Injective chromatic number of lexicographic product of two graphs

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Abstract
The injective chromatic number of a graph $G$, denoted by $i(G)$, is the minimum number of colors needed to color the vertices of $G$ such that two vertices with a common neighbor are assigned distinct colors. In this paper $i(G[H])$ are determined, where $G[H]$ is the composition (or lexicographic product) of $G$ ($P_n$, $C_n$ and $K_n$) with any arbitrary graph $H$.

Keywords
injective chromatic number, lexicographic product.

AMS Subject Classification
05C15, 05C76.

1. Introduction
The basic definitions of graph theory are taken from [2]. The concepts of injective coloring and injective chromatic number $i(G)$ of a graph $G$ are introduced by G. Hahn et al [1]. The authors established some upper and lower bounds for injective chromatic number $i(G)$ and are obtained the injective chromatic number of the hypercubes also. In [5] B. Luzar et al obtained some results on injective coloring of planar graphs with large girth and few colors. Seog-Jin kim et al [3] proved that injective chromatic number of $G$ is at least half of the chromatic number of $G^2$, the square of $G$. The injective chromatic sum $\sum_i(G)$ and injective strength $s_i(G)$ are introduced by A. Kishore and M.S. Sunitha [4]. The authors obtained the injective chromatic sum of some classes of graphs, suggested bounds for injective chromatic sum and established injective chromatic sum of graph complements, join, union, product and corona. The concept of injective chromatic polynomial is also introduced in this paper. J. Song and J. Yue. [6] obtained some sharp bounds (or exact values) of injective chromatic number of Cartesian product, direct product, lexicographic product, union, join, and disjunction of graphs.

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2. Preliminaries
The concepts of injective coloring and injective chromatic number are introduced in 2002 by G. Hahn et al. and it is defined as follows:

Definition 2.1. [1] An injective coloring of a graph $G = (V, E)$ is a coloring of the vertices of $G$ that assigns different colors to any pair of vertices that have a common neighbor.

Definition 2.2. [1] The injective chromatic number $i_i(G)$ of a graph $G = (V, E)$ is the minimum number of colors needed to color the vertices of $G$ such that two vertices with a common neighbor are assigned distinct colors.

Definition 2.3. [2] The composition (or lexicographic product) $G = G_1[G_2]$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ has $V = V_1 \times V_2$ as its vertex set, and $u = (u_1, u_2)$ is adjacent with $v = (v_1, v_2)$ whenever $u_1$ adjacent $v_1$ or $u_1 = v_1$ and $u_2$ adjacent $v_2$.
3. Main Results

In this section the injective chromatic number of lexicographic products are obtained. For any two arbitrary graphs $G$ and $H$, the injective chromatic number of $K_n[G]$ is bounded, it is established in [6].

Remark 3.1. [6] Let $G$ and $H$ be graphs. Then $\chi_i(G[H]) \leq (\Delta + 1)\chi_i(H)$ and the bound is sharp.

The following theorem shows that the injective chromatic number of $K_n[G]$ is the total number of vertices of $K_n[G]$.

Theorem 3.2. For any connected graph $G$ with at least two vertices,

$$\chi_i(K_n[G]) = n|V(G)|, \quad n \geq 2.$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of $K_n$ and $u_1, u_2, \ldots, u_m$ be the vertices of $G$. The vertex set of $K_n[G]$ is $V = \{(v_1, u_1), (v_1, u_2), \ldots, (v_1, u_m), (v_2, u_1), (v_2, u_2), \ldots, (v_2, u_m), \ldots, (v_n, u_1), (v_n, u_2), \ldots, (v_n, u_m)\} = V_1 \cup V_2 \cup \ldots \cup V_n$. Note that any vertex $(v_i, u_j)$, $1 \leq i \leq n$ is a common neighbor of any two vertices in $V_i$, $1 \leq i, j \leq n$, $i \neq j$ and thus $m$ distinct colors are needed for the vertices in $V_i$. Since $G$ is a connected graph, $N(u_i) \neq \emptyset$, $1 \leq i \leq m$ and let $u_j \in N(u_i)$. Then there is a path $(v_i, u_j)(v_i, u_j)(v_i, u_j)$, $1 \leq i, j \leq n$, $i \neq j$ cannot have vertices with same colors. Thus each set $V_i$, $1 \leq i \leq n$ need distinct $m$ colors and hence $\chi_i(K_n[G]) = nm = n|V(G)|$. □

The injective chromatic number of lexicographic product of $P_n$ with any arbitrary connected graph $G$ is obtained as follows.

Theorem 3.3. For any connected graph $G$ with at least two vertices,

$$\chi_i(P_n[G]) = \begin{cases} 2|V(G)| & \text{for } n = 2, \\ 3|V(G)| & \text{otherwise}. \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of $P_n$ and $u_1, u_2, \ldots, u_m$ be the vertices of $G$. The vertex set of $P_n[G]$ is $V = \{(v_1, u_1), (v_1, u_2), \ldots, (v_1, u_m), (v_2, u_1), (v_2, u_2), \ldots, (v_2, u_m), \ldots, (v_n, u_1), (v_n, u_2), \ldots, (v_n, u_m)\} = V_1 \cup V_2 \cup \ldots \cup V_n$. Any vertex in $V_2$ is a common neighbor for the vertices in $V_1$, then $m$ distinct colors are needed to color the vertices in $V_1$. Let $i$ be the color of the vertices $(v_1, u_i)$, $1 \leq i \leq m$. Now consider the vertices in $V_2$. Since $G$ is a connected graph, $N(u_i) \neq \emptyset$ and let $u_j \in N(u_i)$, then $(v_2, u_j)(v_2, u_j)(v_2, u_j)$, $1 \leq i, k \leq m$ form a path of length 2. Thus any color of $(v_1, u_j)$ cannot be the color of vertices in $V_2$. Thus a new set of $m$ colors are needed to color the vertices in $V_2$. Let $m+i$ be the color of $(v_2, u_j)$ $1 \leq i \leq m$. Now consider the vertices in $V_3$. Here $(v_3, u_i)(v_3, u_j)(v_3, u_k)$, $1 \leq i, k \leq m$ form a path of length 2. Thus any color of $V_2$ cannot be the color of the vertices in $V_3$. Also $(v_3, u_j)(v_3, u_k)(v_1, u_k)$, $1 \leq i, j, k \leq m$ form a
total of $5n$ colors are needed for an injective coloring of $C_{3k+2}[G]$.

**Conclusion**

In this paper $\chi_i(G[H])$ are determined, where $G[H]$ is the composition (or lexicographic product) of two graphs $G$ and $H$. Where $G$ is $P_n$, $C_n$ or $K_n$ and $H$ is any arbitrary graph.

**References**


