.5)) $= n-d$.

If $n = 4$ then $d = 3, 2$.
So if $d = 3 = n-1$, $\Leftrightarrow loc(G) = 1$ (by(3.6)) $= n-d$.
And if $d = 2 = n-2 \Rightarrow loc(G) = 2$ (by(3.7)) $= n-(n-2) = n-d$.

If $n = 5$ then $d = 4, 3, 2$.
So if $d = 4 = n-1 \Leftrightarrow loc(G) = 1$ (by(3.6)) $= n-(n-1) = n-d$.
And if $d = 3 = n-2 \Rightarrow loc(G) = 2$ (by(3.7)) $= n-(n-2) = n-d$.
Also if $d = 2$ then $loc(G) = n-2$ (by(3.5)) $= n-d$.
Hence the proof.

□

Theorem 3.9. *For every C_n with $n \geq 3$ has locating number 2.*

Proof. Let $v_1, v_2, ..., v_n$ be the consecutive vertices of C_n. If we take $X_1 = \{v_1\}$ then the locating codes for the vertices of C_n w.r.t X_1 are
$0, 1, 2, ..., n/2, (n/2)-1, ...2, 1$ respectively if n is even and if n is odd

$0, 1, 2, ..., (n-1)/2, (n-1)/2, (n-1)/2-1, ...2, 1$ respectively.
Therefore it is not a locating set (since distinct vertices have same locating code).

Now consider $X_2 = \{v_1, v_2\}$ where v_1 and v_2 are adjacent vertices then the locating codes for the vertices of C_n w.r.t X_2 becomes
$(0,1), (1,2), ...(n/2-1, n/2), (n/2, n/2-1), ...(2,1), (1,0)$ if n is even and $(0,1), (1,2), ...((n-1)/2-1, (n-1)/2), ((n-1)/2, (n-1)/2), ((n-1)/2, (n-1)/2-1), ...(2,1), (1,0)$ if n is odd respectively. Since distinct vertices have distinct locating code, X_2 becomes a locating set and it follows that X_2 is a minimum locating set.
Therefore $loc(C_n) = 2$ for every $n \geq 3$.

□

Corollary 3.10. *Every adjacent vertices of $C_n (n \geq 3)$ are locating vertices of C_n w.r.t some minimum locating set.*

Theorem 3.11. *[2] Let G be a connected graph of order $n \geq 3$. If $loc(G) = 2$, then $3 \leq n \leq 6$, where $n = |V(G)|$.*

Remark 3.12. *But the above theorem fails in the case of trees and cycles. Counter example for this;*

1) Consider the graph $G(V,E)$ with $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_4v_7\}$. This is a tree with $loc(G) = 2$.

2)Next consider a graph $G(V,E)$ with $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_7v_1\}$. This is a cycle, C_7 with $loc(G) = 2$.

Therefore if G be a connected graph of order $n \geq 3$ and $loc(G) = 2$ then $3 \leq n \leq 6$ is not true.

Proposition 3.13. *Let G be a tree or K_n or C_n and if $n = d+2$ then $loc(G) = 2$, where n be the vertices of G and d be the diameter of G.*

Proof. In case of a tree we have, if $d = n-2$ then $loc(G) = 2$(by(3.7)).
$\Rightarrow$ If $n = d+2$ then $loc(G) = 2$.

In case of K_n, K_3 is the one and only complete graph which satisfies $n = d+2$. And we know that $loc(K_n) = n-1$(by(2.4)).
Therefore $loc(K_3) = 3-1 = 2$.

In case of C_n, C_3 and C_4 are the only two cycles which satisfies $n = d+2$. And since $loc(C_n) = 2\ \forall n \geq 3$(by(3.9)), $loc(C_3) = loc(C_4) = 2$.
Hence the proof.

□

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